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Analytic Continuation of the First Kind Associated Legendre Functions.

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Abstract:
The linear and the quadratic transformationsof the hypergeometric function are proven very useful in making various transformations and carryingout the analytic continuation of hypergeometric function into any part of the complex z-plane cut along the real axis from the point z= +1 to the point z= +∞. Here we shall represent the associated Legendre functions (or spherical functions) of the first kindin terms of the hypergeometric function to gain their analytic continuation into any part of the complex z-plane. Furthermore, the hypergeometric representation enables us to develop the theory of spherical functions by implementing the general theory of the hypergeometric function. Obtaining the hypergeometric representation of such functions by means of linear and quadratic transformations is more general and less complicated than the Euler's integral representation which is restricted to certain constraints to the values of the parameters of the hypergeometric function that are essential to make use of the integral definition of the Beta function.
Key words:*Analytic Continuation,Hypergeometricfunction,Hypergeometric series, First kindassociated Legendrefunctions, Spherical functions, Linear and quadratic transformationsof the hypergeometric function.*

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I. Introduction

Motivated by the great importance of special functions in general and the Legendre functions in particular, here we shall relate the associated Legendre functions to the so called Gaussianhypergeometric function (Abramowitz andStegun 1968;Andrews et al. 1999;Laham and Abdallah 1996;Rainville 1960;Lebedev1965;Wang and Chu2014; Wang and Chu2018). The hypergeometric function was introduced by Euler and then studied thoroughly by Gauss (Laham and Abdallah 1996) and plays an important role in mathematical analysis and its applications such as conformal mapping of triangular domains bounded by line segments or circular arcs(Lebedev1965).The Legendrefunctions have been discovered by Laplace and Legendre as early in the 18th century and they are connected with many problems ofmathematical physics, in the potential theory for spheroidal, toroidal andother coordinates (Hobson 1955).The associated Legendre functions of the first and second kinds (Kuipers and Meulenbeld 1957;Virchenko1987) possess high importance in variety of applications to problems of physics, quantum mechanics, and engineering. Many algebraic and transcendental functions that appear in problems of mathematical physics can be expressed in terms of the hypergeometric function, thus the theory of these functions can be considered as a special case of the general theory of the hypergeometric functions (Lebedev1965). The hypergeometric representation of the associated Legendre functions has the great advantage of obtaining the analytic continuation of these functions into any part of the complexz-plane (Lebedev1965). In turn this should allow variety of applications for such functions. The hypergeometric representation of any function can be achieved with the aid of the so called linear as well as nonlinear transformations on the independent variable for the hypergeometric function (Lebedev1965;Wang and Chu 2017). Such transformations were derived extensively in an elaborate manner in (Erdelyi et al.1953-55) and references therein. The linear transformations consist of all the following fractional linear form

$$z' = \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{R}.$$

Since the core idea of deriving the linear transformations comes from the theory of the twenty-four solutions of the hypergeometric differential equationdiscovered by Kummer in 1836, sometimes the linear transformations are called byKummer's relations (Rainville 1964). The nonlinear transformations contain expressions like

$$\frac{1-\sqrt{1-z}}{2}, \qquad \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}, \qquad \frac{1}{(1-z)^2}, \qquad z^2, \ldots$$

which are known as the quadratic transformations of the hypergeometric function. In fact the theory of quadratic transformations of the hypergeometric function is an old topic and can be traced back to Gauss,Kummer, and Goursat (Lebedev 1965). For an extensive list of such transformations the reader is referred to the references(Abramowitz andStegun 1968;Rainville 1964; Lebedev 1965;Erdelyi et al.1953-55).Conceptually, both kinds oftransformations are proven very useful in making various transformations and known as Euler's transformations of the hypergeometric function (Rainville 1964). It is worth mentioning that the linear and nonlinear transformations are among the most important relations in the theory of hypergeometric function. However there are other approaches to investigate the properties of the hypergeometric functions and obtain their analytic continuation.For instance, Hobson(1955)investigated properties of the associated Legendre functions by means ofcontour integralsdefined in terms of Pochhammer symbol (Abramowitz and Stegun1968)and Jordan double contour integrals(Verchenko and Rumiantseva 2008). Such an approach has the privilege of being away from the convergence issue of infinite series.The main results in the theory of the generalized associated Legendre functions have been establishedby Kuipers and Meulenbeld(1957).Also Verchenko and Rumiantseva(2008)considered the generalized hypergeometricfunction (Verchenko 1999;Verchenko et al. 2001;Raoa and Shukla 2013;Malovichko1976; Wang and Chu 2016)to gain an integral representation of the generalized associated Legendre functionsof both kinds (Kuipers and Meulenbeld1957;Verchenko and Rumiantseva 2008;Verchenko and Fedotova 2001). Verchenkoet al. implemented the integral form of the generalized (in the sense of Wright (Verchenko and Fedotova 2001)hypergeometric function which is defined in terms of the so called Fox-Wright functions (Verchenko and Fedotova 2001; Wright 1935) obtaining an integral form of thegeneralizedassociated Legendre functions. A further generalized hypergeometric k-functions is defined andsome properties are established in (Rahman et al. 2016; Miller 2003) using a special case of Wright hypergeometric function. Since the classical Gausshypergeometric function and the associated Legendre functions are respectively, special cases of the generalized hypergeometric function and the generalized associated Legendre functions, one could claim thatVerchenko andRumiantseva (2008) presented a more general approach using the generalization concept of such functions. The Euler's integral representation of the hypergeometricfunction(Rainville 1960) can be obtained using the integral definition of the Beta function(Abramowitz and Stegun1968). Such an integral form of the hypergeometricfunctions less general because is often restricted to certain constraints on the values of the parameters of the hypergeometric functionwhich are essential to make use of the integral definition of the Beta function. We will consider the hypergeometric representation of the considered functions only by means of linear and quadratic transformations of the hypergeometric function, referring the reader elsewhere for integral representations(Lebedev 1965).

This paper is structured as follows: in section one; we briefly set up the notations for the hypergeometric function. After obtaining the solutions of the associated Legendre differential equation in section two,hypergeometric representations of the first kind associated Legendre functions are presented in section four. Some useful linear transformations are derived in section three. Finally, a discussion and conclusion is drawn on the hypergeometric representation of the associated Legendre functionsin sectionsfive and six respectively.

II. Overview on the Gaussianhypergeometric function

In this section we shall introduce some notations that are used in this paper. Consider the series

$$1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)}\frac{z^n}{n!}, \tag{1}$$

where z is a complex variableα or β and γ are parameters, which can take arbitrary real or complex values provided that $\gamma \neq 0,-1,-2,\ldots$. If we let α=1 andβ = γ, then we get the elementary geometric series$\sum_{n=0}^{\infty} z^n$. The series (1) is called the Gauss hypergeometric series, which has great importance in mathematical analysis and its applications. Using the generalized factorial function (Abramowitz and Stegun1968)orPochhammer symbol $(a)_n$ defined as

$$(a)_n=\prod_{k=1}^{n}(a+k-1), (a)_0=1, \quad a\neq 0.$$

By using the Pochhammer symbol we can simplify the series(1) in the form

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}. \tag{2}$$

This series can be written in terms of the gamma function using the following relation between the Pochhammer symbol and the gamma function (Abramowitz and Stegun1968)defined as

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}, \qquad a\neq 0, \qquad n=0,1,2,\ldots \tag{3}$$

Hence, one has

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)\Gamma(n+1)}z^n .$$

By using the ratio test, it can be easily proved that the radius of convergence of the hypergeometric series (1)is unity $|z|<1$, except when the parameters α or β is zero or a negative integer, in which case the series (1)terminates and turns to a polynomial where the convergence has no sense. Also using the Gauss test, it can be shown that the hypergeometric seriesconverges absolutely for $|z|=1$provided that $\Re(\gamma-\alpha-\beta)>0$ (Rainville 1960;Lebedev 1965;Rainville 1964).We shall denote the convergent hypergeometric series by the notation $F(\alpha,\beta;\gamma;z)$ that is,

$$F(\alpha,\beta;\gamma;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}z^n,\qquad |z|<1,\qquad \gamma\neq 0,-1,-2,\dots$$

That is there exists a complex function which is analytic in the complex z-plane cutalong the real axis from the point z = 1 tothe point z = ∞and coincides with $F(\alpha,\beta;\gamma;z)$inside the unit disc. Moreover $F(\alpha,\beta;\gamma;z)$is an analytic function of its parametersαor β and ameromorphic function of its parameter γ, with simple poles at the points $\gamma\neq 0,-1,-2,\dots$(Lebedev 1965).

III. The associated Legendredifferential equation

The linear, second-order, homogeneous, spherical harmonic differential equation

$$(1-z^2)y''-2zy'+\left[\mu(\mu+1)-\frac{\upsilon^2}{2(1-z)}-\frac{\eta^2}{2(1+z)}\right]y(z)=0 ,$$

is called the generalized associated Legendre differential equation where z is a complex variable. The parameters ν and μ are arbitrary complex constants and called the order and the degree of the corresponding generalized associated Legendre functions, respectively. This equation arises from separation of variables in solving Laplace equation Δy = 0 in spherical coordinates. Kuipers and Meulenbeld(1957) obtained solutions of the generalized associated Legendre differential equation which are valid for unrestricted values of the parameters μ, η and ν. Their results have been expressed in terms of contour integrals and the hypergeometric functions. For some applications, it is often necessary to solve the associated Legendre differential equation for real values of the variable z in the interval (-1, +1), that is for z=cosθ, and for integral values of the parameters μ and ν. Hobson (1896) presented definitions of the associated Legendre functions for unrestricted values of the parameters ν, μ and the argument z by means of contour integrals. The generalized associated Legendre differential equationcan be reduced to the following classical associated Legendre differential equation aswe set the parametersη=ν=n, thus one hasfor arbitrary μ and nonnegative integral values of n,

$$(1-z^2)y''-2zy'+\left[\mu(\mu+1)-\frac{n^2}{1-z^2}\right]y(z)=0,\qquad n=0,1,2,\dots \tag{4}$$

This equationyieldsthe ordinary Legendre differential equation asn=0 and itis well known in mathematical physics to solve boundary value problems of potential theory, geodesy and quantum mechanics. To solve the differential equation (4), we assume that the variable z belongs to the complex z-plane cutalong the real axis from the point z = -∞ to the point z = 1,and introduce the following gauge transformation in terms of the new function *w(z)*which is related to the function by the following formula(Lebedev 1965; Bealsand Wong 2010),

$$y(z)=(z^2-1)^{\frac{n}{2}}w(z).$$

The derivatives ofthe function y are obtained as

$$y'=(z^2-1)^{\frac{n}{2}}w'+nzw(z^2-1)^{\frac{n}{2}-1},$$

$$y''=(z^2-1)^{\frac{n}{2}}w''+2nz(z^2-1)^{\frac{n}{2}-1}w'+nw(z^2-1)^{\frac{n}{2}-1}\left[\frac{z^2(n-1)-1}{z^2-1}\right].$$

Substituting these derivatives in equation (4) leads to

$$(1-z^2)w''-2(n+1)zw'+(\mu-n)(\mu+n+1)w=0. \tag{5}$$

Now if *u* be a solution of the ordinary Legendre differential equation, then one has

$$(1-z^2)u'' - 2zu' + \mu(\mu+1)u = 0\,.$$

Differentiating this equation n times withrespectto z , and letting$w = \frac{d^n u}{dz^n}$leads to

$$(1-z^2)w'' - 2(n+1)zw' + (\mu-n)(\mu+n+1)w = 0.$$

Hence we showed that the function W is a solution of equation (5). It follows that the solutions of equation (4) are obtained as the following

$$y_1(z) = (z^2-1)^{\frac{n}{2}}\frac{d^n u_1}{dz^n},\text{and}\, y_2(z) = (z^2-1)^{\frac{n}{2}}\frac{d^n u_2}{dz^n},$$

where $u_1 = P_\mu(z)$, and $u_2 = Q_\mu(z)$are solutions of the ordinary Legendre differential equation. The single-valued functions $y_1(z)$ and $y_2(z)$ are denoted by $P_\mu^n(z)$ and$Q_\mu^n(z)$ respectively, and are called the associated Legendre functions (or spherical functions)of the first and second kinds, respectively, of order μ and degree *n*that is,

$$P_\mu^n(z) = (z^2-1)^{\frac{n}{2}}\frac{d^n P_\mu(z)}{dz^n}, \tag{6}$$

$$Q_\mu^n(z) = (z^2-1)^{\frac{n}{2}}\frac{d^n Q_\mu(z)}{dz^n}. \tag{7}$$

Sometimes equations (6) and (7) are called by Hobson definition of the associated Legendre functions (Whittaker and Watson 1952; Hobson and Barnes 1908). For general values of μ the Legendre functions$P_\mu(z)$and$Q_\mu(z)$of the first and second kinds, respectively, are analytic in the complex z-plane cutalong the segments (-∞, -1] and(-∞, 1], respectively. It follows that from equations (6) and (7) that$P_\mu^n(z)$and$Q_\mu^n(z)$are entire functions of the variable z in the complex z-plane cutalong the the real axis from the pointz=-∞to the point z=+1(Lebedev 1965). We already know that the general solution of the ordinary Legendre differential equation defined as

$$u_\mu(z) = AP_\mu(z) + BQ_\mu(z). \tag{8}$$

Where*A* and*B* are constants. Differentiating relation (8)*n*times and multiplying by the factor $(z^2-1)^{\frac{n}{2}}$leads to

$$(z^2-1)^{\frac{n}{2}}\frac{d^n u_\mu}{dz^n} = A(z^2-1)^{\frac{n}{2}}\frac{d^n P_\mu(z)}{dz^n} + B(z^2-1)^{\frac{n}{2}}\frac{d^n Q_\mu(z)}{dz^n}.$$

Hence the general solution of equation (4) is obtained as

$$y(z) = AP_\mu^n(z) + BQ_\mu^n(z).$$

Some of the associated Legendre functions of the first and second kindsare obtained as,

$$P_0^0(z) = 1, \qquad P_1^1(z) = -(z^2-1)^{\frac{1}{2}}, \qquad P_2^0(x) = \frac{1}{2}(3x^2-1).$$

$$Q_1^0(z) = \frac{z}{2}\ln\left(\frac{z+1}{z-1}\right) - 1,$$

$$Q_1^1(z) = (z^2-1)^{\frac{1}{2}}\left[\frac{1}{2}\ln\left(\frac{z+1}{z-1}\right) - \frac{z}{z^2-1}\right],\;\; Q_2^1(z) = (z^2-1)^{\frac{1}{2}}\left[\frac{3z}{2}\ln\left(\frac{z+1}{z-1}\right) - \frac{3z^2-2}{z^2-1}\right].$$

Theassociated Legendre functions are defined in restricted regions of the complex z-plane forz ∉ (-∞, 1], next we show how they can be continued analytically to other regions by obtaining theirhypergeometric representation.

IV. Linear Transformations of theKummer'ssolutions of the Hypergeometricequation

In this section we shall derive someimportant linear combinations of Kummer's solutionsof the hypergeometricdifferential equation(Lebedev 1965; Rainville 1964) that will be used later in this article.

Theorem: If $|z|<1$ and $|1-z|<1$, if $\mathrm{Re}(\gamma-\alpha-\beta)>0$ and $\mathrm{Re}(1-\gamma)>0$, and if none of $\gamma-\alpha-\beta$, γ , $\gamma-1$ is an integer, then we have

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z)$$

$$+\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z) \tag{9}$$

This transformation gives the analytic continuation of the function $F(\alpha,\beta;\gamma;z)$ from the region $|z| < 1$ in the complex z-plane into the region $|1-z| < 1$ as shown in figure 1 below.

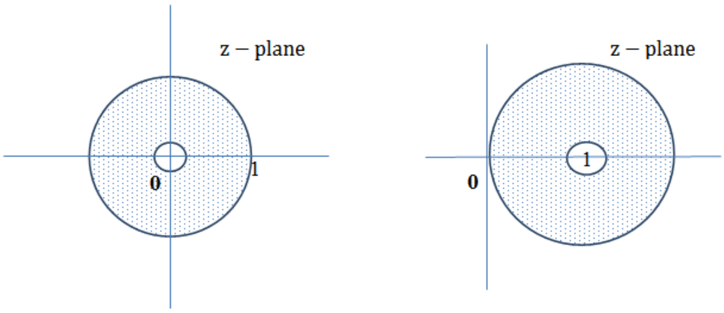

Figure 1: From left to right the dotted regions are $|z| < 1$ and $|1-z| < 1$.

If we replace z by $\dfrac{z}{z-1}$ in the transformation (9), then we get

$$F\left(\alpha,\beta;\gamma;\frac{z}{z-1}\right)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F\left(\alpha,\beta;\alpha+\beta-\gamma+1;\frac{1}{1-z}\right)$$
$$+\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\alpha+\beta-\gamma}F\left(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;\frac{1}{1-z}\right) \tag{10}$$

Now recall the following relations (Lebedev 1965; Rainville 1964),

$$F(\alpha,\beta;\gamma;z)=(1-z)^{-\alpha}F\left(\alpha,\gamma-\beta;\gamma;\frac{z}{z-1}\right),\qquad \left|\frac{z}{z-1}\right|<1, \tag{11}$$

Or

$$F(\alpha,\beta;\gamma;z)=(1-z)^{-\beta}F\left(\gamma-\alpha,\beta;\gamma;\frac{z}{z-1}\right),\qquad \left|\frac{z}{z-1}\right|<1, \tag{12}$$

which give the analytic continuation of $F(\alpha,\beta;\gamma;z)$ into the region $\left|\dfrac{z}{z-1}\right|<1$ as shown in figure 2.

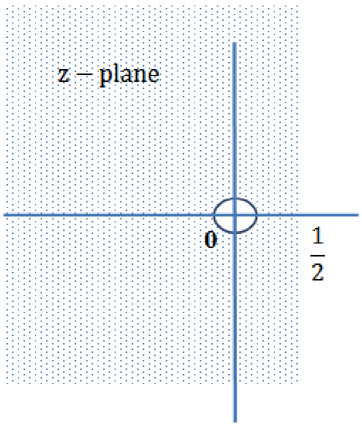

Figure 2: The dotted region $\left|\dfrac{z}{z-1}\right|<1$.

Applying the transformation (11)tothe left hand side of equation (10), multiplying by $(1-z)^{-\alpha}$ and replacing β by $\gamma-\beta$ in equation (10), leads to

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)}(1-z)^{-\alpha}F\left(\alpha,\gamma-\beta;1+\alpha-\beta;\frac{1}{1-z}\right)$$
$$+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(1-z)^{-\beta}F\left(\gamma-\alpha,\beta;1-\alpha+\beta;\frac{1}{1-z}\right), \tag{13}$$

which gives the analytic continuation of $F(\alpha,\beta;\gamma;z)$ into the region $|1-z|>1$ as shown in figure 3.

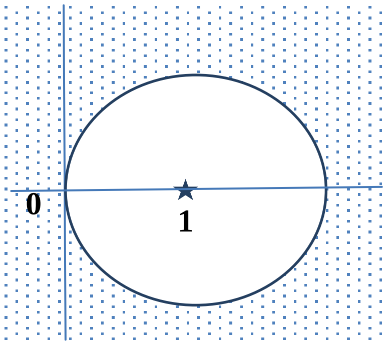

Figure 3: The dotted region $|1-z|>1$.

If we apply the transformations (11) and (12) to the first and second terms of the right hand side of (13) respectively, then we get

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)}(-z)^{-\alpha}F\left(\alpha,1+\alpha-\gamma;1+\alpha-\beta;\frac{1}{z}\right)$$
$$+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-z)^{-\beta}F\left(1+\beta-\gamma,\beta;1-\alpha+\beta;\frac{1}{z}\right), \quad (14)$$

where $|z|>1$, and $\alpha-\beta\neq 0,\pm1,\pm2,\ldots,$ .This transformation gives the analytic continuation of $F(\alpha,\beta;\gamma;z)$ into the region $|z|>1$ as shown in figure 4.

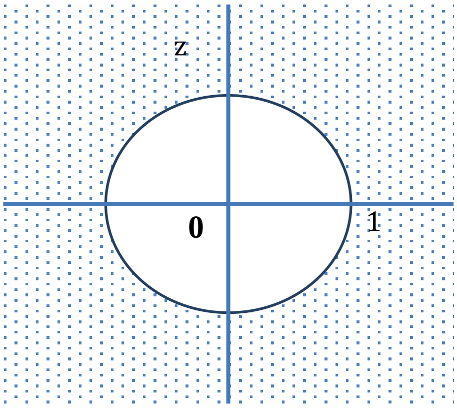

Figure 4: The dotted region $|z|>1$.

Next we will show how to make use of these transformations to gain the analytic continuation of the first kind associated Legendre functions.

V. Analytic Continuations of the First Kind Associated LegendreFunctions $P_\mu^n(z)$

In this section we shall represent the first kind associated Legendre functions$P_\mu^n(z)$in terms of the hypergeometric function to carry out their analytic continuation into different parts of the complex z-plane (Beals and Wong2010). This can be achieved with the aid of some linear as well as quadratic transformations of the hypergeometric functions. Starting by substituting the Murphy's expression of $P_\mu^n(z)$given in (Whittaker and Watson 1952) as a hypergeometric function into equation (6) toobtain

Doing the n-fold differentiation of the hypergeometric functionand then carrying on some tedious calculations involving some relations between gamma function and the Pochammer symbol(Abramowitz and Stegun 1968;Lebedev 1965), we end up at thehypergeometric form of the associated Legendre functions of the first kind as

$$P_\mu^n(z)=\frac{\Gamma(\mu+n+1)}{2^n\Gamma(\mu-n+1)\Gamma(n+1)}\left(z^2-1\right)^{n/2}F\left(n-\mu,n+\mu+1\ ;\ n+1;\ \frac{1-z}{2}\right),\qquad |1-z|<2. \quad (12)$$

Also we can apply another transformationfor the hypergeometricrepresentation of$P_\mu^n(z)$to obtain different forms of $P_\mu^n(z)$defined in different regions in the complex z-plane (Sneddon 1980).
Henceif we apply the following quadratic transformation (Lebedev 1965; Rainville 1964)

$$F\left(\alpha,\beta;\alpha+\beta+\frac{1}{2};\ z\right)=F\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};\ \frac{1-\sqrt{1-z}}{2}\right),\qquad \left|1-\sqrt{1-z}\right|<2, \quad (15)$$

to the hypergeometric function on the right-hand side of equation (12), then one has

$$P_\mu^n(z)=\frac{(\mu+n)!}{2^n(\mu-n)!n!}\left(z^2-1\right)^{n/2}F\left(\frac{n-\mu}{2},\frac{n+\mu+1}{2};n+1\ ;\ 1-z^2\right),\quad \left|1-z^2\right|<1,\quad n\neq-1,-2,\ldots \quad (16)$$

In which we set the parametersof the hypergeometric function in equation (15)as the following:

$$\alpha=\frac{n-\mu}{2},\ \beta=\frac{n+\mu+1}{2}\ \text{,and}\ \alpha+\beta+\frac{1}{2}=n+1.$$

Furthermore if we apply the transformation(13) to the hypergeometric function on the right-hand side of equation (16), then one has

$$P_\mu^n(z)=\frac{(-1)^n(2\mu)!}{2^\mu\ \mu!\,(\mu-n)!}(1-z^2)^{n/2}z^{\mu-n}F\left(\frac{n-\mu}{2},\frac{n-\mu+1}{2};\frac{1}{2}-\mu;\ \frac{1}{z^2}\right),\qquad |z|>1. \quad (17)$$

From the properties of gamma function, the resulting second term in equation (17) is vanishing due to the term $\Gamma(\alpha)\Gamma(\gamma-\beta)$ in the denominator of the pre-factor of the second term in (16) where $\alpha=\frac{n-\mu}{2},\gamma-\beta=\frac{n-\mu+1}{2}$.
Also,if we apply the quadratic transformation (11) which is due to Euler (Beals and Wong2010), to the hypergeometric function on the right-hand side of equation (17), then one has

$$P_\mu^n(z)=\frac{(-1)^{(\mu+n)/2}(2\mu)!}{2^\mu\ \mu!\,(\mu-n)!}(1-z^2)^{n/2}F\left(\frac{n-\mu}{2},-\frac{(n+\mu)}{2};\frac{1}{2}-\mu;\ \frac{1}{1-z^2}\right),\qquad |1-z^2|>1. \quad (18)$$

Another application of the quadratic transformation (11) to equation (18) yields the following new hypergeometric representation as,

$$P_\mu^n(z)=\frac{(-1)^{(\mu+n)/2}(2\mu)!}{2^\mu\ \mu!\,(\mu-n)!}(1-z^2)^{n/2}F\left(n-\mu,-n-\mu;\frac{1}{2}-\mu;\ \frac{\sqrt{z^2-1}-z}{2\sqrt{z^2-1}}\right),$$
$$\left|\frac{\sqrt{z^2-1}-z}{2\sqrt{z^2-1}}\right|<1. \quad (19)$$

Another application of the quadratic transformation (11) to equation (19) yields another new hypergeometric representation as,

$$P_\mu^n(z)=\frac{(-1)^{n/2}(2\mu)!}{2^{2\mu-n}\ \mu!\,(\mu-n)!}(1-z^2)^{n/2}\left(z+\sqrt{z^2-1}\right)^{\mu-n}F\left(n-\mu,\frac{1}{2}+n;\frac{1}{2}-\mu;\ \frac{z-\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right),$$
$$\left|\frac{z-\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right|<1. \quad (20)$$

Further hypergeometric representations of the first kind associated Legendre functionscan be found in the references (Abramowitz andStegun 1968;Laham and Abdallah 1996;Rainville 1960;Lebedev 1965; Rainville 1964; Gradshteynand Ryzhik 2007; Erdelyiet al.1953-55).Primarily, we are not interested in deriving the hypergeometric forms of theassociated Legendre functions rather we aim to show that the transformations approach is simple and convenient because it only calls some transformations and then carry out appropriate passages to the limit.

VI. Discussion

We know that the associated Legendre functions of the first kindare defined for values of the complex variable z lie in the complement of the segment (-∞, 1]. Section 4 presents different forms of the hypergeometric representation of the associated Legendre functions of the first kind $P_\mu^n(z)$which were obtained by means of linear and quadratic transformation of the hypergeometric function and being away from any integral representation.For example, the hypergeometricrepresentation provided by equation (12) gives the analytic continuation of $P_\mu^n(z)$into the complex region$|1-z|<2$,with a cut is made along the real axis from the point z=-∞ to the point z=+1, whereas the hypergeometricrepresentation given by equation (16) analytically continued$P_\mu^n(z)$into the region$|1-z^2|<1$with the same cut mentioned above.Furthermore, the hypergeometricrepresentations given by equation (17) and (18) carries out the analytic continuation of$P_\mu^n(z)$into the complex regions$|z|>1$ and $|1-z^2|>1$. Repeated applications of the linear or the quadratic transformations yields many more hypergeometricrepresentationsof$P_\mu^n(z)$.Thus we obtain different representations of$P_\mu^n(z)$which are defined in different parts of the complex z-plane, for example the hypergeometricrepresentations given by equation (19) and (20) carries out the analytic continuation of$P_\mu^n(z)$into the complex regions$\left|\frac{\sqrt{z^2-1}-z}{2\sqrt{z^2-1}}\right|<1$ and$\left|\frac{z-\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right|<1$.

VII. Conclusion

It was observed that the associated Legendre functions can be expressed by the hypergeometricseries in suitably restricted regions of the complex z-plane cut along the real segment(-∞, +1].By rewriting the associated Legendre functions in terms of thehypergeometric function, more regions in the complex z-planewere obtainedfor the analytic continuation. Therefore, to conclude it is very informative to express the associated Legendre functions in terms of the hypergeometric function as shownin the discussion.It is worth to emphasize that the hypergeometricrepresentation enables us to develop the theory of spherical functions by implementing the general theory of the hypergeometric function. Specifically, this approach is very helpful to gain the generalization of the spherical functions for arbitraryvalues of the degree n. It is remarkable to note that the regions of validity so often pass through a singular point of the differential equation where the regular singular points of the hypergeometric differential equation are z = 0, 1, ∞. To sum up, one could claim that the linear and quadratic transformations approach of obtaining the hypergeometricrepresentation is more convenient and less complicated than the integral representation approach.

References

[1]. AbramowitzM andStegun A.Handbook of mathematical functions. (Ninth edition)New York: Dover Publications Inc.,1968.
[2]. AndrewsGEAskeyRRoy R.Special functions. Cambridge Univ. Press, 1999.
[3]. LahamNM andAbdallahAK.Special functions for scientists and engineers.Yarmouk University, 1996.
[4]. Rainville ED.Special Functions. The Macmillan Co., New York, 1960.
[5]. Lebedev NN.Special Functions and their Applications.In: SilvermanRA (Eds), Dover Publication, Inc., New York, 1972.
[6]. Hobson EW.Spherical and ellipsoidal harmonics. Cambridge University Press, London, 1955.
[7]. KuipersLand Meulenbeld B.On the generalization of Legendre's associated differential equation.Konkl.Nederl.Akad. Wet. A.1957, 60(4): 436-450.
[8]. WangMK and ChuYM.Landen inequalities for a class of hypergeometric functions with applications, Math.Inequal. Appl. 2018, 21(2), 521-537.
[9]. WangMKand ChuYM.Refinements of transformation inequalities for zero-balanced hypergeometric functions, Acta Math. Sci. 2017, 37B(3), 602-622.
[10]. WangMK ChuYM and SongYQ.Asymptotical formulas for Gaussian and generalized hypergeometric functions, Appl. Math.Comput., 2016, 276, 44-60.
[11]. SongYQZhou PGand ChuYM. Inequalities for the Gaussian hypergeometric function, Sci. China Math. 2014, 57(11), 2369-2380.
[12]. Virchenko NA.On some applications of the generalized associated Legendre's function. Ukr.Math.J. 1987, 39(2): 149-156.
[13]. Rainville ED.Intermediate differential equations.The Macmillan Co., New York, 1964.
[14]. Erdelyi A MagnusWOberhettinger F TricomiFG.Highertranscendental functions.Bateman Project, Vols. 1-3, McGraw-Hill Co., New York, 1953-55.
[15]. Verchenko NAand Rumiantseva OV.On the generalized associated Legendre functions.Fractional Calculus & Applied Analysis.2008,2: 175-185.

[16]. Verchenko NA. On some generalizations of the functions of hypergeometric type.Fractional Calculus & Applied Analysis.1999, 2(3): 233-244.
[17]. Verchenko NAKallaSLAl-Zamel A.Some results on a generalized hypergeometric function. Integral Transforms and Special Functions.2001, 12(1): 89-100.
[18]. Raoa SBandShukla AK.Note on generalized hypergeometric function.Integral Transforms and Special Functions.2013, 24:896-904.
[19]. Malovichko, V. On a generalized hypergeometric function and some integral operators.Math. Phys.1976, 19: 99-103.
[20]. VerchenkoNAandFedotova I.Generalized associated Legendre functions and their applications. World Scientific Publishing Co. Pte.Ltd., Singapore-New Jersey-London-Hong Kong, 2001.
[21]. WrightEM.The asymptotic expansion of the generalized hypergeometric function.J. London Math. Soc.1935, 10: 286-293.
[22]. RaoaSBPrajapatiJCShukla AK.Wright type hypergeometric function and its properties.Advances in Pure Mathematics.2013, 3: 335-342.
[23]. RahmanGArshad MMubeen S. Some results on generalized hypergeometric k-functions. Bulletin of Mathematical Analysis and Applications, 2016,8(3): 66-77.
[24]. MillerAR.On a Kummer-type transformation for the generalized hypergeometric function. J. Comput. Appl. Math. 2003, 157(2): 507–509.
[25]. Whittaker ET and Watson GN.A course of modern analysis.(Fourth edition), Cambridge University Press, London, 1952.
[26]. Beals R and Wong R.Special functions: A graduate text. Cambridge studies in advanced mathematics, 2010.
[27]. Hobson EW.On a type of spherical harmonics of unrestricted degree, order, and argument.Phil. Trans.1896, 187: 443-531.
[28]. Hobson EW and BarnesEW.On generalized Legendre functions. Quart. J. Math.1908, 39: 97.
[29]. Gradshteyn ISRyzhikIM.Table of integrals, series, and products. (Seventh edition), Elsevier Academic PressPublications, 2007.
[30]. Sneddon I.N., Special Functions of Mathematical physics and chemistry, 3rd ed. New York: Longman, 1980.

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