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Finite Difference Method for the Biharmonic Equation with Different Types of Mixed Boundary Conditions.

Haniyah A. M. Saed Ben Hamdin
Sirte University, Science Faculty, Mathematics Department, Sirte, Libya

Abstract: *Finite difference method is one of several powerful numerical techniques for obtaining an approximate solution for partial differential equations. It has been provenas an efficient technique to solve initial and boundary value problems for linear and nonlinear partial differential equations for any dimension. Here we implement such numerical technique to obtain the numerical solution for the Helmholtz equation and the biharmonic equation with one spatial variable and time-independent. At first, we formulate the finite difference method to the Helmholtz equation as an eigenvalue problem subject to the Neumann boundary conditions. Then we formulate such numerical techniques to the biharmonicequation as an eigenvalue problemsubject to different combinations of Dirichlet and Neumann boundary conditions which correspond to a variety of physical phenomena. For demonstration purposes, we show a comparison of the approximate eigenfunctions and eigenvalues that are obtained from the finite difference methodagainst the analytical results. Suchcomparisonshows a very good agreement between the two categories. Hence itconfirms the efficiency of the finite difference method to solve the biharmonic equation with different types of mixed boundary conditions which are the clamped, simply supported and free boundary conditions.*

Keywords*: Finite Difference Method, Helmholtz Equation, Modified Helmholtz Equation, BiharmonicEquation, Mixed boundary conditions, Neumann boundary conditions, Dirichlet boundary conditions.*

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I. Introduction

The Partial Differential Equations (PDEs) topic arises in the 18^{th} century as the ordinary differential equations fail to describe many physical phenomena [1, 2]. Then Solving the PDEs subject either analytically or numerically has been a large undertaking. Analytic methods [3] for solving PDEsare not always applicable due to many circumstances for instance if the region of interest is of complex shape or inhomogeneous (different material properties), alsoif the prescribed boundary conditions are time-dependent or of mixed type. Thus one could resort to the well-known approximate techniques [4, 5, 6] for solving PDEs such as the finite Element Method (FEM) [7, 8], the boundary Element Method (BEM)[9, 10, 11, 12] orthe Finite difference method (FDM)[13, 14, 15, 16]. It is worth to mention that the BEM requires a rigours mathematical background to be able to deal with the resulting different types of singularintegrals, whereas the FEM and the FDM are much easier to apply especially for non-mathematicians. The FDM is one of the simple and powerful techniques to solve PDEs; it is applicable to regions of arbitrary shape and for any dimension to solve initial and boundary value problems for linear and nonlinear PDEs. The FDM is based on replacing the differential operator in the PDE by certain approximate difference formulas as it will be shown later. Then similar to the other well-known numerical methods such as the FEM and BEM, the domain is discretized into space and time (if the PDE is time-dependent) points, and finally the solution is computed at each space and time nodal points. Thus the FDM converts the PDE into a system of algebraic equations which can be solved by the known matrix algebra tools. In fact the numerical methods such as the FEM, BEM and the FDM become popular with the widespread of high-speed and huge storagecomputing machines which are powerful in dealing with large matrix operations.

Since the FDM is built on approximating the derivatives in the PDE in terms ofdifference formulas that are coming from truncating the infinite Taylor series to a finite polynomial, this develops an error known as the truncation error. The local truncation errorat each nodal point is proportional to the mesh size on the grid system; that is a finer mesh should considerably decreases the truncation error.The local truncation error can be computed explicitly using the Lagrange form of the remainder of the Taylor series as shown in reference [17].

To prevent the round-off error, one should reasonably balance between the mesh size and the running time (duration) of the simulation, as the round-off error may increase due to a large number of mathematical processes. For the time-dependent PDEs, large time step may cause instability [18].

In this paper, we implement the FDM to numerically solve the time-independent biharmonic equation for a one-dimensional beam, so we can guarantee the existence of the analytic solution for the sake of comparison. The biharmonic equation is the equation of flexural motion of homogeneous plates; it is a fourth-order partial differential equation that arises in many areas. We formulate the problem as an eigenvalue problem with different combinations of Dirichlet and Neumann boundary conditions which correspond to a variety of physical phenomena. Since the biharmonicdifferential operator can be factorized into the Helmholtz operator and the modified Helmholtz operator, so for convenience we first implement the FDM to the Helmholtz equation with Neumann boundary conditions. This paper is structured as, in section one and two; we respectively present the biharmonic equation and the prescribed boundary conditions. Then in section three we briefly present the core idea of FDM and derive all types of the approximate difference formulas. For the sake of comparison, we compare the approximate results against the analytical results in section four and six respectively for the Helmholtz and the biharminic equations. A conclusion is drawn in section seven.

II. The BiharmonicEquation

The biharmonic equation is the equation of flexural motion of homogeneous plates and can be defined as

$$\left(\Delta_r^2 + \frac{2\rho h}{D}\frac{\partial^2}{\partial t^2}\right)\phi(r,t) = 0,$$

where$\Delta = \mathrm{div}(\mathrm{grad})$is the Laplace operator $\frac{\partial^2}{\partial r^2}$ in the Cartesian coordinates. The function$\phi(r,t)$ is the deflection, D is the flexural rigidity given as $D = \frac{Eh^3}{12(1-\nu^2)}$. The constant$E$is the Young's modulus of extension, ν is the Poisson ratio, h is the thickness of the plate, and ρ is the mass density, all these constants characterising the mechanical properties of the plate [19, 20]

In this paper we deal with the time-independent biharmonic equation which can be obtainedby the aid of the separation of variables method where we substitute

$$\phi(r,t) = e^{i\omega t}\phi(r).$$

Then doing some manipulations to obtain the time-independent biharmonic equation as,

$$(\Delta_r^2 - k^4)\phi(r) = 0, \tag{1}$$

Where$k$is the wave-number and obeys the dispersion relation $k^4 = \frac{\rho h\omega^2}{D}$.

It should be noted that; since the biharmonicoperator$(\Delta_r^2 - k^4)$, can be factorised into the Helmholtz and the modified Helmholtz operators which are $(\nabla_r^2 + k^2)$and $(\nabla_r^2 - k^2)$respectively. Thus the solution of biharmonic operator can be written as a sum of the solution of each operator.

Hence, for a one-dimensional case$r = x$, one could easily obtain the analytic solution of the time-independent biharmonic equation(1) as,

$$\phi(x) = A\cosh(\lambda x) + B\sinh(\lambda x) + C\cos(\lambda x) + D\sin(\lambda x). \tag{2}$$

III. Types Of Boundary Conditions

The study of many physical phenomena leads to boundary value problems (BVPs), which take the form,

$$L\phi(r) = f(r). \tag{3}$$

These BVPs require that $\phi(r)$ should be defined for some region $r_1 \le r \le r_2$ (it possible that $r_1 \to -\infty$ or $r_2 \to +\infty$ or both).In order to adequately describe the problem, there are a variety of boundary conditions that can be considered for a boundary value problem. They can be categorised as the following for a boundary point$r \in \partial D$, where D is the region of interest.

1- If the boundary conditions are set for the function$\phi(r)$, that is$\phi(r_1) = f(r)$ and $\phi(r_2) = g(r)$ then it is called Dirichlet Boundary Conditions (DBCs). Example of this type of BCs is a membrane problem.

2- If the boundary conditions are set for $\frac{\partial\phi(r)}{\partial n_r}$, then it is called Neumann Boundary Conditions (NBCs). The operator $\frac{\partial}{\partial n_r}$ denotes the directional derivative along the normal vector $\vec{n}$ at the boundary element$r$, that is,

$$\frac{\partial}{\partial n_r} = \vec{n}.\nabla_r ,$$

where the dot denotes the scalar product, ∇_r is the gradient operator with respect to r.An example of this type of BCs is an acoustic problem, where the acoustic potential (pressure) can not be set to zero, but the velocity can be set to zero on the boundary.

A linear combination of DBCs and NBCs is known as mixed or Robin boundary conditions, that is,

$$A\frac{\partial\phi(r)}{\partial n_r} + B\phi(r) = f(r),$$

where$A$ and B are constants. Examples of this type of BCs occur in heat problems, where the temperature is related to the thermal flux. The following homogeneous boundary conditions are examples of this type,

a. Clamped (fixed) boundary condition at the chosen point $r = r_0$has the displacement and the slope of ϕ zeros, i.e

$$\phi(r_0) = 0 = \phi'(r_0). \tag{4}$$

b. Simply supported BCs at the chosen point $r = r_0$ has the displacement and the bending moment of ϕ set to zero, i.e

$$\phi(r_0) = 0 = \phi''(r_0). \tag{5}$$

c. Free boundary condition at the chosen point $r = r_0$ has the bending moment and the shearing force set to zero, i.e

$$\phi''(r_0) = 0 = \phi'''(r_0). \tag{6}$$

A problem (3) is homogeneous only if $f(r)$ and also the specified initial or boundary conditions are zeros. If any or all of $f(r)$ and the prescribed initial or boundary conditions are non-zeros then it is inhomogeneous. Homogeneity for linear equation implies the superposition principle, which means that any arbitrary linear combinations of solutions are themselves solutions.

IV. Finite Difference Scheme

The principle of the FDM is that the derivatives in the PDE are approximated by a linear combination of functions that are evaluated at certain points on the grid system. Thus, for a one-dimensional problem, uniformly spaced the interval$\Omega = 0 \leq x \leq l$, such that

$$\text{grid points} x_k = (k-1)\Delta x, \qquad \text{mesh size} \Delta x = \left(\frac{l}{N-1}\right), k = 1 \ldots N,$$

where$h = \Delta x$ is the mesh parameter of the discretization process, and N is the total number of the spatial nodes including the boundary points as shown in Figure 1.The accuracy of the computation can be improved by refining the grid to increase the number of grid (nodal) points.

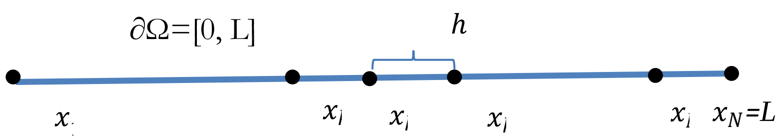

Figure 1: Finite difference mesh for one special variable x on the interval $\Omega = 0 \leq x \leq l$.

To show the core idea of the finite difference scheme; we start at the Taylor expansion of the solution function $\phi(x_k)$ as

$$\phi(x_k + h) = \phi(x_k) + h\phi'(x_k) + \frac{h^2}{2!}\phi''(x_k) + \frac{h^3}{3!}\phi'''(x_k) + \mathcal{O}(h^4), \quad (7)$$

$$\phi(x_k - h) = \phi(x_k) - h\phi'(x_k) + \frac{h^2}{2!}\phi''(x_k) - \frac{h^3}{3!}\phi'''(x_k) + \mathcal{O}(h^4), \quad (8)$$

where the term$\mathcal{O}(h^4)$ denoted to the error developed by truncating the Taylor expansion at a finite part. For instance, by truncating the Taylor expansion (7) and (8) at the first two terms we obtain respectively the so-called forward and backward difference formulas for the first derivative as,

$$\phi'(x_k) = \frac{\phi(x_k + h) - \phi(x_k)}{h} + \mathcal{O}(h), (9)$$

$$\phi'(x_k) = \frac{\phi(x_k) - \phi(x_k - h)}{h} + \mathcal{O}(h). (10)$$

Where the former and the latter uses the points x_k, x_{k+1} and x_{k-1}, x_k, respectively.
Also by subtracting the Taylor expansion (8) from (7), we obtain the so-called central (centred) difference formula for the first derivative as,

$$\phi'(x_k) = \frac{\phi(x_k + h) - \phi(x_k - h)}{2h} + \mathcal{O}(h^2), (11)$$

which is called the first-order central difference formula.
Also by summing up the Taylor expansion (1) and (2), we obtain the so-called central difference formulafor the second derivative as,

$$\phi''(x_k) = \frac{\phi(x_k + h) - 2\phi(x_k) + \phi(x_k - h)}{h^2} + \mathcal{O}(h^2), (12)$$

which is called the second-order central differenceformula. The term$\mathcal{O}(h^2)$denotes to the local truncation error which goes to zero much faster than the one for forward and backward difference approximations, that is

$$\mathcal{O}(h^2) \to 0 \; as \; h \to 0.$$

Hence the central difference formula should give best approximation of the first derivative as shown in Figure 2below.

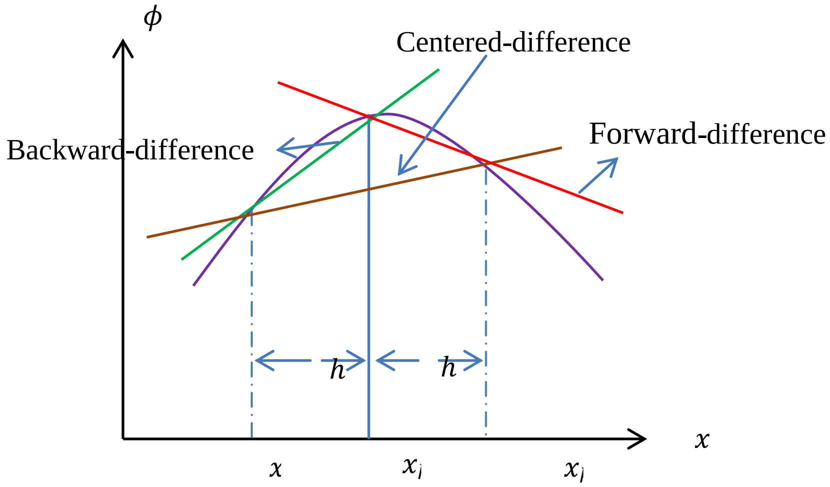

Figure 2: Estimation of the first derivative of $\phi(x_k)$ by using the forward, backward and central difference formulas.

Finite difference formulas for higher-order derivatives can be approximated by taking more terms in the Taylor expansions (7) and (8). It should be mentioned that forward and backward difference approximating formulas are of h order, whereas the central difference approximating formula is ofh^2 order. Therefore, we could claim that the central difference approximating formula should yields more accurate results. Hereafter we shall adopt the notation x_{k+1} for $x_k + h$ and $x_{k-1}$for $x_k - h$. Next we show how to formulate the FDM to solve the one-dimensional Helmholtz Equation**.**

V. Finite Difference Formulations for Helmholtz Equation in One-Dimension

Helmholtz equation is one of the most important wave equation, which describes the membrane vibration and acoustics pressure waves in fluids and gases, and can be defined in the following form

$$(\nabla^2 + k^2)\phi(x) = 0, \tag{13}$$

where$k$ is the wave-number. We consider this equation for a one-dimensionalbeam and subject to NBCs at the end points. To obtain the approximate solution of equation (13) we implement the FDMby replacing the second order derivative in equation (13) by the approximate central difference formula (12), thus we obtain an eigenvalue problem. Then the beam is discretized into nodal points, and the solution is computed at these points. Then we have built up a Matlab code to evaluate the eigenvectors and eigenvaluesof the resulting matrix for a mesh size $(N = 700)$. A tabular comparison of the exact and computed eigenvalues is shown in Table 1. Furthermore, the computed eigenfunctions are compared to the exact eigenfunctions as shown in Figure 3.

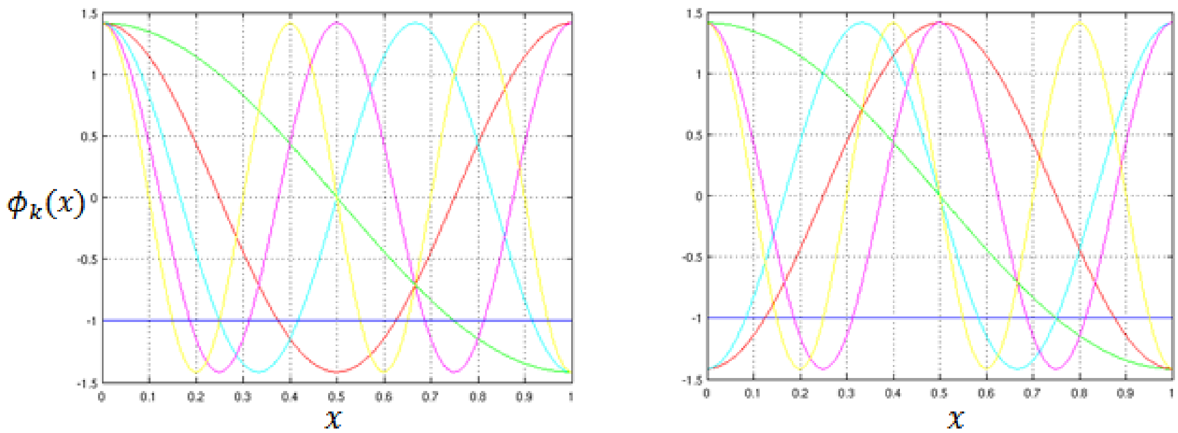

Figure3: From left to the right the exact and numerical eigenfunctions respectively of Helmholtz equation, in 1-dimension with NBCs for $(N = 700)$.

Table 1: Exact and numerical eigenvalues of Helmholtz equation with NBCs.

k	Exact Eigenvalues	Numerical Eigenvalues
1	9.8696	9.8695
2	39.4784	39.4778
3	88.8264	88.8238
4	157.9136	157.9053
5	246.7401	246.7198
6	355.3057	355.2636
7	483.6106	483.5326
8	631.6546	631.5216
9	799.4379	799.2249

Next we show how to formulate the FDM to solve the one-dimensional Biharmonic Equation**.**

VI. Finite Difference formulations for One-Dimensional Biharmonic Equation

The one-dimensional and time-independent biharmonic eigenvalue problem is

$$\frac{d^4\phi}{dx^4} = \lambda\phi(x); \qquad \lambda = k^4, \tag{14}$$

subject to a variety of mixed boundary conditions such as simply supported, clamped, free boundary conditions. To implement the FDM, we replace the biharmonic operator by the corresponding central difference approximation (which can be obtained by considering the Taylor expansion up to the forth order). For shortness we will use ϕ_k for $\phi(x_k)$, one obtains

$$\frac{\phi_{k+2} - 4\phi_{k+1} + 6\phi_k - 4\phi_{k-1} + \phi_{k-2}}{h^4} + \mathcal{O}(\mathrm{h}^2) = \lambda_k \phi(x), \tag{15}$$

which is an eigenvalue problem. Then, we built up a Matlab code to evaluate the approximate eigenvalues and the eigenfunctionsof equation (15).

For clarity, we shall give a brief description of the calculations of the exact eigenvalues and the eigenfunctions of equation (14) subject to simply supported BCs (5) at both edges of the beam.

By employing the simply supported BCs (5) into the analytic solution (2) of equation (14) and then call the following orthogonality property,

$$\int_0^l \phi_n(x)\phi_m^*(x)\,dx = \delta_{nm} \ ,$$

whereδ_{nm} denotes to the kronker delta operator. Thus, we obtain analytic eigenfunctions as,

$$\phi_n(x) = \sqrt{\frac{2}{l}}D\sin\left(\frac{n\pi x}{l}\right). \tag{16}$$

In addition, we obtain the frequency equation,

$$\sin(\lambda l)\sinh(\lambda l) = 0,$$

which gives the exact eigenvalues as,

$$\lambda_n = \left(\frac{n\pi}{l}\right). \tag{17}$$

Next section shows a comparison of the computed results against the exact ones.

VII. Results and Discussion

By solving the eigenvalue problem (14) analytically and numerically by implementing the FDM subject to thesimply supportedBCs (5)as shown in the previous section, here we show a comparison of the analytic results (16) and (17)against the approximate ones obtained by the FDM. We run out the comparison for both the eigenvalues and the eigenfunction for a mesh size$(\mathrm{N} = 700)$. Figure 4 shows the first six exact and approximate eigenfunction$\phi_k(x)$for the biharmonic equation of a one-dimensional beam with simply supported BCs at both edges. As shown in Figure 4 the first eigenfunction$\phi_1(x)$(either computed analytically or numerically)which appears in blue colour does not cross the x-axis, whereas the second eigenfunction$\phi_2(x)$which appears in green colour crosses the x-axis once. Also, the third eigenfunction$\phi_3(x)$which appears in red colour crosses the x-axis twice, and so on for the rest of the eigenfunctions. Thus the eigenfunction$\phi_k(x)$coresses the x-axis $(k-1)$ times, for this reason there is no need to put a legend in the graph. A tabular comparison of the exact (16)and computed eigenvalues $\lambda_n$is shown in Table 2.

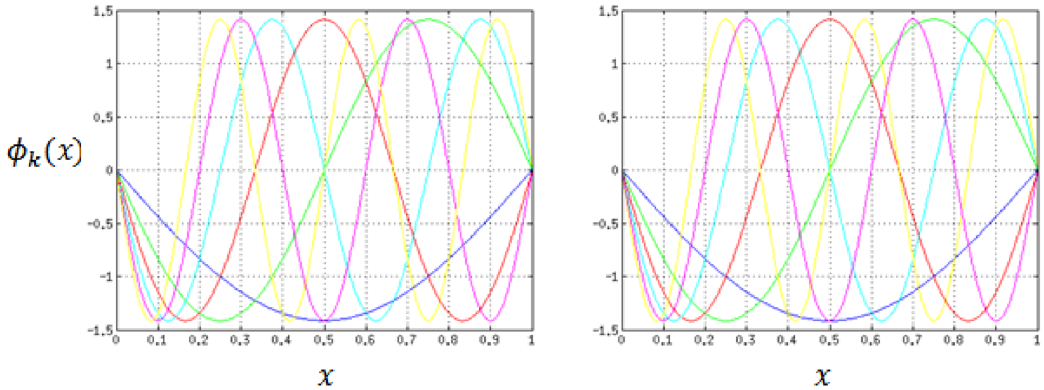

Figure 4 From left to the right the exact and numerical eigenfunctions respectively of a biharmonic equation of a beam with simply supported BCs for (N = 700) in 1-dimension.

Table 2: Exact and numerical eigenvalues of biharmonic equation with simply supported BCs for $(N = 700)$.

k	Exact Eigenvalues$\lambda_k = \left(\frac{k\pi}{l}\right)^4$	Numerical Eigenvalues
1	0.380	0.382
2	6.088	6.122
3	30.820	30.996
4	97.409	97.960

To prevent the repetition;for the other two cases that are considered below we shall neglect the details for the exact and numerical computationsof the eigenvalues and the eigenfunctionssubject to different type ofBCs. The boundary conditions for the first case are that the beam is clamped (fixed)(4) at the left edge and simply supported at the right one. Whereas, the boundary conditions for the second case are the free boundary conditions (6) at both edges of the beam.

A tabular comparison of the exact and numerical eigenvalues λ_n of biharmonic equation of a one-dimensional beam that is fixed at the left edge and simply supported at the right edge for a mesh size (N = 700) are shown in Table 3.

Table 3: Exact and numerical eigenvalues of biharmonic equation of a beam in 1-dimension fixed at the left edge and simply supported at the right edge for (N = 700).

k	Exact Eigenvalues	Numerical Eigenvalues
1	0.3803	0.3818
2	3.9943	4.0102
3	17.3881	17.4567
4	50.8481	51.0465

Figure 5 shows the first six numerical eigenfunctions of the biharmonic equation of a beam in 1-dimension fixed at the left edge and simply supported at the right edge for a mesh size (N = 600) on the grid system for the domain $\Omega = 0 \leq x \leq l$. For a verification purposes, the boundary conditions are satisfiedproperlyat the both edges of the beam in Figure 5 where one can see how the left edge is fixed and the right edge is simply supported.

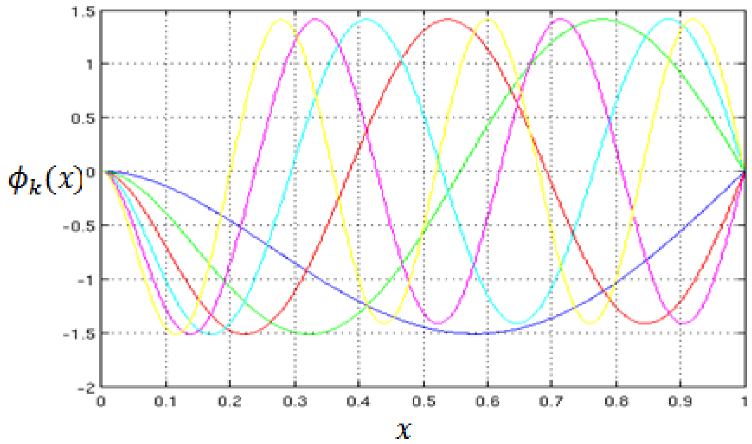

Figure 5: The numerical eigenfunctions of thebiharmonic equation of a beam in 1-dimension fixed at the left edge and simply supported at the right edge for $(\mathrm{N} = 600)$.

By a similar way of computing the analytic and the approximate eigenfunctions of equation (14), Figure 6 shows the first six numerical eigenfunctions of a biharmonic equation of a one-dimensional beam with free boundary conditions (4) at both edges of the beam for a mesh size $(\mathrm{N} = 600)$ on the grid system for the domain $\Omega = 0 \leq x \leq l$. As shown in Figure 6 the first eigenfunction$\phi_1(x)$ which appears in blue colour does not fluctuate on the x-axis, whereas the second eigenfunction$\phi_2(x)$ which appears in green colour fluctuates once on the x-axis. Also, the third eigenfunction$\phi_3(x)$ which appears in red colour fluctuates twice on the x-axis, and so on for the rest of the eigenfunctions. Thus the eigenfunction$\phi_k(x)$fluctuates on the x-axis $(k-1)$ times, for this reason there is no need to put a legend in the graph.

For a verification purposes, one should notice that the boundary conditions are satisfiedproperly at the both edges of the beam in Figure 6 where one can see that both edges of the bean are moving up and downfreely.

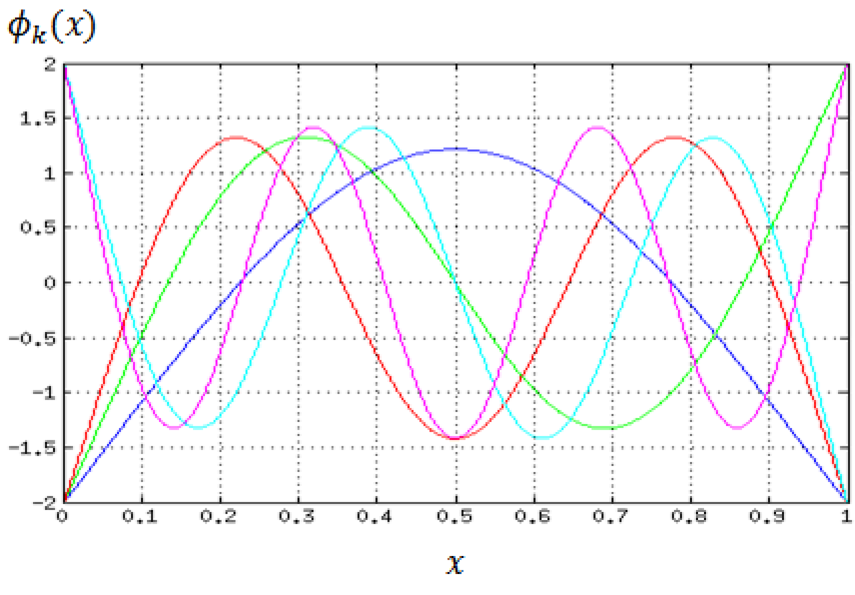

Figure 6: The numerical eigenfunctions of thebiharmonic equation of a one-dimensional beam with free BCs at both edgesfor $(\mathrm{N} = 600)$.

VIII. Conclusions

At a starting point, we have solved the time-independent Helmholtz equation as an eigenvalue problem analytically and numerically by implementing the FDM subject to theNBCs prescribed at the two edges of a one-dimensional beam. The comparison of the exact solution against the numerical one shows a very good agreement between the two categories.Then we have solved the time-independent biharmonic equation as an eigenvalue problem analytically and numerically by implementing the FDM subject to thedifferent types ofRobin BCsprescribed at the two edges of a one-dimensional beam. The comparison of the analytic results against the approximate results shows a very good agreement between the exact and the approximate methods. Therefore, one could claim that the FDM is an efficient technique for solving differential equations with a variety of mixed BCs, where sometimes the exact solution is rather cumbersome to obtain (if not exist).

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