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When Is an Ellipse Inscribed In a Quadrilateral Tangent at the Midpoint of Two or More Sides

Alan Horwitz
Corresponding Author:Alan Horwitz

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I. Introduction

Among all ellipses inscribed in a triangle, *T*, the midpoint, or Steiner, ellipse is interesting and well-known [2]. It is the unique ellipse tangent to *T* at the midpoints of all three sides of *T* and is also the unique ellipse of maximal area inscribed in *T*. What about ellipses inscribed in quadrilaterals, *Q* ? Not surprisingly, perhaps, there is not always a midpoint ellipse-i.e., an ellipse inscribed in *Q* which is tangent at the midpoints of all four sides of *Q*; In fact, in [1] it was shown that if there is a midpoint ellipse, then *Q* must be a parallelogram. That is, if *Q* is not a parallelogram, then there is no ellipse inscribed in *Q* which is tangent at the midpoint of all four sides of *Q*; But can one do better than four sides of *Q* ? In other words, if *Q* is not a parallelogram, is there an ellipse inscribed in *Q* which is tangent at the midpoint of three sides of *Q* ? In Theorem 1 we prove that the answer is no. In fact, unless *Q* is a trapezoid(a quadrilateral with at least one pair of parallel sides), or what we call a midpoint diagonal quadrilateral(see the definition below), then there is not even an ellipse inscribed in *Q* which is tangent at the midpoint of two sides of *Q*(see Lemmas 3 and 4).

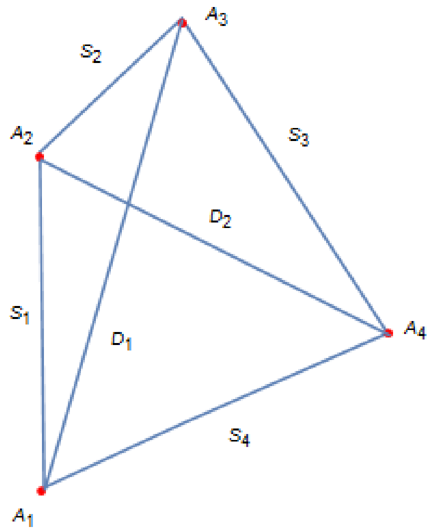
Definition: A convex quadrilateral, *Q*, is called a midpoint diagonal quadrilateral(mdq) if the intersection point of the diagonals of *Q*coincides with the midpoint of at least one of the diagonals of *Q*.

A parallelogram, Þ, is a special case of an mdq since the diagonals of Þ bisect one another. In [5] we discussed mdq's as a generalization of parallelograms in a certain sense related to tangency chords and conjugate diameters of inscribed ellipses.

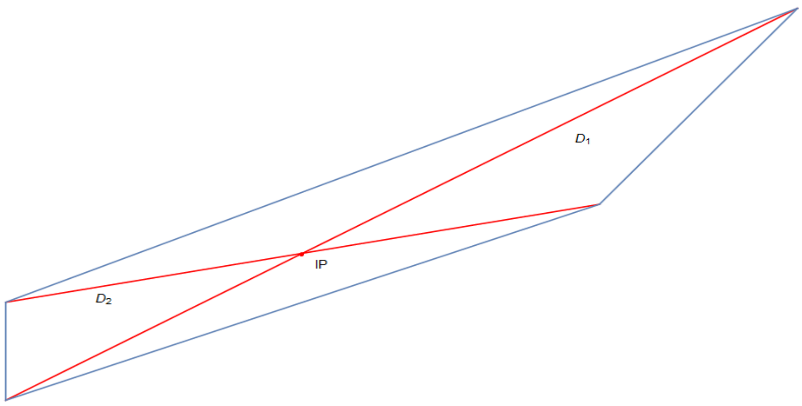
What about uniqueness ? If *Q* is an mdq, then the ellipse inscribed in *Q* which is tangent at the midpoint of two sides of *Q* is not unique. Indeed we prove(Lemma 3) that in that case there are two such ellipses. However, if *Q* is a trapezoid, then the ellipse inscribed in *Q* which is tangent at the midpoint of two sides of *Q* is unique (Lemma 4).

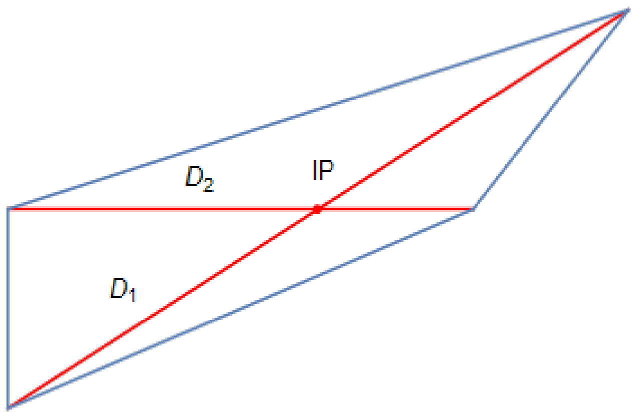
Is there a connection with tangency at the midpoint of the sides of *Q* and the ellipse of maximal area inscribed in *Q* as with parallelograms ? In [3] we showed that the midpoint ellipse for a parallelogram also turns out to be the unique ellipse of maximal area inscribed in *Q*. For trapezoids, we prove((Lemma 4) that the unique ellipse of maximal area inscribed in *Q* is the unique ellipse tangent to *Q* at the midpoint of two sides of *Q*. However, for mdq's, the unique ellipse of maximal area inscribed in *Q* need not be tangent at the midpoint of any side of *Q*.

We use the notation $Q(A_1, A_2, A_3, A_4)$ to denote the quadrilateral with vertices A_1, A_2, A_3, and A_4 ,starting with $A_1 =$ lower left corner and going clockwise. Denote the sides of $Q(A_1, A_2, A_3, A_4)$ by S_1, S_2, S_3, and S_4, going clockwise and starting with the leftmost side, S_1, and denote the diagonals of $Q(A_1, A_2, A_3, A_4)$ by $D_1 = A_1A_3$ and $D_2 = A_2A_4$.



We note here that there are two types of mdq's: Type 1, where the diagonals intersect at the midpoint of D_2 and Type 2, where the diagonals intersect at the midpoint of D_1; Mdq's of types 1 and 2, respectively, are illustrated below.

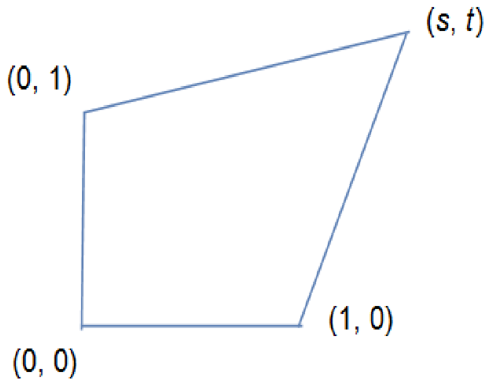




Given a convex quadrilateral, $Q = Q(A_1, A_2, A_3, A_4)$, which is not a parallelogram, it will simplify our work below to consider quadrilaterals with a special set of vertices. In particular, there is an affine transformation which sends A_1, A_2, and A_4 to the points $(0,0),(0,1)$, and $(1,0)$, respectively. It then follows that $A_3 = (s,t)$ for some $s,t > 0$; Summarizing:

$$Q_{s,t} = Q(A_1, A_2, A_3, A_4), \qquad (1)$$

$$A_1 = (0,0), A_2 = (0,1), A_3 = (s,t), A_4 = (1,0).$$



Since $Q_{s,t}$ is convex, $s+t>1$; Also, if Q has a pair of parallel vertical sides, first rotate counterclockwise by 90°, yielding a quadrilateral with parallel horizontal sides. Since we are assuming that Q is not a parallelogram, we may then also assume that $Q_{s,t}$ does not have parallel vertical sides and thus $s \neq 1$. Note that any trapezoid which is not a parallelogram may be mapped, by an affine transformation, to the quadrilateral $Q_{s,1}$; Thus we may assume that $(s,t) \in G$, where

$$G = \{(s,t) : s,t > 0, s+t > 1, s \neq 1\}. \qquad (2)$$

The following result gives the points of tangency of any ellipse inscribed in $Q_{s,t}$ (see [4] where some details were provided). We leave the details of a proof to the reader.

For the rest of the paper we work with the quadrilateral $Q_{s,t}$ defined above.

Proposition 1: (i) E_0 is an ellipse inscribed in $Q_{s,t}$ if and only if the general equation of E_0 is given by

$$t^2x^2 + \left(4q^2(t-1)t + 2qt(s-t+2) - 2st\right)xy +$$
$$\left(q(t-s)+s\right)^2 y^2 - 2qt^2x - 2qt\left(q(t-s)+s\right)y + q^2t^2 = 0 \qquad (3)$$

for some $q \in J = (0,1)$. Furthermore, (3) provides a one-to-one correspondence between ellipses inscribed in $Q_{s,t}$ and points $q \in J$.

(ii) If E_0 is an ellipse given by (3) for some $q \in J$, then E_0 is tangent to the four sides of $Q_{s,t}$ at the points

$$P_1 = \left(0, \frac{qt}{q(t-s)+s}\right) \in S_1,\ P_2 = \left(\frac{(1-q)s^2}{q(t-1)(s+t)+s}, \frac{t\left(s+q(t-1)\right)}{\left(q(t-1)(s+t)+s\right)}\right) \in S_2,$$

$$P_3 = \left(\frac{s+q(t-1)}{q(s+t-2)+1}, \frac{(1-q)t}{q(s+t-2)+1}\right) \in S_3, \text{ and } P_4 = (q,0) \in S_4.$$

Remark: Using Proposition 1, it is easy to show that one can always find an ellipse inscribed in a quadrilateral, Q, which is tangent to Q at the midpoint of at least one side of Q, and this can be done for any given side of Q.

The following lemma gives necessary and sufficient conditions for $Q_{s,t}$ to be an mdq.

Lemma 1: (i) $Q_{s,t}$ is a type 1 midpoint diagonal quadrilateral if and only if $s=t$.

(ii) $Q_{s,t}$ is a type 2 midpoint diagonal quadrilateral if and only if $s+t=2$.

Proof: The diagonals of $Q_{s,t}$ are $D_1 : y = \frac{t}{s}x$ and $D_2 : y = 1-x$, and they intersect at the point $P = \left(\frac{s}{s+t}, \frac{t}{s+t}\right)$; The midpoints of D_1 and D_2 are $M_1 = \left(\frac{s}{2}, \frac{t}{2}\right)$ and $M_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$, respectively.Now

$M_2 = P \Leftrightarrow \frac{s}{s+t} = \frac{1}{2}$ and $\frac{t}{s+t} = \frac{1}{2}$, both of which hold if and only if $s = t$; That proves (i); $M_1 = P \Leftrightarrow \frac{s}{s+t} = \frac{1}{2}s$ and $\frac{t}{s+t} = \frac{1}{2}t$, both of which hold if and only if $s + t = 2$. That proves (ii).

The following lemma shows that affine transformations preserve the class of mdq's. We leave the details of the proof to the reader.

Lemma 2: Let $T : R^2 \to R^2$ be an affine transformation and let Q be a midpoint diagonal quadrilateral. Then $Q' = T(Q)$ is also a midpoint diagonal quadrilateral.

II. Main Results

The following result shows that among non-trapezoids, the only quadrilaterals which have inscribed ellipses tangent at the midpoint of two sides are the mdq's.

Lemma 3: Let Q be a convex quadrilateral in the *xy* plane which is not a trapezoid.

(i) There is an ellipse inscribed in Q which is tangent at the midpoint of two or more sides of Q if and only if Q is a midpoint diagonal quadrilateral, in which case there are two such ellipses.

(ii) There is no ellipse inscribed in Q which is tangent at the midpoint of three sides of Q.

Proof: By Lemma 2 and standard properties of affine transformations, we may assume that $Q = Q_{s,t}$, the quadrilateral given in (1) with $(s,t) \in G$; The midpoints of the sides of $Q_{s,t}$ are given by $MP_1 = \left(0, \frac{1}{2}\right) \in S_1$, $MP_2 = \left(\frac{s}{2}, \frac{1+t}{2}\right) \in S_2$, $MP_3 = \left(\frac{1+s}{2}, \frac{t}{2}\right) \in S_3$, and $MP_4 = \left(\frac{1}{2}, 0\right) \in S_4$; Now let E_0 denote an ellipse inscribed in $Q_{s,t}$, and let $P_j \in S_j$, $j = 1,2,3,4$ denote the points of tangency of E_0 with the sides of $Q_{s,t}$; By Proposition 1(ii),

$$P_1 = MP_1 \Leftrightarrow \frac{qt}{q(t-s)+s} = \frac{1}{2}. \quad (4)$$

$$P_2 = MP_2 \Leftrightarrow \frac{(1-q)s}{q(t-1)(s+t)+s} = \frac{1}{2} \quad (5)$$

$$\text{and } \frac{t(s+q(t-1))}{(q(t-1)(s+t)+s)} = \frac{1+t}{2}. \quad (6)$$

$$P_3 = MP_3 \Leftrightarrow \frac{s+q(t-1)}{q(s+t-2)+1} = \frac{1+s}{2} \quad (7)$$

$$\text{and } \frac{(1-q)t}{q(s+t-2)+1} = \frac{t}{2}. \quad (8)$$

$$P_4 = MP_4 \Leftrightarrow q = \frac{1}{2}. \quad (9)$$

Equations (4) and (9) each have the unique solutions $q_1 = \frac{s}{s+t}$ and $q_4 = \frac{1}{2}$, respectively.The system of equations in (5) and (6) has unique solution $q_2 = \frac{s}{t^2 + st + s - t}$, and the system of equations in (7) and (8) has unique solution $q_3 = \frac{1}{s+t}$; It is trivial that $q_1, q_3, q_4 \in J = (0,1)$; Since $(s,t) \in G$, $t(s+t-1) > 0$, which implies that $q_2 \in J$. We now check which pairs of midpoints of sides of $Q_{s,t}$ can be points of tangency of E_0; Note that different values of q yield distinct inscribed ellipses by the one-to-one correspondence between ellipses inscribed in $Q_{s,t}$ and points $q \in J$.

(a) S_1 *and* S_2: $q_1 = q_2 \Leftrightarrow \frac{s}{t^2+st+s-t} - \frac{s}{s+t} = 0 \Leftrightarrow \frac{st(s+t-2)}{(ts+t^2+s-t)(s+t)} = 0 \Leftrightarrow s+t=2.$

(b) S_1 *and* S_3: $q_1 = q_3 \Leftrightarrow \frac{1}{s+t} - \frac{s}{s+t} = 0 \Leftrightarrow \frac{s-1}{s+t} = 0,$ which has no solution since $s \neq 1.$

(c) S_2 *and* S_3: $q_2 = q_3 \Leftrightarrow \frac{1}{s+t} - \frac{s}{t^2+st+s-t} = 0 \Leftrightarrow \frac{(t-s)(s+t-1)}{(ts+t^2+s-t)(s+t)} = 0 \Leftrightarrow s=t.$

(d) S_1 *and* S_4: $q_1 = q_4 \Leftrightarrow \frac{1}{2} - \frac{s}{s+t} = 0 \Leftrightarrow \frac{1}{2}\frac{s-t}{s+t} = 0 \Leftrightarrow s=t.$

(e) S_2 *and* S_4: $q_2 = q_4 \Leftrightarrow \frac{1}{2} - \frac{s}{t^2+st+s-t} = 0 \Leftrightarrow \frac{1}{2}\frac{(s+t)(t-1)}{t^2+st+s-t} = 0,$ which has no solution since $s+t \neq 0$ and $t \neq 1.$

(f) S_3 *and* S_4: $q_3 = q_4 \Leftrightarrow \frac{1}{2} - \frac{s}{s+t} = 0 \Leftrightarrow \frac{1}{2}\frac{s+t-2}{s+t} = 0 \Leftrightarrow s+t=2.$

That proves that there is an ellipse inscribed in $Q_{s,t}$ which is tangent at the midpoints of S_1 *and* S_2 or at the midpoints of S_3 *and* S_4 if and only if $s+t=2,$ and there is an ellipse inscribed in $Q_{s,t}$ which is tangent at the midpoints of S_2 *and* S_3 or at the midpoints of S_1 *and* S_4 if and only if $s=t;$ Furthermore, if $s \neq t$ or if $s+t \neq 2,$ then there is no ellipse inscribed in $Q_{s,t}$ which is tangent at the midpoint of two sides of $Q_{s,t}$. That proves (i) by Lemma 1. To prove (ii), to have an ellipse inscribed in $Q_{s,t}$ which is tangent at the midpoint of three sides of $Q_{s,t}$, those three sides are either $S_1, S_2,$ and S_3; $S_1, S_2,$ and S_4; $S_1, S_3,$ and S_4; or $S_2, S_3,$ and S_4; By (a)-(f) above, that is not possible.

For *trapezoids* inscribed in Q we have the following result.

Lemma 4: Assume that Q is a trapezoid which is not a parallelogram. Then

(i) There is a unique ellipse inscribed in Q which is tangent at the midpoint of two sides of Q, and that ellipse is the unique ellipse of maximal area inscribed in Q.

(ii) There is no ellipse inscribed in Q which is tangent at the midpoint of three sides of Q.

Proof: Again, by affine invariance, we may assume that $Q = Q_{s,1}$, the quadrilateral given in (1) with $t=1$; Note that $0 < s \neq 1$; Now let E_0 denote an ellipse inscribed in $Q_{s,1}$. Letting $MP_j \in S_j$, $j = 1,2,3,4$ denote the corresponding midpoints of the sides and using Proposition 1(ii) again, with $t=1$, we have

$$P_1 = MP_1 \Leftrightarrow \frac{q}{(1-s)q+s} = \frac{1}{2}, \quad (10)$$

$$P_2 = MP_2 \Leftrightarrow (1-q)s = \frac{s}{2}, \quad (11)$$

$$P_3 = MP_3 \Leftrightarrow \frac{s}{(s-1)q+1} = \frac{1+s}{2} \quad \text{and (12)}$$

$$\frac{1-q}{(s-1)q+1} = \frac{1}{2}, \quad (13)$$

$$P_4 = MP_4 \Leftrightarrow q = \frac{1}{2}. \quad (14)$$

The unique solution of the equations in (11) and in (14) is $q = \frac{1}{2} \in J$; The unique solution of the equation in (10) is $q = \frac{s}{1+s} \in J$, and the unique solution of the system of equations in (12) and (13) is $q = \frac{1}{1+s} \in J$;

We now check which pairs of midpoints of sides of $Q_{s,1}$ can be points of tangency with E_0 :

(a) $q=\frac{1}{2}$ gives tangency at the midpoints of S_2 and S_4.

(b) S_1 and S_2 or S_1 and S_4: $\frac{s}{1+s}=\frac{1}{2}\Leftrightarrow s=1.$

(c) S_3 and S_2 or S_3 and S_4: $\frac{1}{1+s}=\frac{1}{2}\Leftrightarrow s=1.$

(d) S_1 and S_3: $\frac{s}{1+s}=\frac{1}{1+s}\Leftrightarrow s=1.$

Since we have assumed that $s\neq 1,$ the only way to have an ellipse inscribed in $Q_{s,1}$ which is tangent at the midpoint of two sides of $Q_{s,1}$ is if those sides are S_2 and S_4 and $q=\frac{1}{2}.$ That proves that there is a unique ellipse inscribed in $Q_{s,1}$ which is tangent at the midpoint of two sides of $Q_{s,1}$. Now suppose that E_0 is *any* ellipse with equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0,$ and let a and b denote the lengths of the semi-major and semi-minor axes, respectively, of E_0. Using the results in [7], it can be shown that $a^2b^2=\frac{4\delta^2}{\Delta^3}$, where $\Delta=4AC-B^2$ and $\delta=CD^2+AE^2-BDE-F\Delta.$ By Proposition 1(i), then, with $t=1$ and after some simplification, we have $a^2b^2=f(q)=\frac{s}{4}q(1-q)$; $q=\frac{1}{2}$ clearly maximizes $f(q)$ and thus gives the ellipse of maximal area inscribed in $Q_{s,1}$. That proves the rest of (i). (ii) now follows easily and we omit the details.

Remark: It can be shown [5] that if Q is a trapezoid which is not a parallelogram, then Q cannot be an mdq. Thus the only quadrilaterals Lemmas 3 and 4 have in common are parallelograms.

Since a convex quadrilateral which is not a parallelogram either has no two sides which are parallel, or is a trapezoid, the following theorem follows immediately from Lemma 3(ii) and Lemma 4(ii).

Theorem: Suppose that Q is a convex quadrilateral which is not a parallelogram. Then there is no ellipse inscribed in Q which is tangent at the midpoint of three sides of Q.

Examples: (1) Let Q be the quadrilateral with vertices $(0,0),(0,1),(2,4),$ and $(1,1)$; It follows easily that Q is a type 1 midpoint diagonal quadrilateral. The ellipse with equation

$10\left(x-\frac{2}{3}\right)^2-10\left(x-\frac{2}{3}\right)\left(y-\frac{4}{3}\right)+4\left(y-\frac{4}{3}\right)^2=\frac{5}{3}$ is tangent to Q at $\left(0,\frac{1}{2}\right)$ and at $\left(\frac{1}{2},\frac{1}{2}\right)$, the midpoints of S_1 and S_4, respectively. The ellipse with equation

$54\left(x-\frac{4}{5}\right)^2-54\left(x-\frac{4}{5}\right)\left(y-\frac{8}{5}\right)+16\left(y-\frac{8}{5}\right)^2=\frac{27}{5}$ is tangent to Q at $\left(1,\frac{5}{2}\right)$ and at $\left(\frac{3}{2},\frac{5}{2}\right)$, the midpoints of S_2 *and* S_3, respectively. One can show that neither of these ellipses is the ellipse of maximal area inscribed in Q. See Figure 1 below.

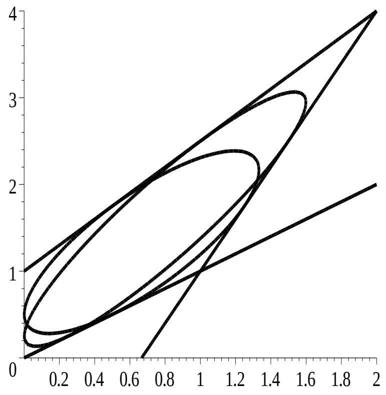


Figure 1

(2) Let Q be the trapezoid with vertices $(0,0),(0,1),(2,1)$, and $(1,1)$; The ellipse with equation $\left(x-\frac{5}{4}\right)^2-3\left(x-\frac{5}{4}\right)\left(y-\frac{1}{2}\right)+\frac{25}{4}\left(y-\frac{1}{2}\right)^2=1$ is tangent to Q at $(2,1)$ and at $\left(\frac{1}{2},0\right)$, the midpoints of S_2 and S_4, respectively. See Figure 2 below.

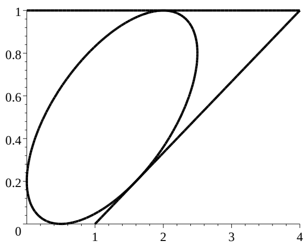


Figure 2

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