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High Order Hybrid Method for the Solution of Ordinary Differential Equations

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Abstract: *In this paper a new three-step consistent and zero stable implicit High Order Hybrid Methods has been developed for the solution of initial valued problems of ODEs. This work is based on the general High Order Hybrid algorithm developed by Shokri. The three-step method developed is shown to be of order 8, consistent, zero-stable and convergent. Simpson's block method was used to generate starting values for the implementation of the new methods. Results of numerical experiments to test the efficiency of the new methods compared to the exact solutions reveals that the new hybrid scheme computes favorably with most numerical methods and even better than some of the existing methods in literatures. Hence the new hybrid method proposed in this work is suitable for good approximation of the solution of initial valued problems of first order ordinary differential equations.*

Keywords: *Offstep point, High Order, Consistent, Ordinary Differential Equation, Hybrid Methods.*

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I. Introduction

Mathematical model are developed as prototype of real life scenario for better understanding and experiments. Most of these real life phenomena arises from science and engineering, physical system, psychology, medicine economics and virtually all aspect of human endeavors. This mathematical models mostly results into differential equations of one or more derivatives of unknown functions (i.e. as either Partial differential Equations PDEs or Ordinary Differential Equations ODEs). Unfortunately, the analytical solutions of most of these differential equations either does not exist or are cumbersome, hence the need for numerical solution.

Consider a first order ODEs of the form

$$y' = f(x,y), y(a) = y_0,\ x\epsilon[a,b], x, y \epsilon R^n \ and\ f \epsilon C^1[a,b] \quad \ldots\ldots\ldots\ldots\ldots\ldots.. \qquad (1)$$

where we seek the solution of (1) not on the entire interval $x\epsilon[a,b]$, but on the grid points $\{x_i | x_i = a + ih\}_0^k$ and some off grid points $v \notin \{0,1,2,\cdots,k\}$ provided (1) is well-possed.

The development of several numerical methods for solving (1) abound in many literatures, and can broadly be categorize into extrapolation (linear multi-steps) and substitution (One-step)/Runge-kuta methods [16]. The Linear Multi-Step Methods (L.M.M) though more accurate than the one-step methods, have to their disadvantage the sacrifices of the one-step nature of the algorithm in order to achieve higher order, this makes them not efficient in terms of function evaluations per changing step size due to dependence on predictor starting values, unlike the one-step counterparts that are self starting [13]. Hence, the challenges of developing numerical schemes of high order with low stepsize.

The Runge–Kutta Methods been a one-step method tends to achieve higher order by sacrificing linearity while preserving the one-step nature of the algorithm. But to its disadvantage, error analysis of the Runge-Kutta methods is considerably more difficult than in the case of LMMs, thus time consuming, hence further increase the dimension of the problem. To achieve the goal of developing numerical schemes of high order with low step size, led to the advent of what is christen 'Hybrid Methods' or simply 'Modified LMM'.

The hybrid methods are modified LMM which incorporates function evaluation at offstep points. The classical hybrid algorithms (2) on which many modern hybrid schemes are hinged on is accredited to the work of [2, 7, 8].

$$\sum_{j=1}^{k} \propto_j y_{n+j} = h \sum_{j=1}^{k} \beta_j f_{n+j} + h\beta_v f_{n+v} \qquad (2)$$

where k = +1, α_0 and β_0 should not both be zero, $v \notin$ {0, 1, . . . , k}, and $f_{n+v} = f(x_{n+v}, y_{n+v})$. These methods are a modification of the predictor-corrector methods achieved by introducing an additional predictor at an offstep points. The are generally low step methods that can achieved high order by increasing the number of offstep points rather than the stepsize. This allows the consequences of the Dahlquist barrier [6] to be avoided leading to convergent k-step methods with order $2k$ + 1 up to k = 7 [3,6, 7, 14, 15, 16]. Shokri(2014) added the off-step point y_{n+v}, (0 < v<1) in the right hand side of the classical hybrid methods together with slight modification to give a new high order k-step algorithms defined as

$$y_{n+1} = \sum_{j=0}^{k} a_j y_{n-j+1} + \sum_{j=1}^{v} b_j y_{n-\theta_j+1} + h\sum_{j=o}^{k} c_j f_{n-j+1} + h\sum_{j=1}^{v} d_j f_{n-\theta_j+1} \qquad (\mathbf{3})$$

$where\ \ a_j\ ,b_j\ ,c_j\ ,d_j\ are\ arbitrary\ constants/coefficients, with\ \ 0 < \theta_j < \ k\ such\ that\ \theta_j \notin \{0,1,2,\cdots,k\}, j\ =\ 1,2,\ldots,v\ \ as\ free\ parameter(s)\ .$

$\propto_k = +1, \propto_0\ \ and\ \beta_0\ \ are\ not\ both\ zero, v = \{0,1,2,\ldots,k\}\ and\ \ f_{n+v} = f(x_{n+v}, y_{n+v}).$

This new algorithm achieves order 2(v+k) leading to methods with order 2 higher than the order achieved by the classical k-step algorithm. The implementation of (3) been an implicit method can only be possible if we know the values of the solution at $y(x)$ and $y'(x)$ at k successive points., this can be gotten from 'Block Methods'. The Block Methods were first proposed by Milne [12], who advocated their use only as a means of obtaining starting values for predictor-corrector algorithms. In other words the unknown y_{n+1} cannot be calculated directly since it is contained within y'_{n+1}. Today the advantages of the block methods has surpasses just their use for obtaining starting values, it has made implement of schemes and other theoretical analysis of numerical schemes like stability analysis, finding order , plotting of region of stability/interval of stability etc, a lot easier as demonstrated in the work of [1, 4, 5, 10, 11, 13]. According to [13] the work of Onumanyi in 2004 where they used multistep collocation approach of Lie and Norset (1986) gave the major insight into the exploration of what is christen today as block linear multi-step methods. In this paper we used the Shokri (2014) algorithm (1.3) to develop a new Three-Step High order implicit hybrid methods for the solution of ODEs..

II. Methodology

In this section the new high order hybrid multi-step method (3) was derived using the difference equation(2) which we assumed has a unique solution $y(x)$ on $[a, b]$, and suppose that $y(x) \in C^{(p+1)}[a,b]\ for\ p \geq 1$. Then the deference operator L for the algorithm (2) can be written as.

$$L[y(x),h] = y(x + h) - \sum_{j=0}^{k} a_j y(x + (1-j)h) - h\sum_{j=0}^{k} c_j f(x + (1-j)h)$$
$$-\sum_{j=o}^{v} [b_j y\big(x + \big(1-\theta_j\big)h\big) + hd_j fy\big(x + \big(1-\theta_j\big)h\big) \qquad (\mathbf{4})$$

Using Taylor series expansion to determine the coefficients of (1.3),then (2.1) can be expressed as

$$L[y(x),h] = C_0 y(x_n) + C_0 h y^{(1)}(x_n) + \cdots + C_q h^q y^{(q)}(x_n) + \cdots \qquad (\mathbf{2.2})$$

Where

$$C_q = \frac{1}{q!} - \sum_{j=1}^{k} \frac{(1-j)^q a_j}{q!} - \sum_{j=1}^{v} \left[\frac{(1-\theta_j)^q b_j}{q!} - \frac{(1-\theta_j)^{(q-1)} d_j}{(q-1)!}\right] - \sum_{j=1}^{k} \frac{(1-j)^{(q-1)} c_j}{(q-1)!}, q = 2,3,4\ldots \qquad (\mathbf{5})$$

2.1. DERI VATION OF THE NEW HIGH ORDER HYBRID METHODS (HOHM)

By choosing $k = 3\ \ and\ v = 1$ in $(\mathbf{4})$, we get

$$a_1 y_n + a_2 y_{n-1} + a_3 y_{n-2} + b_1 y_{n-\theta_1+1} + hc_0 f_{n+1} + hc_1 f_n + hc_2 f_{n-1} + hc_3 f_{n-2} + hd_1 f_{n-\theta_1+1}$$

where $a_1, a_2, a_3, b_1, c_0, c_1\ , c_2, c_3, d_1\ and\ 0 \leq \theta_1 \leq 1$ are 9 arbitrary parameters to be determine, with θ_1 as a free parameter.

By solving for the arbitrary parameters with choices of the free parameter $\theta_1 = \frac{1}{2}$ will yields

$$y_{n+3} = \frac{67}{2875} y_n + \frac{7}{23} y_{n+1} + \frac{81}{23} y_{n+2} - \frac{8192}{2875} y_{n+\frac{5}{2}} + \frac{3}{575} hf_n + \frac{3}{23} hf_{n+1} + \frac{27}{23} hf_{n+2} + \frac{768}{575} hf_{n+\frac{5}{2}} + \frac{3}{23} hf_{n+3}$$

As the new three step hybrid method for the solution of (1)

2.2 NUMERICAL ANALYSES OF THE HYBRID SCHEME

In this section we analyze the order, error constants, zero-stability, convergence and interval of absolute stability of the new hybrid scheme.

2.2.1 Order and Error Constant of the Hybrid Scheme

Using the Linear difference operator (5) with $\boldsymbol{\theta_1 = \frac{1}{2}}$ with the aid of Maple software,

we have $C_0 = C_1 = C_2 = \cdots = C_8 = 0, \quad but\ C_9 \neq 0,$

Hence our Hybrid scheme is of order 8 with error constant of $-\frac{1}{309120}$

2.2.2 Zero Stability and Convergence of the Hybrid Scheme

The first characteristic polynomial of the method of the hybrid scheme is computed as

$$\rho(r) = r^3 + \frac{8192}{2875} r^{\frac{5}{2}} - \frac{81}{23} r^2 - \frac{7}{23} r - \frac{67}{2875}$$

Using the boundary locus method , the hybrid scheme has one principal root r = 1 and two distinct spurious roots r_2 and r_3 as complex with modulus less than 1, hence the new hybrid scheme is zero stable. Since the Hybrid scheme has order greater than 2 and is also zero stable, hence is convergent.

III. Numerical Experiments

The new hybrid method is experimented in solving the below questions to test their efficiency and we adopt the block of Simpsons One-third rule to generate simultaneously starting values at $y(x)$ and $y'(x)$ t for the integration

1. $y' = -y,\ y(0) = 1, h = 0.1, (0 \le x \le 1),\ y(x) = e^{-x}$
2. $6\, y'_1 = -100y_1 + 9.901y_2, y'_2 = 0.1y_1 - y_2,\ y_1(0) = 1, y_2(0) = 0, h = 0.1,$ $(0 \le x \le 1), Exact\ Solution: y_1(x) = e^{-0.99x}, y_2(x) = 10e^{-0.99x}$

X	EXACT SOLUTION	Hybrid Scheme with, K=3, $\boldsymbol{\theta_1 = \frac{1}{2}}$
0.1	**0.9048374180**	**3.5814X10^{-6}**
0.2	**0.8187307531**	**3.5815X10^{-6}**
0.3	**0.7408182207**	**1.62917X10^{-5}**
0.4	**0.7408182207**	**1.20881X10^{-5}**
0.5	**0.6065306597**	**1.3608X10^{-5}**
0.6	**0.5488116361**	**2.141386X10^{-5}**
0.7	**0.4965853038**	**1.987559X10^{-5}**
0.8	**0.4493289641**	**1.99633X10^{-5}**
0.9	**0.4065696597**	**2.68240x10^{-5}**
1.0	**0.3678794412**	**2.28151X10^{-5}**

Table 1: Showing results of Example 1

X	Exact Soln of y_1	**Hybrid Scheme with, K=3, $\theta_1 = \frac{1}{2}$**	Exact Soln of y_2	**Hybrid Scheme with, K=3, $\theta_1 = \frac{1}{2}$**
0.1	**0.9057427080**	**$3.48X10^{-6}$**	**9.057427080**	**$3.01X10^{-6}$**
0.2	**0.8203698531**	**$3.22X10^{-7}$**	**8.203698531**	**$2.58X10^{-6}$**
0.3	**0.7430440124**	**$1.25X10^{-5}$**	**7.430440124**	**$1.00X10^{-4}$**
0.4	**0.6730066959**	**$5.56X10^{-5}$**	**6.730066959**	**$6.15\ X10^{-4}$**
0.5	**0.6095709073**	**$1.00X10^{-5}$**	**6.095709073**	**$3.00X10^{-4}$**
0.6	**0.5521144043**	**$6.94X10^{-5}$**	**5.521144043**	**$3.14X10^{-4}$**
0.7	**0.5000735957**	**$8.20X10^{-5}$**	**5.000735957**	**$3.01X10^{-4}$**
0.8	**0.4529380128**	**$6.33X10^{-5}$**	**4.529380128**	**$9.09X10^{-4}$**
0.9	**0.4102453023**	**$1.00X10^{-5}$**	**4.102453023**	**$4.09X10^{-4}$**
1.0	**0.3715766910**	**$6.11X10^{-5}$**	**3.715766910**	**$3.11X10^{-4}$**

Table 2: Showing results of Example 2

IV. Discussion And Conclusion

A three-step consistent and zero stable implicit High Order Hybrid Methods has been developed for the solution of initial valued problems of ODEs from the new k-step hybrid algorithms of Shokri (2014) . The inclusion of offstep points allowed the adoption of linear multistep procedure which circumvent the zero stability barrier and consequently upgraded the order (i.e. order 8) and accuracy of the method. It is observed that, he order of the methods increases as the number of offstep points increases. The new numerical scheme has very low error constants of $-\frac{1}{309120}$. Numerical experiments performed with the new hybrid scheme on some samples of first order initial value problems of ODEs, reveals that the new hybrid scheme computes favourably with most numerical methods and even better than some of the existing methods. Hence the new hybrid method proposed in this work is suitable for good approximation of the solution of initial valued problems of first order ordinary differential equations.

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