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Pell Sequences in Rings with Identity and Their Applications

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Abstract: *In this paper, we define Pell sequences $\{\mathcal{P}_n\}_{n\geq 0}$ over rings with identity and investigate the their properties. We obtain the generating functions and Binet's formulas for these sequences. Then we present some Hessenberg matrices with applications to these sequences and show that the determinants and permanents of these Hessenberg matrices are equal to nth term of Pell sequences in rings with identity. Also, the terms of these sequences are derivated by the matrix.*

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I. Introduction

Sequences of integer number such as Fibonacci, Lucas, Pell, Jacobsthal are the most well-known second order recurrence sequences. For rich applications of these sequences in science and nature, one can see the citations in [14]. Fibonacci sequence is defined by recurrence relation $f_n = f_{n-1} + f_{n-2}$ with the initial values $f_0 = 0$ and $f_1 = 1$ for $n \geq 2$. The Fibonacci numbers f_n are the terms of the sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ Wall first proposed the notion of Wall number in 1960 and gave some fundamental properties concerning Wall number of the Fibonacci sequence.

The Pell sequence is defined by recurrence relation

$$P_n = 2P_{n-1} + P_{n-2}\,,\ n \geq 2$$

where $P_0 = 0$ and $P_1 = 1$ [3]. The first few terms of the Pell sequence are

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \ldots$$

For $n \geq 0$, the nth Pell number is

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where $\alpha = 1 + \sqrt{2}$ ve $\beta = 1 - \sqrt{2}$ are roots of the characteristic equation $x^2 - 2x - 1 = 0$.

Sequences of integer number have been intensively studied for many years and have become into an interesting topic in Applied Mathematics. Most of the study associated with sequences of integer number is done in groups (see, for example, [6], [8], [12]). However, very little is done in rings and Pell sequences in ring has never been studied. Let R be a ring with ideality 1. The sequence $\{M_n\}_{n\geq 0}$ of elements of R, recursively defined by

$$M_n = BM_{n-1} + AM_{n-2}\,, \quad n \geq 2 \qquad (1)$$

where $M_0\,, M_1\,, A$ and B are abritrary elements of R. DeCarli gave a generalized Fibonacci sequence over an arbitrary ring in 1970. DeCarli begins by considering a special case of equation (1), denoted by $\{F_n\}_{n\geq 0}$ and defined by

$$F_n = BF_{n-1} + AF_{n-2}, \quad n \geq 2$$

where $F_0 = 0, F_1 = 1$ and $A,\ B$ are abritrary elements of R [5]. Special cases of Fibonacci sequence over an arbitrary ring have been considered by Buschman [13], Horadam [2] and Vorobyov [10] where this ring was taken to be the set of integers. Wyler [11] also worked with such a sequence over a particular commutative ring with identity. Tasyurdu and Gültekin [15], [16] obtained the period of generalized Fibonacci sequence in finite rings with identity and fields of order p^2. Also, it was obtained the period of generalized Fibonacci sequence was defined over an arbitrary ring and the terms of this sequence are derivated by determinant of Tridiagonal matrix [17]. In [18], it was defined the Jacobsthal sequences over arbitrary rings with identity and obtained the some properties of these sequences.

In the literature many research that concern about the sequences of integer number contribute significantly to mathematics, especially to the field of matrix algebra. Many authors studied on determinantal and permanental representations of the sequences of integer number and investigated the relationships between the Hessenberg matrices and the these sequences (see, for example, [1], [7], [9], [14]).

A nxn matrix $M_n = (m_{ij})$ is called lower Hessenberg matrix if $m_{ij} = 0$ when $j - i > 1$, i.e.,

$$M_n = \begin{pmatrix} m_{1,1} & m_{1,2} & 0 & 0 & \cdots & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & 0 & \cdots & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & m_{n-1,4} & \cdots & m_{n-1,n} \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & \cdots & m_{n,n} \end{pmatrix}.$$

In [9], for $n \geq 2$, it was given the following formula

$$det(M_n) = m_{n,n}\, det(M_{n-1}) + \sum_{i=1}^{n-1} \left[(-1)^{n-i} m_{n,i} \prod_{j=i}^{n-1} m_{j,j+1} det(M_{i-1}) \right]. \qquad (2)$$

with $det(M_0) = 1$ and $det(M_1) = m_{11}$. Also in [1], for $n \geq 2$, it was given the following formula

$$per(M_n) = m_{n,n}\, per(M_{n-1}) + \sum_{i=1}^{n-1} \left[m_{n,i} \prod_{j=i}^{n-1} m_{j,j+1} per(M_{i-1}) \right]. \qquad (3)$$

with $per(M_0) = 1$ and $per(M_1) = m_{11}$.

II. The Pell Sequences in Rings with Identity and Their Generating Functions and Binet's Formulas

In this section, we define Pell sequences in rings with identity and give generating functions and Binet's fomulas for them. Throughout this paper, $\mathcal{P}_0 = 0$ is zero of a ring and $\mathcal{P}_1 = 1$ is identity of a ring.

Definition 1. The Pell sequences $\{\mathcal{P}_n\}_{n\geq 0}$ in rings with identity are defined by recurrence relation

$$\mathcal{P}_n = 2A_1\mathcal{P}_{n-1} + A_0\mathcal{P}_{n-2}, \qquad n \geq 2 \qquad (4)$$

with initial conditions $\mathcal{P}_0 = 0$, $\mathcal{P}_1 = 1$ and $A_0,\ A_1$ are arbitrary elements of the ring, $\mathcal{P}_n$ is n-th term of these sequences.

Using Definition 1, we can write the first terms of sequence $\{\mathcal{P}_n\}_{n\geq 0}$ as follows:

$$0, 1, 2A_1, 4A_1^2 + A_0, 8A_1^3 + 4A_1A_0, 16A_1^4 + 12A_1^2A_0 + A_0^2 \ldots$$

Now we present generating functions of Pell sequences in rings with identity.

Theorem 1. The generating functions for Pell sequences in rings with identity are

$$g(t) = \frac{t}{1 - 2A_1t - A_0t^2} \qquad (5)$$

where $A_0,\ A_1$ are arbitrary elements of the ring.

Proof. Define $g(t) = \sum_{n=0}^{\infty} \mathcal{P}_n t^n$. Then we have

$$\begin{aligned} g(t) &= \mathcal{P}_0 + \mathcal{P}_1 t + \mathcal{P}_2 t^2 + \cdots + \mathcal{P}_n t^n + \cdots \\ -2A_1 g(t)t &= -2A_1\mathcal{P}_0 t - 2A_1\mathcal{P}_1 t^2 - 2A_1\mathcal{P}_2 t^3 - \cdots - 2A_1\mathcal{P}_n t^{n+1} - \cdots \\ -A_0 g(t)t^2 &= -A_0\mathcal{P}_0 t^2 - A_0\mathcal{P}_1 t^3 - A_0\mathcal{P}_2 t^4 - \cdots - A_0\mathcal{P}_n t^{n+2} - \cdots \end{aligned}$$

If we add the equations by side by, we get

$$(1 - 2A_1t - A_0t^2)g(t) = \mathcal{P}_0 + \mathcal{P}_1 t - 2A_1\mathcal{P}_0 t$$

where the coefficients of t^n for $n \geq 2$ are equal to zero by using recurrence relation $\mathcal{P}_n = 2A_1\mathcal{P}_{n-1} + A_0\mathcal{P}_{n-2}$ from equation (4). Since $\mathcal{P}_0 = 0$, $\mathcal{P}_1 = 1$, the generating functions for Pell sequences in rings with identity are

$$g(t) = \frac{t}{1 - 2A_1t - A_0t^2}. \qquad \blacksquare$$

Next, we present Binet's formulas for Pell sequences in rings with identity. The Binet's formula give us to find the n-th Pell number in rings without having to know all the terms before it. The corresponding characteristic equation of the equation (4) is

$$x^2 - 2A_1x - A_0 = 0.$$

The roots of this equation are

$$\alpha = A_1 + \sqrt{{A_1}^2 + A_0}$$
$$\beta = A_1 - \sqrt{{A_1}^2 + A_0}.$$

Also, the roots α and β verify the following relations:

$$\alpha + \beta = 2A_1$$
$$\alpha - \beta = 2\sqrt{{A_1}^2 + A_0}$$
$$\alpha\beta = -A_0$$
$$\alpha^2 = 2A_1\alpha + A_0$$
$$\beta^2 = 2A_1\beta + A_0.$$

Theorem 2. For $n \geq 0$, the nth Pell number in rings with identity is

$$\mathcal{P}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Proof. We can use generating functions of the sequences $\{\mathcal{P}_n\}_{n\geq 0}$. Using the roots of the equation $1 - 2A_1t - A_0t^2 = 0$

$$1 - 2A_1t - A_0t^2 = -A_0\left(t + \frac{\alpha}{A_0}\right)\left(t + \frac{\beta}{A_0}\right)$$

where $\alpha = A_1 + \sqrt{{A_1}^2 + A_0}$ and $\beta = A_1 - \sqrt{{A_1}^2 + A_0}$. We can write the generating functions of the sequences $\{\mathcal{P}_n\}_{n\geq 0}$ as

$$g(t) = \frac{t}{1 - 2A_1t - A_0t^2}$$
$$-A_0g(t) = \frac{t}{\left(t + \frac{\alpha}{A_0}\right)\left(t + \frac{\beta}{A_0}\right)}$$

Next, we find K and L which satisfy:

$$-A_0g(t) = \frac{t}{\left(t + \frac{\alpha}{A_0}\right)\left(t + \frac{\beta}{A_0}\right)} = \frac{K}{\left(t + \frac{\alpha}{A_0}\right)} + \frac{L}{\left(t + \frac{\beta}{A_0}\right)}.$$

We need to find do K and L, so the following system of equations should be solved:

$$K + L = 1$$
$$K\beta + L\alpha = 0$$

We find that:

$$K = \frac{\alpha}{\alpha - \beta}$$
$$L = \frac{-\beta}{\alpha - \beta}.$$

Substituting into the equation above gives the partial fractions expansion of $-A_0g(t)$

$$-A_0g(t) = \frac{\frac{\alpha}{\alpha-\beta}}{\left(t + \frac{\alpha}{A_0}\right)} + \frac{\frac{-\beta}{\alpha-\beta}}{\left(t + \frac{\beta}{A_0}\right)}$$
$$= \frac{\alpha}{\alpha - \beta}\left(\frac{1}{t + \frac{\alpha}{A_0}}\right) - \frac{\beta}{\alpha - \beta}\left(\frac{1}{t + \frac{\beta}{A_0}}\right)$$
$$= \frac{\alpha}{\alpha - \beta}\left(\frac{1}{t - \frac{1}{\beta}}\right) - \frac{\beta}{\alpha - \beta}\left(\frac{1}{t - \frac{1}{\alpha}}\right)$$

$$= \frac{\alpha}{\alpha - \beta}\frac{1}{\frac{1}{\beta}}\frac{1}{\beta t - 1} - \frac{\beta}{\alpha - \beta}\frac{1}{\frac{1}{\alpha}}\frac{1}{\alpha t - 1}$$

$$= \frac{\alpha\beta}{\alpha-\beta}\left(\sum_{n=0}^{\infty}\alpha^n t^n - \sum_{n=0}^{\infty}\beta^n t^n\right)$$
$$= \frac{-A_0}{\alpha-\beta}\left(\sum_{n=0}^{\infty}\frac{1}{\alpha-\beta}(\alpha^n-\beta^n)t^n\right)$$

Equation coefficients, we get

$$g(t) = \sum_{n=0}^{\infty}\frac{\alpha^n-\beta^n}{\alpha-\beta}t^n$$

we conlude that

$$\mathcal{P}_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}.$$

and so proof is completed. ∎

Alternatif Proof of Theorem 2: We can use induction method on n to prove $\mathcal{P}_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$. We can obviously see the true of the hypothesis for our two base cases:

$$n = 0;\ \frac{\alpha^0-\beta^0}{\alpha-\beta} = 0 = \mathcal{P}_0$$
$$n = 1;\ \frac{\alpha^1-\beta^1}{\alpha-\beta} = 1 = \mathcal{P}_1$$

Let us suppose that the Binet's formula holds for arbitrary integer $n \geq 0$. We shall show that the Binet's formula holds for $n+1$. Using equation (4) we obtain

$$\begin{aligned}\mathcal{P}_{n+1} &= 2A_1\mathcal{P}_n + A_0\mathcal{P}_{n-1}\\ &= 2A_1\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) + A_0\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right)\\ &= \frac{1}{\alpha-\beta}(\alpha^{n-1}\alpha^2-\beta^{n-1}\beta^2)\\ &= \frac{(\alpha^{n+1}-\beta^{n+1})}{\alpha-\beta}\end{aligned}$$

where $\alpha^2 = 2A_1\alpha + A_0$ and $\beta^2 = 2A_1\beta + A_0$. So proof is completed. ∎

Some Properties of Pell Sequences in Rings with Identity

Theorem 3. If $\mathcal{P}_{n+2} = 2A_1\mathcal{P}_{n+1} + A_0\mathcal{P}_n$ then $\mathcal{P}_{n+2} = \mathcal{P}_{n+1}2A_1 + \mathcal{P}_nA_0$.

Proof. We can complete the proof by induction on n. For $n = 0$, if $\mathcal{P}_2 = 2A_1\mathcal{P}_1 + A_0\mathcal{P}_0$ then

$$\mathcal{P}_2 = 2A_1\mathcal{P}_1 + A_0\mathcal{P}_0 = 2A_1(1) + A_0(0) = (1)2A_1 + (0)A_0 = \mathcal{P}_12A_1 + \mathcal{P}_0A_0.$$

We assume that it is true $n \in \mathbb{Z}^+$. That is, if $\mathcal{P}_{n+2} = 2A_1\mathcal{P}_{n+1} + A_0\mathcal{P}_n$ then $\mathcal{P}_{n+2} = \mathcal{P}_{n+1}2A_1 + \mathcal{P}_nA_0$. Then we shall show that it is true for $n+1$. Using induction's hypothesis if $\mathcal{P}_{n+3} = 2A_1\mathcal{P}_{n+2} + A_0\mathcal{P}_{n+1}$ then

$$\begin{aligned}\mathcal{P}_{n+3} &= 2A_1\mathcal{P}_{n+2} + A_0\mathcal{P}_{n+1}\\ &= 2A_1(2A_1\mathcal{P}_{n+1} + A_0\mathcal{P}_n) + A_0(2A_1\mathcal{P}_n + A_0\mathcal{P}_{n-1})\\ &= 2A_1(\mathcal{P}_{n+1}2A_1 + \mathcal{P}_nA_0) + A_0(\mathcal{P}_n2A_1 + \mathcal{P}_{n-1}A_0)\\ &= 2A_1(\mathcal{P}_{n+1}2A_1) + 2A_1(\mathcal{P}_nA_0) + A_0(\mathcal{P}_n2A_1) + A_0(\mathcal{P}_{n-1}A_0)\\ &= (2A_1\mathcal{P}_{n+1})2A_1 + (2A_1\mathcal{P}_n)A_0 + (A_0\mathcal{P}_n)2A_1 + (A_0\mathcal{P}_{n-1})A_0\\ &= (2A_1\mathcal{P}_{n+1} + A_0\mathcal{P}_n)2A_1 + (2A_1\mathcal{P}_n + A_0\mathcal{P}_{n-1})A_0\\ &= \mathcal{P}_{n+2}2A_1 + \mathcal{P}_{n+1}A_0\end{aligned}$$

Thus the proof is completed. ∎

Theorem 4. (Cassini's identity) For $n \geq 2$,

i. $\mathcal{P}_{n+1}\mathcal{P}_{n-1} - \mathcal{P}_n^2 = \mathcal{P}_{n-1}A_0\mathcal{P}_{n-1} - \mathcal{P}_nA_0\mathcal{P}_{n-2}$
ii. $\mathcal{P}_{n-1}\mathcal{P}_{n+1} - \mathcal{P}_n^2 = \mathcal{P}_{n-1}A_0\mathcal{P}_{n-1} - \mathcal{P}_{n-2}A_0\mathcal{P}_n$.

Proof. Using Theorem 3 if $\mathcal{P}_{n+2} = 2A_1\mathcal{P}_{n+1} + A_0\mathcal{P}_n$ then $\mathcal{P}_{n+2} = \mathcal{P}_{n+1}2A_1 + \mathcal{P}_nA_0$. Then

i. $$\begin{aligned}\mathcal{P}_{n+1}\mathcal{P}_{n-1} - \mathcal{P}_n^2 &= (\mathcal{P}_n2A_1 + \mathcal{P}_{n-1}A_0)\mathcal{P}_{n-1} - \mathcal{P}_n(2A_1\mathcal{P}_{n-1} + A_0\mathcal{P}_{n-2})\\ &= \mathcal{P}_n2A_1\mathcal{P}_{n-1} + \mathcal{P}_{n-1}A_0\mathcal{P}_{n-1} - \mathcal{P}_n2A_1\mathcal{P}_{n-1} - \mathcal{P}_nA_0\mathcal{P}_{n-2}\\ &= \mathcal{P}_{n-1}A_0\mathcal{P}_{n-1} - \mathcal{P}_nA_0\mathcal{P}_{n-2}.\end{aligned}$$

ii. $\mathcal{P}_{n-1}\mathcal{P}_{n+1} - \mathcal{P}_n^2 = \mathcal{P}_{n-1}(2A_1\mathcal{P}_n + A_0\mathcal{P}_{n-1}) - (\mathcal{P}_{n-1}2A_1 + \mathcal{P}_{n-2}A_0)\mathcal{P}_n$
$= \mathcal{P}_{n-1}2A_1\mathcal{P}_n + \mathcal{P}_{n-1}A_0\mathcal{P}_{n-1} - \mathcal{P}_{n-1}2A_1\mathcal{P}_n - \mathcal{P}_{n-2}A_0\mathcal{P}_n$
$= \mathcal{P}_{n-1}A_0\mathcal{P}_{n-1} - \mathcal{P}_{n-2}A_0\mathcal{P}_n$.
So the proof is completed. ■

There is a relation between the sequences $\{M_n\}_{n\geq 0}$ and the sequences $\{\mathcal{P}_n\}_{n\geq 0}$. We present the following theorem and corollary for this relation;

Theorem 5. For $n \geq 1, r \geq 0$,
$$M_{n+r} = \mathcal{P}_r A_0 M_{n-1} + \mathcal{P}_{r+1} M_n.$$

Proof. We can complete the proof by induction method on r. For $r = 0$,
$$\begin{aligned} M_{n+0} &= M_n \\ &= (0)A_0M_{n-1} + (1)M_n \\ &= \mathcal{P}_0A_0M_{n-1} + \mathcal{P}_1M_n. \end{aligned}$$
That is, $M_{n+0} = \mathcal{P}_0A_0M_{n-1} + \mathcal{P}_1M_n$. For $r = 1$,
$$\begin{aligned} M_{n+1} &= 2A_1M_n + A_0M_{n-1} \\ &= (1)A_0M_{n-1} + 2A_1(1)M_n + A_0(0)M_n \\ &= \mathcal{P}_1A_0M_{n-1} + (2A_1\mathcal{P}_1 + A_0\mathcal{P}_0)M_n \\ &= \mathcal{P}_1A_0M_{n-1} + \mathcal{P}_2M_n \end{aligned}$$
where $2A_1 = B$ and $A_0 = A$ in equation (1). That is, $M_{n+1} = \mathcal{P}_1A_0M_{n-1} + \mathcal{P}_2M_n$. We assume that it is true for $r = k$, namely

$$M_{n+k} = \mathcal{P}_kA_0M_{n-1} + \mathcal{P}_{k+1}M_n.$$

Then we shall show that
$$M_{n+k+1} = \mathcal{P}_{k+1}A_0M_{n-1} + \mathcal{P}_{k+2}M_n.$$

Using induction's hypothesis we obtain
$$\begin{aligned} M_{n+k+1} &= M_{(n+1)+k} \\ &= \mathcal{P}_kA_0M_n + \mathcal{P}_{k+1}M_{n+1} \\ &= \mathcal{P}_kA_0M_n - \mathcal{P}_{k+1}2A_1M_n + \mathcal{P}_{k+1}2A_1M_n + \mathcal{P}_{k+1}M_{n+1} \\ &= \mathcal{P}_{k+1}(M_{n+1} - 2A_1M_n) + (\mathcal{P}_kA_0 + \mathcal{P}_{k+1}2A_1)M_n \\ &= \mathcal{P}_{k+1}A_0M_{n-1} + \mathcal{P}_{k+2}M_n \end{aligned}$$
where $2A_1 = B$ and $A_0 = A$ in equation (1). That is,
$$M_{n+k+1} = \mathcal{P}_{k+1}A_0M_{n-1} + \mathcal{P}_{k+2}M_n$$

and so the proof is completed. ■

Corollary 1. For $n \geq 1$,
$$M_n = \mathcal{P}_nM_1 + \mathcal{P}_{n-1}A_0M_0.$$

Proof. Interchange r and n, replace n by $n - 1$ and $r = 1$ in Theorem 5, we get
$$M_{n+r} = \mathcal{P}_rA_0M_{n-1} + \mathcal{P}_{r+1}M_n$$
and
$$\begin{aligned} M_{r+n} &= M_{1+(n-1)} \\ &= \mathcal{P}_{n-1}A_0M_{1-1} + \mathcal{P}_{(n-1)+1}M_1 \\ &= \mathcal{P}_{n-1}A_0M_0 + \mathcal{P}_nM_1. \end{aligned}$$
That is, $M_n = \mathcal{P}_nM_1 + \mathcal{P}_{n-1}A_0M_0$ and the proof is completed. ■

For the sequences $\{\mathcal{P}_n\}_{n\geq 0}$, Theorem 5 becomes
$$\mathcal{P}_{n+r} = \mathcal{P}_rA_0\mathcal{P}_{n-1} + \mathcal{P}_{r+1}\mathcal{P}_n, \qquad n \geq 1. \tag{6}$$
If we replace n by $n + 1$ and r by n in equation (6), then we have
$$\mathcal{P}_{n+1}^2 + \mathcal{P}_nA_0\mathcal{P}_n = \mathcal{P}_{2n+1}.$$

Theorem 6. For $n \geq 1, r \geq 1$
$$\mathcal{P}_n\mathcal{P}_{n+r} - \mathcal{P}_{n+r}\mathcal{P}_n = \mathcal{P}_n\mathcal{P}_rA_0\mathcal{P}_{n-1} - \mathcal{P}_{n-1}A_0\mathcal{P}_r\mathcal{P}_n.$$

Proof. If we replace n by $r+1$ and r by $n-1$ in equation (6), we have

$$\begin{aligned}\mathcal{P}_{n+r} &= \mathcal{P}_{(r+1)+(n-1)} \\ &= \mathcal{P}_{n-1}A_0\mathcal{P}_{r+1-1} + \mathcal{P}_{n-1+1}\mathcal{P}_{r+1} \\ &= \mathcal{P}_{n-1}A_0\mathcal{P}_{r} + \mathcal{P}_{n}\mathcal{P}_{r+1}.\end{aligned}$$

That is,

$$\mathcal{P}_{n+r} = \mathcal{P}_{n-1}A_0\mathcal{P}_{r} + \mathcal{P}_{n}\mathcal{P}_{r+1} \tag{7}$$

From equations (6), (7) and the fact that a ring satisfies the associative law for multiplication, we have

$$\mathcal{P}_n(\mathcal{P}_{r+1}\mathcal{P}_n) = (\mathcal{P}_n\mathcal{P}_{r+1})\mathcal{P}_n$$

$$\mathcal{P}_n(\mathcal{P}_{r+1}\mathcal{P}_n + \mathcal{P}_rA_0\mathcal{P}_{n-1} - \mathcal{P}_rA_0\mathcal{P}_{n-1}) = (\mathcal{P}_n\mathcal{P}_{r+1} + \mathcal{P}_{n-1}A_0\mathcal{P}_r - \mathcal{P}_{n-1}A_0\mathcal{P}_r)\mathcal{P}_n$$

$$\mathcal{P}_n\left(\underbrace{\mathcal{P}_rA_0\mathcal{P}_{n-1} + \mathcal{P}_{r+1}\mathcal{P}_n}_{\mathcal{P}_{n+r}} - \mathcal{P}_rA_0\mathcal{P}_{n-1}\right) = \left(\underbrace{\mathcal{P}_{n-1}A_0\mathcal{P}_r + \mathcal{P}_n\mathcal{P}_{r+1}}_{\mathcal{P}_{n+r}} - \mathcal{P}_{n-1}A_0\mathcal{P}_r\right)\mathcal{P}_n$$

$$\mathcal{P}_n(\mathcal{P}_{n+r} - \mathcal{P}_rA_0\mathcal{P}_{n-1}) = (\mathcal{P}_{n+r} - \mathcal{P}_{n-1}A_0\mathcal{P}_r)\mathcal{P}_n$$

$$\mathcal{P}_n\mathcal{P}_{n+r} - \mathcal{P}_n\mathcal{P}_rA_0\mathcal{P}_{n-1} = \mathcal{P}_{n+r}\mathcal{P}_n - \mathcal{P}_{n-1}A_0\mathcal{P}_r\mathcal{P}_n$$

$$\mathcal{P}_n\mathcal{P}_{n+r} - \mathcal{P}_{n+r}\mathcal{P}_n = \mathcal{P}_n\mathcal{P}_rA_0\mathcal{P}_{n-1} - \mathcal{P}_{n-1}A_0\mathcal{P}_r\mathcal{P}_n$$

and so the proof is completed. ■

Theorem 7. (d'Ocagne's identity) For $n \geq 1$, $m \geq 1$ and $m \geq n$, we have

$$\mathcal{P}_m\mathcal{P}_{n+1} - \mathcal{P}_{m+1}\mathcal{P}_n = \mathcal{P}_mA_0\mathcal{P}_{n-1} - \mathcal{P}_{m-1}A_0\mathcal{P}_n.$$

Proof. Using equations (6) and (7), we have

$$\begin{aligned}\mathcal{P}_m\mathcal{P}_{n+1} - \mathcal{P}_{m+1}\mathcal{P}_n &= \mathcal{P}_m(\mathcal{P}_1A_0\mathcal{P}_{n-1} + \mathcal{P}_2\mathcal{P}_n) - (\mathcal{P}_{m-1}A_0\mathcal{P}_1 + \mathcal{P}_m\mathcal{P}_2)\mathcal{P}_n \\ &= \mathcal{P}_m\mathcal{P}_1A_0\mathcal{P}_{n-1} + \mathcal{P}_m\mathcal{P}_2\mathcal{P}_n - \mathcal{P}_{m-1}A_0\mathcal{P}_1\mathcal{P}_n - \mathcal{P}_m\mathcal{P}_2\mathcal{P}_n \\ &= \mathcal{P}_m(1)A_0\mathcal{P}_{n-1} - \mathcal{P}_{m-1}A_0(1)\mathcal{P}_n \\ &= \mathcal{P}_mA_0\mathcal{P}_{n-1} - \mathcal{P}_{m-1}A_0\mathcal{P}_n.\end{aligned}$$

and

$$\mathcal{P}_m\mathcal{P}_{n+1} - \mathcal{P}_{m+1}\mathcal{P}_n = \mathcal{P}_mA_0\mathcal{P}_{n-1} - \mathcal{P}_{m-1}A_0\mathcal{P}_n.$$

Thus the proof is completed. ■

Theorem 8. For $n \geq 2$,

$$\mathcal{P}_n\mathcal{P}_{n+1} - \mathcal{P}_{n-1}\mathcal{P}_{n+2} = \mathcal{P}_{n-2}A_0\mathcal{P}_{n+1} - \mathcal{P}_{n-1}A_0\mathcal{P}_n.$$

Proof. Using Definition 1 and Theorem 3 we have

$$\begin{aligned}\mathcal{P}_n\mathcal{P}_{n+1} - \mathcal{P}_{n-1}\mathcal{P}_{n+2} &= (\mathcal{P}_{n-1}2A_1 + \mathcal{P}_{n-2}A_0)\mathcal{P}_{n+1} - \mathcal{P}_{n-1}(2A_1\mathcal{P}_{n+1} + A_0\mathcal{P}_n) \\ &= \mathcal{P}_{n-1}2A_1\mathcal{P}_{n+1} + \mathcal{P}_{n-2}A_0\mathcal{P}_{n+1} - \mathcal{P}_{n-1}2A_1\mathcal{P}_{n+1} - \mathcal{P}_{n-1}A_0\mathcal{P}_n \\ &= \mathcal{P}_{n-2}A_0\mathcal{P}_{n+1} - \mathcal{P}_{n-1}A_0\mathcal{P}_n.\end{aligned}$$

and

$$\mathcal{P}_n\mathcal{P}_{n+1} - \mathcal{P}_{n-1}\mathcal{P}_{n+2} = \mathcal{P}_{n-2}A_0\mathcal{P}_{n+1} - \mathcal{P}_{n-1}A_0\mathcal{P}_n.$$

Thus the proof is completed. ■

III. Applications of Pell Sequences in Rings with Identity in Matrices

In this section, we consider Hessenberg matrices with applications to Pell sequences in rings with identity. We investigate the relationships between the Hessenberg matrices and Pell sequences in rings with identity. We then define two type Hessenberg matrices and show that the determinants and permanents of these Hessenberg matrices are nth term of Pell sequences in rings with identity. Also, the terms of Pell sequences in rings with identity are derivated by the matrix.

The definition of Pell sequences in rings with identity show that pairs of successive term are significant $\{\mathcal{P}_n\} = \{\dots, \mathcal{P}_n, \mathcal{P}_{n+1}, \mathcal{P}_{n+1}, \dots\}$. It is therefore naturel to consider pairs as 2- vectors, e.g $(\mathcal{P}_{n+1}, \mathcal{P}_n)^T$. In fact, the recursive relation above can be described in this manner:

$$(2A_1, A_0)\begin{pmatrix}\mathcal{P}_{n+1} \\ \mathcal{P}_n\end{pmatrix} = 2A_1\mathcal{P}_{n+1} + A_0\mathcal{P}_n$$

where A_1, A_0 are arbitrary elements of the ring. More useful is considering two successive vectors in a matrix:

$$\begin{pmatrix}\mathcal{P}_{n+1} & \mathcal{P}_n \\ A_0\mathcal{P}_n & A_0\mathcal{P}_{n-1}\end{pmatrix}$$

so that when $n = 1$ we have

$$\begin{pmatrix}\mathcal{P}_2 & \mathcal{P}_1 \\ A_0\mathcal{P}_1 & A_0\mathcal{P}_0\end{pmatrix} = \begin{pmatrix}2A_1 & 1 \\ A_0 & 0\end{pmatrix}$$

call this matrix Q.

Theorem 9. The Pell sequences in rings with identity are generated by a matrix $Q = \begin{pmatrix} 2A_1 & 1 \\ A_0 & 0 \end{pmatrix}$, then

$$Q^n = \begin{pmatrix} \mathcal{P}_{n+1} & \mathcal{P}_n \\ A_0\mathcal{P}_n & A_0\mathcal{P}_{n-1} \end{pmatrix} \qquad (8)$$

where $n \in \mathbb{Z}^+$.

Proof. We shall use the induction method on n. If $n = 1$ and $n = 2$ then

$$\begin{aligned} Q &= \begin{pmatrix} 2A_1 & 1 \\ A_0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{P}_2 & \mathcal{P}_1 \\ A_0\mathcal{P}_1 & A_0\mathcal{P}_0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} Q^2 &= \begin{pmatrix} 2A_1 & 1 \\ A_0 & 0 \end{pmatrix}\begin{pmatrix} 2A_1 & 1 \\ A_0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 4A_1^2 + A_0 & 2A_1 \\ A_0 2A_1 & A_0(1) \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{P}_3 & \mathcal{P}_2 \\ A_0\mathcal{P}_2 & A_0\mathcal{P}_1 \end{pmatrix}. \end{aligned}$$

So the proof is completed for $n = 1$ and $n = 2$. Let the equation (8) be hold for $n \in \mathbb{Z}^+$. Then we shall show that the equation (8) holds for $n + 1$. Indeed we have

$$Q^{n+1} = Q^n Q.$$

Using induction's hypothesis we obtain

$$\begin{aligned} Q^{n+1} &= Q^n Q \\ &= \begin{pmatrix} \mathcal{P}_{n+1} & \mathcal{P}_n \\ A_0\mathcal{P}_n & A_0\mathcal{P}_{n-1} \end{pmatrix}\begin{pmatrix} \mathcal{P}_2 & \mathcal{P}_1 \\ A_0\mathcal{P}_1 & A_0\mathcal{P}_0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{P}_{n+1}\mathcal{P}_2 + \mathcal{P}_n A_0\mathcal{P}_1 & \mathcal{P}_{n+1}\mathcal{P}_1 + \mathcal{P}_n A_0\mathcal{P}_0 \\ A_0\mathcal{P}_n\mathcal{P}_2 + A_0\mathcal{P}_{n-1}A_0\mathcal{P}_1 & A_0\mathcal{P}_n\mathcal{P}_1 + A_0\mathcal{P}_{n-1}A_0\mathcal{P}_0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{P}_{n+1}2A_1 + \mathcal{P}_n A_0(1) & \mathcal{P}_{n+1}(1) + \mathcal{P}_n A_0(0) \\ A_0\left(\mathcal{P}_n 2A_1 + \mathcal{P}_{n-1}A_0(1)\right) & A_0\mathcal{P}_n(1) + A_0\mathcal{P}_{n-1}A_0(0) \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{P}_{n+2} & \mathcal{P}_{n+1} \\ A_0\mathcal{P}_{n+1} & A_0\mathcal{P}_n \end{pmatrix}. \end{aligned}$$

which is as desired. ∎

Theorem 10. For $n \geq 1$, if $P_n(A_1, A_0) = (p_{ij})$ is a $n \times n$ Hessenberg matrix as follows

$$P_n(A_1, A_0) = \begin{pmatrix} 1 & -A_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2A_1 & -A_0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 2A_1 & -A_0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2A_1 & -A_0 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2A_1 & -A_0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2A_1 \end{pmatrix}$$

with $P_0(A_1, A_0) = 0$ then

$$det\left(P_n(A_1, A_0)\right) = \mathcal{P}_n$$

where $\mathcal{P}_n$ is the nth term of Pell sequences in rings with identity.

Proof. We can use the mathematical induction method on n to prove $det\left(P_n(A_1, A_0)\right) = \mathcal{P}_n$. Then,

$n = 1$, $det\left(P_1(A_1, A_0)\right) = |1| = \mathcal{P}_1$

$n = 2$, $det\left(P_2(A_1, A_0)\right) = \begin{vmatrix} 1 & -A_0 \\ 0 & 2A_1 \end{vmatrix} = 2A_1 = \mathcal{P}_2$

$n = 3$, $det\left(P_3(A_1, A_0)\right) = \begin{vmatrix} 1 & -A_0 & 0 \\ 0 & 2A_1 & -A_0 \\ 0 & 1 & 2A_1 \end{vmatrix} = 4{A_1}^2 + A_0 = \mathcal{P}_3$

$$\vdots$$

We assume that it is true for $n \in \mathbb{Z}^+$, namely

$$det(P_n(A_1,A_0)) = \mathcal{P}_n, det(P_{n-1}(A_1,A_0)) = \mathcal{P}_{n-1}, det(P_{n-2}(A_1,A_0)) = \mathcal{P}_{n-2}, \ldots$$

and we shall show that it is true for $n+1$. Using equation (2), we have

$$\begin{aligned}
det(P_{n+1}(A_1,A_0)) &= p_{n+1,n+1}det(P_n(A_1,A_0)) + \sum_{i=1}^{n}\left[(-1)^{n+1-i}p_{n+1,i}\prod_{j=i}^{n}p_{j,j+1}det(P_{i-1}(A_1,A_0))\right] \\
&= 2A_1 det(P_n(A_1,A_0)) + \sum_{i=1}^{n-1}\left[(-1)^{n+1-i}p_{n+1,i}\prod_{j=i}^{n}p_{j,j+1}(P_{i-1}(A_1,A_0))\right] \\
&\quad +(-1)p_{n+1,n}p_{n,n+1}det(P_{n-1}(A_1,A_0)) \\
&= 2A_1 det(P_n(A_1,A_0)) + 0 + (-1)(1)(-A_0)det(P_{n-1}(A_1,A_0)) \\
&= 2A_1\mathcal{P}_n + A_0\mathcal{P}_{n-1} \\
&= \mathcal{P}_{n+1}.
\end{aligned}$$

Thus the proof is completed. ■

Theorem 11. For $n \geq 1$, if $R_n(A_1,A_0) = (r_{ij})$ is a $n \times n$ Hessenberg matrix as follows

$$R_n(A_1,A_0) = \begin{pmatrix}
1 & A_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 2A_1 & A_0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & 2A_1 & A_0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2A_1 & A_0 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2A_1 & A_0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2A_1
\end{pmatrix}$$

with $R_0(A_1,A_0) = 0$ then

$$per(R_n(A_1,A_0)) = \mathcal{P}_n$$

where $\mathcal{P}_n$ is the nth term of Pell sequences in rings with identity.

Proof. We can use the mathematical induction method on n to prove $per(R_n(A_1,A_0)) = \mathcal{P}_n$. Then

$$n=1, per(R_1(A_1,A_0)) = |1| = \mathcal{P}_1$$

$$n=2, per(R_2(A_1,A_0)) = \begin{vmatrix} 1 & A_0 \\ 0 & 2A_1 \end{vmatrix} = 2A_1 = \mathcal{P}_2$$

$$n=3, per(R_3(A_1,A_0)) = \begin{vmatrix} 1 & A_0 & 0 \\ 0 & 2A_1 & A_0 \\ 0 & 1 & 2A_1 \end{vmatrix} = 4{A_1}^2 + A_0 = \mathcal{P}_3$$

$$\vdots$$

We assume that it is true for n $\in \mathbb{Z}^+$, namely

$$per(R_n(A_1,A_0)) = \mathcal{P}_n, per(R_{n-1}(A_1,A_0)) = \mathcal{P}_{n-1}, per(R_{n-2}(A_1,A_0)) = \mathcal{P}_{n-2}, \ldots$$

and we shall show that it is true for $n+1$. Using equation (3), we have

$$\begin{aligned}
per(R_{n+1}(A_1,A_0)) &= r_{n+1,n+1}\, per(R_n(A_1,A_0)) + \sum_{i=1}^{n}\left[r_{n+1,i}\prod_{j=i}^{n}r_{j,j+1}per(R_{i-1}(A_1,A_0))\right] \\
&= 2A_1 per(R_n(A_1,A_0)) + \sum_{i=1}^{n-1}\left[r_{n+1,i}\prod_{j=i}^{n}r_{j,j+1}per(R_{i-1}(A_1,A_0))\right] \\
&\quad + r_{n+1,n}r_{n,n+1}per(R_{n-1}(A_1,A_0)) \\
&= 2A_1 per(R_n(A_1,A_0)) + 0 + (1)(A_0)per(R_{n-1}(A_1,A_0)) \\
&= 2A_1\mathcal{P}_n + A_0\mathcal{P}_{n-1} \\
&= \mathcal{P}_{n+1}.
\end{aligned}$$

Thus the proof is completed. ■

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