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Some Deductions from the Factorization of Finite Simple Groups

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Abstract: *This paper looked into the factorization of minimal normal subgroups of innately transitive groups. Some deductions from these theorems are presented. Some results about normalizers of subgroups of characteristically simple groups were proved and some implications of these results examined. It further extended these results to that of the centralizers of subgroups of characteristically simple groups. Some applications of the results obtained are also presented.*

Keywords: *Minimal normal subgroups, finite simple groups, centralizers, normalizers.*

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I. Introduction

A group is said to be simple if and only if it has no non-trivial normal subgroup. The classification of all finite simple groups is said to be the greatest achievement in finite group theory in recent times. Amongst the numerous open problems in group theory that the results of this classification has solved is the factorization of finite simple groups. Finite simple groups are essential since they act as the basic building blocks of all finite groups, somewhat analogous to the prime numbers which are the basic building blocks of the natural numbers.

A group G is said to be factorizable if it can be written in the form $G=AB$, where A and B are proper subgroups of G. Of course quite a good number of questions in mathematics are related to group factorizations and consequently a good number of research works have been carried out in this area. This paper is motivated by the work of R.W. Baddeley, C.E. Praeger and C. Schneider on factorization of simple and characteristically simple groups.

In this paper we deduce some results about normalizers of subgroups of characteristically simple groups from the work of Baddeley *et al* (2008). We further prove some results about centralizers of subgroups of characteristically simple groups.

In the next section we give definitions and quote some theorems from Baddeley *et al* (2008) that are of importance to us in this paper.

II. Factorization of Characteristically Simple Groups

Definition 2.1 (Baddeley *et al* (2008))

Suppose $M = F_1 \times F_2 \times \cdots \times F_k$ is a finite, non-abelian, characteristically simple group where the F_i's $(i = 1, 2, .., k)$ are pairwise isomorphic, simple normal subgroups, and suppose J_1 and J_2 are proper subgroups of M. We let $\sigma_i: M \to F_i$ denote the natural projection map from M to F_i. Then we say that a group factorization $M = J_1 J_2$ is said to be a full factorization if, for each $i = 1, 2, \ldots, k$,

(a) the subgroups of $\sigma_i(J_1)$, $\sigma_i(J_2)$ are proper subgroups of F_i;

(b) the orders of the proper subgroups $|\sigma_i(J_1)|$, $|\sigma_i(J_2)|$, and $|F_i|$ are divisible by the same set of primes.

Theorem 2.2 (Baddeley *et al* (2008))

For pair wise isomorphic, finite, non-abelian simple groups, $F_1, F_2, \ldots, F_k$ where $k \geq 1$, we let $M = F_1 \times F_2 \times \cdots \times F_k$. Given that $M = J_1 J_2$ is a full factorization, then we have that

$$\sigma_1(J_s)' \times \ldots \times \sigma_k(J_s)' \leq J_s \quad for \ s = 1,2.$$

Moreover, the pair $(F_i, \{\sigma_i(J_1), \sigma_i(J_2)\})$ is a full factorization for each $i = 1, 2, .., k$ and hence appears as $(F, \{A, B\})$ as given in Table 1.

Table 1: Full Factorizations $\{A, B\}$ of Finite Simple Groups

	F	A	B
1	A_6	A_5	A_5
2	M_{12}	M_{11}	$M_{11}, PSL_2(11)$
3	$P\Omega_8^+(q), q \geq 3$	$\Omega_7(q)$	$\Omega_7(q)$
4	$P\Omega_8^+(2)$	$Sp_6(2)$	$A_7, A_8, S_7, S_8, Sp_6(2), \mathbb{Z}_2^6 \rtimes A_7, \mathbb{Z}_2^6 \rtimes A_8$
		A_9	$A_8, S_8, Sp_6(2), \mathbb{Z}_2^6 \rtimes A_7, \mathbb{Z}_2^6 \rtimes A_8$
5	$Sp_4(q), q \geq 4\ even$	$Sp_2(q^2).2$	$Sp_2(q^2).2,\ Sp_2(q^2)$

Source: Baddeley *et al* (2008))

III. Normalizers and Centralizers in Direct Products

Next, we gather some facts about normalizers as well as centralizers of subgroups in direct products.
Let $G = G_1 \times \dots \times G_k$ be a direct product of groups and $H \leq G$. The normalizer of H in G is contained in $N_{G_1}(\sigma_1(H)) \times \dots \times N_{G_k}(\sigma_k(H))$, that is to say that $N_{G_1 \times \dots \times G_k}(H) \leq N_{G_1}(\sigma_1(H)) \times \dots \times N_{G_k}(\sigma_k(H))$.
Furthermore, if $H = \sigma_1(H) \times \dots \times \sigma_k(H)$, then
$N_{G_1 \times \dots \times G_k}(H) = N_{G_1}(\sigma_1(H)) \times \dots \times N_{G_k}(\sigma_k(H))$.
The lemma below generalizes the observation above.

Lemma 3.1 (Baddeley *et al.*, 2008)
Let $G = G_1 \times \dots \times G_k$ be a direct product of groups G_i for $i = 1, \dots, k$, and H_i a subgroup of G_i such that $H_1 \times \dots \times H_k \lhd H$.
The factor $N_G(H_1 \times \dots \times H_k)/(H_1 \times \dots \times H_k)$ is abelian and $N_{G_i}(\sigma_i(H)) = N_{G_i}(H_i)$. Therefore, $N_G(H) = N_G(H_1 \times \dots \times H_k) = N_{G_1}(H_1) \times \dots \times N_{G_k}(H_k)$

Proposition 3.2 (Baddeley *et al* (2008))
Let $M = F_1 \times \dots \times F_k \cong F^k$ be a characteristically simple group and $(M, \{J_1, J_2\})$ a full factorization such that, for all $i \in \{1, \dots, k\}$, the pair $(F_i, \{\sigma_i(J_1),\ \sigma_i(J_2)\})$ is a full factorization and thus is as $(F, \{A, B\})$ in Table 2.
(a) If $F \cong A_6$, M_{12} or $P\Omega_8^+(q)$, then it implies that J_1, J_2 and $J_1 \cap J_2$ are self-normalizing in M.
(b) *If $(Sp_4(q), \{Sp_4(q^2) \cdot 2,\ Sp_4(q^2) \cdot 2\})$ is a full factorization, then for $s = 1, 2$, we have $N_M(J_s) = \prod_i \sigma_i(J_s)$ and $N_M(J_1 \cap J_2) = N_M(J_1) \cap N_M(J_2)$.*

TABLE 2: Factorizations of Finite Simple Groups in Proposition 3.2

	F	A	B
1	A_6	A_5	A_5
2	M_{12}	M_{11}	$M_{11}, PSL_2(11)$
3	$P\Omega_8^+(q)$	$\Omega_7(q)$	$\Omega_7(q)$
4	$Sp_4(q), q \geq 4\ even$	$Sp_2(q^2) \cdot 2$	$Sp_2(q^2) \cdot 2$

Source: Praeger and Schneider (2002)

Example 3.3
Let G be an innately transitive group with a minimal normal subgroup M. where, $M \cong M_{11} \times M_{11}$. Let $J_{1,} = M_{11}$ and $J_2 = M_{11}$ be proper subgroups of M. Then

$$\begin{aligned} N_M(J_1) = N_M(M_{11}) &= N_{M_{11} \times M_{11}}\left(\left(\sigma_1(M_{11}) \times \sigma_2(M_{11})\right)\right) \\ &= N_{M_{11}}(\sigma_1(M_{11})) \times N_{M_{11}}(\sigma_2(M_{11})) \\ &= (\sigma_1(M_{11})) \times (\sigma_2(M_{11})) \\ &= M_{11} = J_1 \end{aligned}$$

This implies that M_{11} is self-normalizing.

We also review a few facts about centralizers of subgroups in direct products .
Let $G = G_1 \times G_2 \times \dots \times G_n$ be a direct product of groups. If $g = (g_1, g_2, \dots, g_n) \in G$, then the centralizer of g is simply the product of the centralizers of g, that is,

$$C_{G_1 \times G_2 \times \dots \times G_n}(g_1, g_2, \dots, g_n) = C_{G_1}(g_1) \times C_{G_2}(g_2) \times \dots \times C_{G_n}(g_n)$$

Proposition 3.4
The centre of a direct product is the direct product of the centres that is.
$Z(G_1 \times G_2 \times \dots \times G_n) = Z(G_1) \times Z(G_2) \times \dots \times Z(G_n)$

Next we state our main results or propositions which are mainly deduction from the work of Baddeley *et al* (2008).

IV. Main Theorems

Now, suppose G is an innately transitive permutation group acting on a set or G-space Ω with plinth M. M is a minimal normal subgroup of G, and so M is a nonabelian characteristically simple group written in the form $M = F_1 \times F_2 \times \cdots \times F_k$ where the F_i's (i.e. $F_1, \cdots F_k$) are finite simple normal groups each isomorphic to the same simple group F. We therefore deduce that M is isomorphic to F^k that is to say, $M \cong F^k$. Let $\sigma_i : M \to F_i$ represent the ith projection map from M to each F_i for $i = 1, 2, .., k$. It is important to note that the group G acts transitively by conjugation on the set $\{F_1, F_2, \cdots F_k\}$.

Theorem 4.1

Let π_l, $l = 1, 2, .., m$, be finite sets of primes. If M is a non-abelian, characteristically simple group with simple normal subgroups $F_1 \ldots, F_k$ and $(M, \{J_1, J_2\})$ is a full factorization then

(a) the pairs $(F_i, \{\sigma_i(J_1), \sigma_i(J_2)\})$ $(for\ i = 1, \ldots, k)$ forms π_l-groups.

(b) the subgroups $\sigma_i(J_1)$ and $\sigma_i(J_2)$cannot have normal π-complements in F_i. In particular $\sigma_i(J_1)$ and $\sigma_i(J_2)$ are not π- complements of each other.

Proof

(a) From the idea of full factorization, $\sigma_i(J_1)$ and $\sigma_i(J_2)$ are subgroups of F_i. We see also from Table 3 that the order $|F_i|$ of F_i, the order $|\sigma_i(J_1)|$ of $\sigma_i(J_1)$, and the order $|\sigma_i(J_2)|$ of $\sigma_i(J_2)$ are divisible by the same set of unique primes. Hence, it follows readily that the pair $(F_i, \{\sigma_i(J_1), \sigma_i(J_2)\})$ forms π_j*-groups for* $i = 1, \ldots, k$

(b) We proceed to prove that of (b) by contradiction. Suppose that $\sigma_i(J_1)$ is the π-complement group of $\sigma_i(J_2)$ then, $\sigma_i(J_1)$ is not divisible by any primes in π (which is a contradiction). Hence $\sigma_i(J_1)$ is not a π-complement of $\sigma_i(J_2)$. Similarly, $\sigma_i(J_2)$ is not a π - complement of $\sigma_i(J_1)$.

For different sets of unique primes say π_l, $l = 1,2, .., m$ the pair $(F_i, \{\sigma_i(J_1), \sigma_i(J_2)\})$ are said to be π_l – groups if the order $|F_i|$ of F_i, the order $|\sigma_i(J_1)|$ of $\sigma_i(J_1)$, and the order $|\sigma_i(J_2)|$ of $\sigma_i(J_2)$ are divisible by the same set of unique primes.

Example 4.2

From table 1 about the full factorization of finite simple and characteristically simple groups the following deductions were obtained:

If $(F, \{A, B\})$ is a full factorization of a finite simple group F with subgroups A and B then we have that

(i) $F = AB$, such that given any $f \in F$, there exists $a \in A$ and $b \in B$ such that $f = ab$.

(ii) The order $|F|$ of F , the order $|A|$ of A and the order $|B|$ of B are all divisible by the same set of unique prime numbers.

Using the above point and table 1 we have the following interpretations in table 3 of full factorizations of specific finite simple groups with the set of unique primes playing a vital role.

Table 3: Interpretations

	F	***A***	***B***	***UNIQUE PRIMES***
1	A_6 $\lvert A_6\rvert = 360$ $= 2^3 \times 3^2 \times 5$	A_5 $\lvert A_5\rvert = 60$ $= 2^2 \times 3 \times 5$	A_5 $\lvert A_5\rvert = 60$ $= 2^2 \times 3 \times 5$	2,3 *and* 5
2	M_{12} $\lvert M_{12}\rvert = 95040$ $= 2^6 \times 3^2 \times 5 \times 11$	M_{11} $\lvert M_{11}\rvert = 7920$ $= 2^4 \times 3^2 \times 5 \times 11$	M_{11} $\lvert M_{11}\rvert = 7920$ $= 2^4 \times 3^2 \times 5 \times 11$ $PSL_2(11)$ $\lvert PSL_2(11)\rvert = 660$ $= 2^2 \times 3 \times 5 \times 11$	2,3,5 *and* 11
3	$Sp_4(q), q \geq 4\ even$ $\lvert Sp_4(q)\rvert = 979200$ $= 2^8 \times 3^2 \times 5^2 \times 17$	$Sp_2(q^2).2$ $\lvert Sp_2(q^2).2\rvert = 16320$ $= 2^6 \times 3 \times 5^2 \times 17$	$Sp_2(q^2).2$ $\lvert Sp_2(q^2).2\rvert = 16320$ $= 2^6 \times 3 \times 5^2 \times 17$ $Sp_2(q^2)$ $\lvert Sp_2(q^2)\rvert = 8160$ $= 2^5 \times 3 \times 5^2 \times 17$	2,3,5 *and* 17

For instance, from Table 3;

Let $\pi_1 = \{2, 3, 5\}$, $\pi_2 = \{2, 3, 5, 11\}$, $\pi_3 = \{2, 3, 5, 17\}$

Then the pair $(A_6, \{A_5, A_5\})$ is a π_1- group, since $|A_6|$, $|A_5|$, $|A_5|$ are divisible by the same set of primes π_1.

In the same way, the pair $(M_{12}, \{M_{11}, PSL_2(11)\})$ is a π_2- group, and so on.

Theorem 4.3
From the results of Baddeley and Praeger in Proposition 3.2, we make the following deductions:
(i) J_1 and J_2 are full diagonal subgroups of M
(ii) The group G is quasiprimitive
(iii) If $J_1, J_2 \ncong A_5,\ PSL_2(11),\ M_{11},\ \Omega_7(q),\ Sp_2(q^2).2,\ Sp_2(q^2)$; then, since J_1 and J_2 are self-normalizing in M, it follows that
(a) J_1 and J_2 are maximal nilpotent subgroups of M and J_1 and J_2 are conjugates and subsequently
(b) J_1 and J_2 are solvable groups.

Proof
Let $M = F_1 \times \ldots \times F_k$ be a finite, non-abelian, characteristically simple group, where $F_1, \ldots, F_k$ are the simple normal subgroups of M each isomorphic to the same simple group F. Let $\sigma_i: M \to \prod_{i=1}^{k} F_i$ be a natural projection map from M to $\prod_{i=1}^{k} F_i$. Also let $J_1 = \sigma_1(J_1) \times \ldots \times \sigma_k(J_1)$ and $J_2 = \sigma_1(J_2) \times \ldots \times \sigma_k(J_2)$. Let $\phi: J_1 \to M$ be an embedding such that $\phi \circ \sigma_i: J_1 \to F_i$ is an epimorphism for $i \in \{1, .., k\}$. The projection map σ_i is an isomorphism.
(i) Since J_1 and J_2 are subgroups of M (which is a direct product of isomorphic simple non-abelian groups F_i's) and ϕ is an inclusion map, then by an important result of Bamberg (2008) we conclude that J_1 and J_2 are subdirect subgroups of M
Since σ_i is an isomorphism, then $\sigma_i|_{J_1}$ and $\sigma_i|_{J_2}$ will also be isomorphisms. Hence J_1 and J_2 are diagonal subgroups of M.
Since J_1 and J_2 are both subdirect and diagonal subgroups of M it follows that they are full diagonal subgroups. J_1 and J_2 are direct products of their projections; that is, $J_1 = \sigma_1(J_1) \times \ldots \times \sigma_k(J_1)$ and $K_2 = \sigma_1(J_2) \times \ldots \times \sigma_k(J_2)$.
(ii) This follows from the fact that J_1 and J_2 are subdirect subgroups of the minimal simple non-abelian group M.
(iii) (a) From the fact that M is a non-solvable group and J_1 and J_2 are self-normalizing in M, it then follows readily that J_1 and J_2 are maximal, nilpotent subgroups of M. This is because it is only maximal, nilpotent subgroups of a non-solvable group that can be self-normalizing in the non-solvable group.
Furthermore, it follows readily using a deep result of Thompson in 1960, that any two such maximal nilpotent subgroups of M are conjugates. Therefore J_1 and J_2 are conjugates.
(b) J_1 and J_2 are nilpotent subgroups and therefore solvable, this follows from the fact that every nilpotent group is solvable.
Also, every minimal simple group has proper subgroups that are all solvable. Now M is a minimal non-abelian simple group with J_1 and J_2 as its proper subgroups; therefore, J_1 and J_2 are solvable groups.

Theorem 4.4
Suppose that G is a finite simple group with a transitive minimal normal subgroup M. As said M is a characteristically simple group and can be expressed as $M = F_1 \times \ldots \times F_k \cong F^k$. Suppose J_1 and J_2 are proper subgroups of M and $(M, \{J_1, J_2\})$ is a full factorization such that for all i, the pair $(F_i, \{\sigma_i(J_1), \sigma_i(J_2)\})$ is a full factorization and thus is as $(F, \{A, B\})$ in one of the rows of Table 2.
If F is as in one of the rows 1 – 3 of Table 2 (that is if $F \cong A_6,\ M_{12}$, or $P\Omega_8^+(q)$) then
(a) $\sigma_i(J_1)$ and $\sigma_i(J_2)$ are conjugates
(b) $J_1 \cap J_2 = 1$
(c) $C_M(J_1 \cap J_2) = 1$
(d) $Z(J_1 \cap J_2) = C_M(J_1 \cap J_2) \cap (J_1 \cap J_2) = 1$

Proof
(a) $\sigma_i(J_1)$ and $\sigma_i(J_2)$ are conjugates since A_6 has two conjugacy classes of subgroups isomorphic to A_5 (Line 1 of Table 2). J_1 and J_2 are conjugates hence their projections $\sigma_i(J_1)$ and $\sigma_i(J_2)$ are also conjugates.
(b) J_1 and J_2 are maximal distinct subgroups of M. Hence the only element in their intersection is the identity element, 1.
(a) Recall that $\sigma_i(J_s)$ are perfect and $J_s = \prod_i \sigma_i(J_s)$ for $s = 1, 2$ (by theorem 2.2). This implies that
$J_1 = \prod_{i=1}^{k} \sigma_i(J_1) = \sigma_1(J_1) \times \ldots \times \sigma_k(J_1)$
$J_2 = \prod_{i=1}^{k} \sigma_i(J_2) = \sigma_1(J_2) \times \ldots \times \sigma_k(J_2)$
Therefore,

$$J_1 \cap J_2 = \prod_{i=1}^{k} \sigma_i(J_1 \cap J_2) = \prod_{i=1}^{k} \sigma_i(J_1) \cap \sigma_i(J_2).$$

Now,

$$C_M(J_1 \cap J_2) = C_M\left(\prod_{i=1}^{k} \sigma_i(J_1) \cap \sigma_i(J_2)\right)$$

Recall that $M = F_1 \times \ldots \times F_k$, Hence we have

$$C_M(J_1 \cap J_2) = \prod_{i=1}^{k} C_{T_i}\big(\sigma_i(J_1) \cap \sigma_i(J_2)\big)$$

$$= C_{F_1}\big(\sigma_1(J_1) \cap \sigma_1(J_2)\big) \times \ldots \times C_{F_k}\big(\sigma_k(J_1) \cap \sigma_k(J_2)\big)$$

Hence, from (Lemma 4.2 in Baddeley *et al* (2008)), we have that

$$C_M(J_1 \cap J_2) = C_{F_1}(\sigma_1(J_1)' \cap \sigma_1(J_2)') \times \ldots \times C_{F_k}(\sigma_k(J_1)' \cap \sigma_k(J_2)')$$

$= 1 \times .. \times 1 \ \ = 1$

$Hence,\ \ C_M(J_1 \cap J_2) = 1$ as required.

(d) This flows from the fact that the centre of a simple non-abelian simple group must be trivial.

Example 4.5

Let G be an innately transitive group with a minimal normal subgroup M. where, $M \cong A_5 \times A_5$. Let $J_{1,} = A_5$ and $J_2 = A_5$ be proper subgroups of M. Then

$$C_M(J_1 \cap J_2) = C_M(A_5 \cap A_5) = C_{A_5 \times A_5}\Big(\big(\sigma_1(A_5) \cap \sigma_1(A_5)\big) \times \big(\sigma_2(A_5) \cap \sigma_2(A_5)\big)\Big)$$

$$= C_{A_5}\big(\sigma_1(A_5) \cap \sigma_1(A_5)\big) \times C_{A_5}\big(\sigma_2(A_5) \cap \sigma_2(A_5)\big)$$

$$= C_{A_5}\big(\sigma_1(A_5)\big) \times C_{A_5}\big(\sigma_2(A_5)\big)$$

$$= C_{A_5}\big(\sigma_1(A_5) \times \sigma_2(A_5)\big)$$

$$= C_{A_5}(A_5)$$

$$= Z(A_5) = 1$$

V. Conclusion

The factorization of finite simple and characteristically simple groups are very important in the study of finite groups. We used the result to explain some facts about normalizers and centralizers of the subgroups that occur. The results obtained in this paper helps to bring to limelight some important facts concerning minimal simple groups.

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