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Cardinality and Hasse Diagram of Cofinite and Non-Cofinite Topological Spaces for $n \leq 4$

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Abstract: In this paper, the number of cofinite topologies and non-cofinite topologies are determined for $1 \leq n \leq 4$. Let X be a finite set having $n \leq 4$ *elements. The number of cofinite topologies are considered from a linear recurrence relation* $T_{cf}(n) = (n-1)T_{Cf}(n-1) + 1$,*and extended to the sum of a sequences of a falling factorial to obtain the cardinality of non-cofinite topologies as* $\sum_{k=1}^{2^n} T(n,k) - \sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!}$. *for non-cofinite topologies and* $\sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!}$ *for cofinite topologies. Also, the graph of the inter relationship between cofinite topologies and non-cofinite topologies was discussed for* $1 \leq n \leq 3$.
Keywords: *Topological Space, Cofinite, non-cofinite, Cardinility, Hasse Diagram*

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I. Introduction

Efforts have been made by many researchers to establish an explicit formula for the computation of the number of topologies defined on a finite set. Although, despite these feat of successes, no formula have been put in place. Nevertheless, there exist partial solutions to the problem. Benoumhani in [1], computed the number of topologies having *k* open sets *T*(*n*,*k*) on the finite set *X*, having *n* elements for 2 ≤ *k* ≤ 12. Ern´e and Stege [3] provided the best methods, and gave the number of topologies on an *n* element set up to *n* = 14 Kolli [4] and Sharp [6], enumerated the topologies on a finite set. Stanley studied the number of open set of finite topologies [5].The number of chain topologies on *X*, having *k* open sets take the form $C(n,k) = \sum_{r-1}^{n-1} \binom{n}{r} C(r, k-1) =$ *k*− *1!S(n,k*− *1)* with *Sn,k=1k!r=0k*− *1rkr(k*−*r)n* as proved by Stephen in his work [7]. Kamel in [8] studied the concept of partial chain topology on any finite set with respect to the given subset, and study some properties with respect to some concepts of topological spaces. They also, find the number of all partial chain topologies with respect to the given subset A which help in obtaining the values of number of all chain topologies on it. Francis and Adeniji [2], obtain the number of elements defined on each topological space for open and clopen topologies on $n \leq 4$, which was called k-element. And also discuss the graphical relationship between topologies for open and clopen set for $n \leq 3$.

In this work, we focus on obtaining the cardinality of cofinite and non-cofinite topologies on set for $n \leq 4$. And, we- also study the graphical relationship between topologies which are defined on cofinite and non-cofinite topologies on set with $n \leq 3$.

II. Preliminaries

Throughout this paper (X, ρ) shall means topological spaces on which no separation are assumed unless otherwise stated. Let T(n) denote the number of topologies on X with $T(n,k)$ as the number of all topologies having k-elements that can be defined on topologies on X . There are many concepts related to topological space. But we consider cardinality of cofinite, and non-cofinite topological spaces in this paper. These are vital elements, relevant to the work. However, throughout this paper cofinite topologies and non-cofinite topologies are denoted by the notations: $T_{cof}(n)$ and $T_{ncof}(n)$ respectively, the following examples and definitions are important to sequel.

2.1 Definition

A topology ρ on a set X is collection of subsets of X such that
I. $\emptyset, X \in \rho$
II. If $M_i \in \rho$ for each $i \in I$, then $\bigcup_{i \in I} M_i \in \rho$
III. If $M_1, \ldots, M_n \in \rho$, then $M_1 \cap \ldots \cap M_n \in \rho$
A set $M \subseteq X$ is called open if $M \epsilon \rho$, The pair (X, ρ) is called a topological space.

2.2 Definition
Let X be the non-empty set and let ρ be a collection of a null set and all subset of X, whose compliments are finite, then ρ is cofinite topology on X if ρ is cofinite topology on X, then the pair (X, ρ) is called cofinite topology space.
Example 2.3
Consider the set
$X = \{a, b, c, d\}$ (2.1)
with the topology
$\rho_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$ (2.2)
Equation (2.1) is a set with three elements. The complement of each of the element of (2.2) gives:
$\emptyset^c = X - \emptyset \Rightarrow X \in \rho_1$
$X^c = X - X \Rightarrow \emptyset \in \rho$
$\{a\}^c = X - \{a\} \Rightarrow \{b, c, d\} \in \rho_1$
$\{c\}^c = X - \{c\} \Rightarrow \{a, b, d\} \in \rho_1$
$\{a, c\}^c = X - \{a, c\} \Rightarrow \{b, d\} \in \rho_1$
$\{b, d\}^c = X - \{b, d\} \Rightarrow \{a, c\} \in \rho_1$
$\{a, b, d\}^c = X - \{a, b, d\} \Rightarrow \{c\} \in \rho_1$
$\{b, c, d\}^c = X - \{a\} \Rightarrow \{b, c, d\} \in \rho_1$
According to this example ρ_1 is cofinite topology in X since ρ_1 is satisfies the properties of a topological space and the complement of element of ρ_1 is contain in ρ_1. The following Figure explains the relationship between elements of ρ in example 2.3

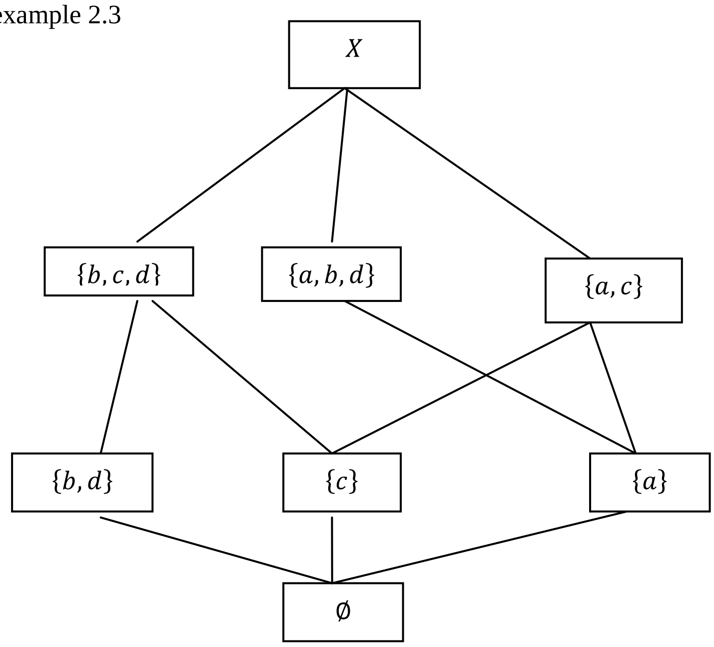

Figure-2.1. Hasse Diagram of Cofinite Topology

III. Cardinality on Cofinite and Non-cofinite Topological Spaces for $n \leq 4$

3.1 Preposition
A topology ρ on X is a cofinite topology on X if and only if every complement of the subset of ρ are contained in ρ
Proof follows from the definition 2.2

3.2 Preposition
Let X be a non-empty set and $m \subseteq X$. Then $m \epsilon \rho$ if and only if m is empty or m^c is finite
Proof
Verify that cofinite topology is a topological space, and is finite. By, a close check on the axioms of a topology.
(T_1) $\emptyset \in \rho$ and $X \in \rho$
Since $X^c = \emptyset$ And $\emptyset^c = X$
(T_2)Let $m_1 \in \rho$, $m_2 \in \rho$

Case (i) if at least one of m_1, m_2 is empty. Then $m_1 \cap m_2 = \emptyset \in \rho$
Case (ii) if none of m_1, m_2 is empty. That is $m_1 \neq \emptyset$ and $m_2 \neq \emptyset$
Therefore m_1^c is finite and m_2^c is also finite. Following De Morgan's law:
$m_1^c \cup m_2^c$ is finite, implies that $(m_1 \cap m_2)^c$ is finite and $m_1 \cap m_2 \in \rho$ is finite.
(T_3)Let $m_i \in \rho$ for every$i \in I$ such that $m_i \neq \emptyset \; \forall \; i$.
$m_i = (m_1 \cup m_2 \cup m_3 \cup m_4 \ldots) = \bigcup_{i\epsilon I} m_i$, for every$i \in I$
$\Rightarrow (m_1^c \cap m_2^c \cap m_3^c \cap m_4^c \ldots)$, this implies that $\bigcap_{i\in I} m_i^c$ is finite. By De Morgan statement
$\Rightarrow (m_1^c \cup m_2^c \cup m_3^c \cup m_4^c \ldots)^c$ is finite
That is, $X - m = X - \bigcup_{i\epsilon\Delta} m_i$
$\Rightarrow \bigcap_{i\epsilon\Delta}(X - m_i) \subset (X - m_i)$ for every $i \in I$. therefore, $m = \bigcup_{i\epsilon I} m_i \in \rho$
$\Rightarrow m_i^c$ is finite for every $i \in I$
Hence, $\bigcup_{i\in I} m_i \in \rho$ is satisfied.
The next theorem gives the Cardinality on Cofinite topological Spaces for $n \leq 4$

3.3 Theorem
Let X be a non-empty set and $T_{cof}(n)$ cofinite topology then
(i) The arbitrary union of any finite number of cofinite is cofinite
(ii) finite intersection of any cofinite is cofinite
(iii) any arbitrary union of cofinite is cofinite

3.4 Theorem
Let $T_{Cof}(n)$ be cofinite topological space then

$$T_{Cof}(n) = \sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!}$$

Proof
Cofinite topological spaces, was established from a recurrence relation
$T_{Cof}(n) = (n-1)T_{Cof}(n-1) + 1$ where $T_{Cof}(0) = 0$

$$for\; n = 1\; and\; T_{Cof}(0) = 0: \quad T_{Cof}(1) = (0)T_{Cof}(0) + 1 = 1 = \frac{1}{1} = \frac{0!}{0!} = 1$$

$$for\; n = 2: \quad T_{Cof}(2) = (1)T_{Cof}(1) + 1 = 1 + 1 \Rightarrow \frac{1}{1} + \frac{1}{1} = \frac{1!}{0!} + \frac{1!}{1!} = 2$$

$$for\; n = 3: \quad T_{Cof}(3) = (2)T_{Cof}(2) + 1 = 4 + 1 \Rightarrow 2 + 2 + 1 = \frac{2}{1} + \frac{2}{1} + \frac{2}{2} = \frac{2!}{0!} + \frac{2!}{1!} + \frac{2!}{2!} = 5$$

$$for\; n = 4: \quad T_{Cof}(4) = (3)T_{Cof}(3) + 1 = 15 + 1 \Rightarrow 6 + 6 + 3 + 1 = \frac{6}{1} + \frac{6}{1} + \frac{6}{2} + \frac{6}{6} = \frac{3!}{0!} + \frac{3!}{1!} + \frac{3!}{2!} + \frac{3!}{3!} = 16$$

Similarly

$$for\; n = 5: \quad T_{Cof}(5) = (4)T_{Cof}(4) + 1 = 64 + 1 \Rightarrow 24 + 24 + 12 + 4 + 1 = \frac{24}{1} + \frac{24}{1} + \frac{24}{2} + \frac{24}{8} + \frac{24}{24} = \frac{4!}{0!} + \frac{4!}{1!} + \frac{4!}{2!} + \frac{4!}{3!} + \frac{4!}{4!} = 65$$

$for\; n = r$: $T_{Cof}(r) = (r-1)T_{Cof}(r-1) + 1$ We split result into r sequence places and take the factorial form of it, such that we have r factorials and a recurrence denominator, that is

$$for\; n = r: \quad T_{Cof}(r) = (r-1)T_{Cof}(r-1) + 1 = \frac{r!}{0!} + \frac{r!}{1!} + \frac{r!}{2!} + \frac{r!}{3!} + - - - - - - + \frac{r!}{r!} = \sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!}$$

Hence $T_{Cof}(n) = \sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!}$

3.5 Theorem
Let X be a non-empty set and $T_{ncof}(n)$ non-cofinite topology then
(i) The arbitrary union of any finite number of non-cofinite is non-cofinite
(ii) finite intersection of any non-cofinite is non-cofinite
(iii) any arbitrary union of non-cofinite is non-cofinite
The next theorem gives the Cardinality on Non-Cofinite topological Spaces for $n \leq 4$

3.6 Theorem
Let $T_{ncof}\ (n)$ be non-cofinite topological space then

$$T_{ncof}\ (n) = \sum_{k=1}^{2^n} T(n,k) - \sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!}$$

Proof
For non-cofinite topologies, the formula $\sum_{k=1}^{2^n} T(n,k)$ in table 4.1 of [2] is sequel for the proof, and therefore the result is obtained as the differences between open sets on topological space $T(n,k)$and co-finite topological space on value of n. The proof for non-co-finite topology is as follows:
If $n = 1$

$$T_{ncof}\ (1) = \{(T(1,1)\ + T(1,2)) - ((0)T_{Cof}\,(0) + 1)\} = \left\{(0+1) - \left(\frac{0!}{0!}\right)\right\} = 1 - 1 = 0$$

$$T_{ncof}\ (1) = \sum_{k=1}^{2} T(1,k) - \sum_{k=1}^{0} \frac{0!}{(k-1)!} = \sum_{k=1}^{2^1} T(2,k) - \frac{0!}{(k-1)!} = 0$$

If $n = 2$ the non-cofinite topology is

$$T_{ncof}\ (2) = \{(T(2,1) + T(2,2) + T(2,3) + T(2,4)) - ((1)T_{Cf}\,(1) + 1)\} = \left\{(0+1+2+1) - \left(\frac{1!}{0!} + \frac{1!}{1!}\right)\right\} = 4 - 2 = 2$$

$$T_{nCof}\ (2) = \sum_{k=1}^{4} T(2,k) - \sum_{k=1}^{1} \frac{1!}{(k-1)!} = \sum_{k=1}^{2^2} T(2,k) - \sum_{k=1}^{1} \frac{1!}{(k-1)!} = 2$$

Similarly, the non-cofinite topology for $n = 3$

$$T_{ncof}\ (3) = \{(T(3,1) + T(3,2) + T(3,3) + T(3,4) + T(3,5) + T(3,6) + T(3,7) + T(3,8)) - ((2)T_{Cf}\,(2) + 1)\}$$
$$= \left\{(0+1+2+1) - \left(\frac{2!}{0!} + \frac{2!}{1!} + \frac{2!}{2!}\right)\right\} = 24$$

$$T_{ncof}\ (3) = \sum_{k=1}^{8} T(3,k) - \sum_{k=1}^{2} \frac{2!}{(k-1)!} = \sum_{k=1}^{2^3} T(3,k) - \sum_{k=1}^{1} \frac{2!}{(k-1)!} = 24$$

Hence, the non-cofinite topology for n

$$T_{ncof}\ (n) = \left\{\left((T(n,1) + T(n,2) + T(n,3) + \cdots. + T(n,k)) - \left(\frac{r!}{0!} + \frac{r!}{1!} + \frac{r!}{2!} + \frac{r!}{3!} + - - - - - - + \frac{r!}{r!}\right)\right)\right\}$$

$$T_{ncof}\ (n) = \sum_{k=1}^{2^n} T(n,k) - \sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!}$$

Hence, this satisfies the required proof.

IV. Corresponding graph of relations for $n \leq 3$

In this section, we introduce the concept of the graph of relationship between cofinite topologies on sets with elements as $n \leq 3$. And, non-cofinite topologies on the same point of set
For $n = 1$: that $X = \{a\}$

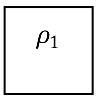

Figure 4.1: Hasse Diagram of Cofinite Topologies on a 1-point set

4.1 Theorem
ρ has one cofinite topology $T_{cof}\ (n)$ and no non-cofinite topology $T_{ncof}\ (n)$ on X. If and only if X is one-point set

Proof
Since the number of topology formed a single point is one. Suppose, $x \in \rho$ such that $\rho \in X$ is a topology on X. Then there exist $\rho \in T_{cof}\ (n)$ and $\rho \notin T_{ncof}\ (n)$

Remark
For $n = 1$ the only topology form is trivial. The topology is the power set of itself.
For $n = 2$: that $X = \{a, b\}$

Figure 4.1: Hasse Diagram of Non-Cofinite Topologies on a 2-point set

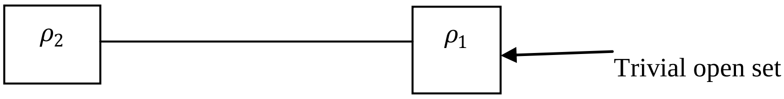

Figure 4.3: Hasse Diagram of Cofinite Topologies on a 2-point set

4.2 Theorem
If ρ_i and ρ_{i+1} are two non-cofinite topologies with two points set form on X. Then, the non-cofinite topologies formed on X are disconnected
Proof
Let ρ_i and ρ_{i+1} be two non-cofinite topologies on two points set form on X. Suppose ρ_i and ρ_{i+1} are two distinct points. Where $x \neq \emptyset$ and $y \neq \emptyset$ such that $x \cap y = \emptyset$ for $x \in \rho_i$ and $y \in \rho_{i+1}$ then $\rho_i \cap \rho_{i+1} = 0$ therefore ρ_i and ρ_{i+1} disconnected.
4.3 Theorem
Let ρ_i and ρ_{i+1} be two cofinite topologies on two points set form on X. If ρ_i and ρ_{i+1} are two distinct points on ρ. Then $\rho_i \subseteq \rho_{i+1}$and $\rho_{i+1} \subseteq \rho_i$ then Then $\rho_i = \rho_{i+1}$
Proof
Suppose $x \in \rho_i$ and $y \in \rho_i$ then $x \cap y \in \rho_i$ and $x \cup y \in \rho_i$. Similarly, if $x \in \rho_{i+1}$ and $y \in \rho_{i+1}$ then $x \cap y \in \rho_{i+1}$ and $x \cup y \in \rho_{i+1}$. Since, $x \cap y \in \rho_i$ and $x \cap y \in \rho_{i+1}$ it implies that $x \cap y \in \rho_i = \rho_{i+1}$ and also $x \cup y \in \rho_i = \rho_{i+1}$.
For $\boldsymbol{n} = 3$: that $X = \{a, b, c\}$

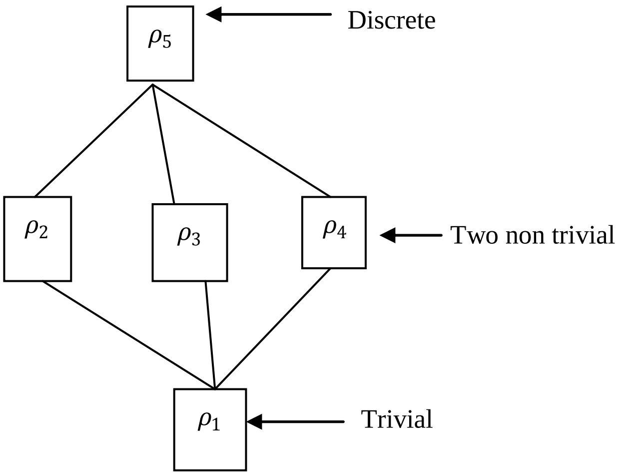

Figure 4.4: Hasse Diagram of Cofinite Topologies on a 3-point set

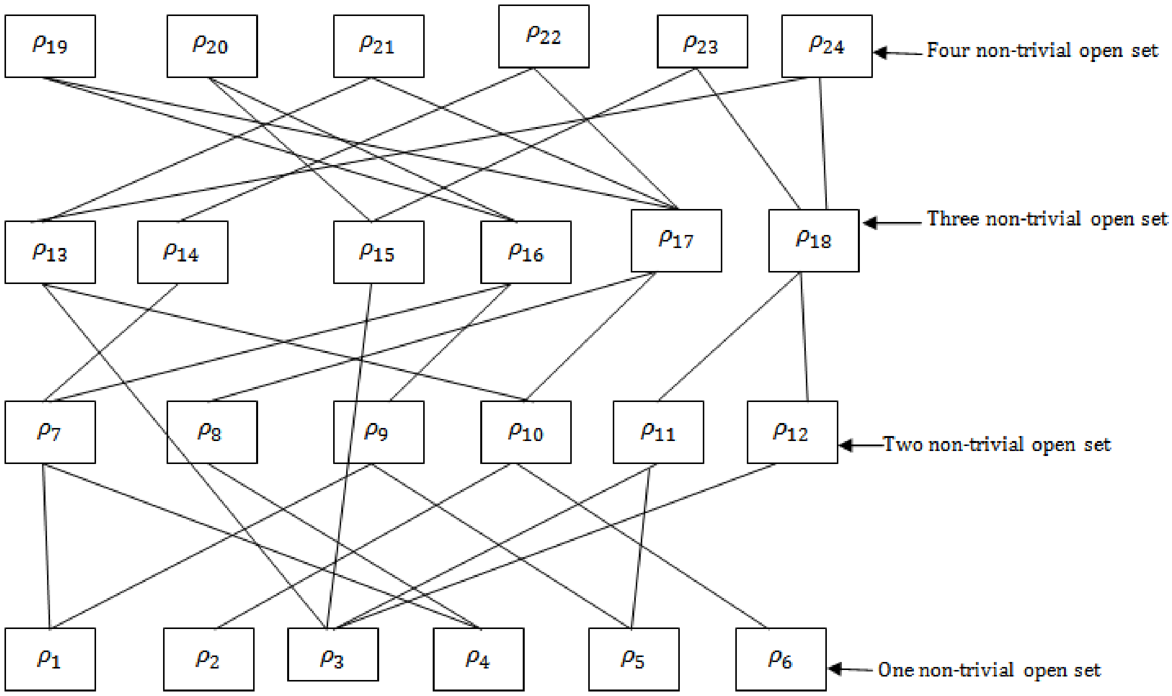

Figure 4.5: Hasse Diagram of Non-Cofinite Topologies on a 3-point set

Lemma: 4.4
Let X be the set having three elements, and let $A \subset X$ where $|A|$ is the number of elements of A on a cofinite topological space $T_{cof}(3)$. Then there exist three topological classes on X.

Proof
The classes of topologies are the trivial class that is $\{\emptyset, X\}$, the two non-trivial class which are $\{\emptyset, \{a, b\}, \{c\}X\}$, $\{\emptyset, \{a, c\}, \{b\}X\}$ and $\{\emptyset, \{b, c\}, \{a\}X\}$ and lastly the discrete class. These gives the five cofinite topologies. on X

Lemma: 4.5
Let X be the set having three elements, and let $A \subset X$ where $|A|$ is the number of elements of A on a non-cofinite topological spaces $T_{cof}(3)$. Then there exist four topological classes on X with non-trivial topologies and non- discrete topologies.

Proof
The classes are as shown in the Fig 4.5 with their corresponding topologies for each class.

V. Results

Table 5.1 gives the Cardinility of cofinite and non-cofinite topologies for $n \leq 4$

Table 5.1 Cardinality on cofinite and non-cofinite topological Spaces for $n \leq 4$

n	Co-finite	Non Co-finite	Total
1	1	0	1
2	2	2	4
3	5	24	29
4	16	339	355

VI. Conclusion

There are lots of computations on formulas for the numbers of topological spaces in finite set. In this paper, we formulated special case for computing the number of cofinite and non-cofinite topological spaces on small value of n, we also, illustrate our results with using Hasse diagram for $n \leq 3$ only. Our results may be refined for $n \geq 4$ and the Hasse diagram for $n \geq 3$, These results are basically obtained by hand.

References

[1]. Benoumhani M. The number of topologies on finite set. Journal of interger sequences, vol (2006); (9), article 06.2.6.
[2]. Francis M.O. and Adeniji A.O. On Number of k-Element in Open and Clopen Topological Space with Corresponding Graph for $n \leq 4$. Academic Journal of Applied Mathematical Sciences. (2019); 5(9): 130-135.
[3]. Ern´e, M. &Stege, K. Counting finite posets and topologies. Order, (1991); 8(3): 247-265.
[4]. Kolli M. On the cardinality of the T_0-topologies on a finite sets. Internatonal journal of combinatories. Article ID (2014). 798074, 7.
[5]. Stanley R. P On the number of open sets of finite topologies, J. Combinatorial Theory (1971); (10): 75-79.
[6]. Sharp H. Jr. Cardinality of finite topologies. Journal of combinatorial theory (1968); (5): 82 – 86.
[7]. Stephen D. Topology on finite sets, Amer. Math. Monthly (1968). 75, 739-741.
[8]. Kamel G. A. Partial chain topologies on finite sets. Computational and applied Mathematics Journal 1 (2015); (4): 174 -179

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