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A Possible Non Negative Lower Bound on the Li-Keiper Coefficients (A high temperature limit for the Riemann ξ Function)

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Abstract*: We investigate the relation between some spin l/2 ferromagnetic models with long range interaction of Statistical Mechanics (in the presence of the Lee-Yang and others theorems on the zeros of the partition functions) and polynomial truncations of the Riemann ξ function, especially in a high temperature region. We obtain a new possible periodic lower "bound" on the Li-Keiper coefficients valid for all N.*

Key Words: *Ferromagnetic spin ½ models, Lee-Yang theorem, non trivial zeros, ξ function, Li-Keiper coefficients, Koebe function, periodic function, background Riemann wave, Riemann Hypothesis(RH).*

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I. Introduction

This work is a continuation with extensions of some analytical as well as computational treatments in some of our works [1] (where the Mehta-Dyson Polynomials were introduced [2, 3, 4, 5]). From one hand, we consider a spin ½ lattice model with 2N particles on a circle C_1;on the other hand we consider a truncation of the expansion of the Riemann ξ function and its properties at the same level 2N;i.e. a relative partition function associated toξ;for the relation between the two partition functions we extend some of our previous contributions in a series of works centered about the Lee-Yang theorem on the zeros of some spin ½ models in an external magnetic field $z=e^{-2h}$, and some on the partition functions related to the truncation of the ξ function in the variable z=1-1/s with the properties of the Li-Keiper coefficients and their tiny oscillations.

Concerning exactly solvable model in Statistical Mechanics the reader may consult [6] and [7] (the last especially for the spin model with long range interaction); for general results on the Riemann Equivalences and related problems the reader may consult [8].For important works related to the Li-Keiper coefficients of interest here we refer to [9,10,11,12,13] and many others. More References are given in [2, 5].

II. Ferromagnetic models and Polynomial truncations of the Riemann ξ function

2.1 The partition functions of a ferromagnetic spin ½ model defined on a circle, with two-body long range interaction of strength K, K = β·J = (1/kT)·J, where T is the absolute temperature, k the Boltzmann constant and J the interaction between two spins(the same here for all couples of spins variable σ_i, σ_j) and in presence of a magnetic field H (a one body interaction) β·H = h (up to an immaterial factor in $X = e^{(-2K)}$ and in $z=e^{(-2h)}$)is given by [3] :

$$Z(z,X) = \sum_{i=1}^{N} \binom{2\cdot N}{i} \cdot X^{i(2N-i)} \cdot (z^i + z^{2N-i}) \tag{1}$$

Eq.(1) is the partition function for a system of 2N interacting spin ½ variables on the circle (N=1,2,...) where the two-body interaction strength is here K, independent of the position of the spin variables.
For later use we will be concerned with only two of the terms in the summation above, i.e. the term i=1 and i=N given by:

$$i=1 \qquad \binom{2\cdot N}{1} \cdot X^{(2N-1)} \cdot z^1 \tag{2}$$

$$i=N \qquad \binom{2\cdot N}{N} \cdot X^{(N\cdot N)} \cdot z^N \tag{3}$$

2.2 The polynomial truncation of the ξ function of order 2N in the variable $z\rightarrow 1-1/s$, i.e. $s = 1/(1-z)$ where s is the usual complex variable $s=\sigma+i\cdot t$ (the critical line being $\frac{1}{2}+i.t$, $t \in R$), obtained from

$$log\left(\xi\left(\frac{1}{1-z}\right)\right) = log\left(\frac{1}{2}\right) + \sum_{i=1}^{\infty}\left(\frac{\lambda(i)}{i}\right)\cdot z^i \tag{4}$$

($\lambda(i)$ be the i-th Li-Keiper coefficient), is given by

$$\xi^*(z,\{\lambda_i\},N) = \sum_{i=0}^{N}\psi_i\cdot(z^i + z^{2N-i}) \tag{5}$$

with $\psi_i = \sum_{k=0}^{i}\binom{2\cdot N}{N-k}\cdot(-1)^k\cdot\varphi_k$
where

$$2\cdot e^{\left[2\cdot N\cdot log(1+z)+\sum_{i=1}^{N'}\left(\frac{\lambda(i)}{i}\right)\cdot z^i\right]} = \sum_{j=0}^{N}\varphi_i\cdot z^j + \cdots \tag{6}$$

Notice that the factor 2 compensate $e^{(log(1/2))}$ and in Eq.(6)the term $log(1-z)^{2N}$ was added [3] (to obtain the truncation)and that z was then changed in -z . (Notice that $\varphi_0 = 1$).
Thus, with the definition: $\xi^*(z, \{\lambda_i\},N) = Z_{2N}(z, \{\lambda_i\},N)$ we have for N=1, $Z_2 = 1\cdot(1+z^2) - 2\cdot\varphi_1 = 1+z^2 -2\cdot z\cdot\lambda_1$ ($\varphi_1 = \lambda_1$) where $\lambda_1 = (1+\gamma/2 - \log(4\cdot\pi)/2) = 0.0230957...$ is the first Li-Keiper coefficient and so on. Also for N>1.
Notice that in the approach, in order to have the same "accumulation point" as in the Ising model (z=1) i.e. $z=e^{(-2\cdot h)}$ at h=0 (zero field) the change of z in -z was introduced so that

$$z \rightarrow (-1)\cdot\left(((\sigma-1)^2 + t^2)/(\sigma)^2 + t^2)\right)^{1/2}\cdot e^{i\cdot(\arctan(t/(\sigma-1))-\arctan(t/\sigma))}$$

i.e. on the critical line $z = -e^{(-2\cdot i\cdot \arctan(2t))}\rightarrow 1$ for $t\rightarrow\infty$ as in the Ising model where the phase transition take place at h= 0,i.e.z=1.

III. Comparison between the partition functions of the two systems, spin model and truncation (especially for i=1 and i =N).

The first Equation for i =1 is given by:

$$2\cdot N- \lambda_1 = 2\cdot N\cdot X^{2\cdot N-1} = 2\cdot N\cdot\exp(-2\cdot K\cdot(2\cdot N-1)) \tag{7}$$

and for small K we have:

$$\lambda_1 = 2\cdot N\cdot(2\cdot N-1)\cdot 2\cdot K = \varphi_1 \tag{8}$$

In fact we know that $\varphi_1 = 1\cdot\lambda_1$ since $e^{(\lambda 1\cdot z)} \sim 1 + \lambda_1\cdot z + ... = \varphi_0 + \varphi_1\cdot z + ...$ from the definition.
For φ_2 we then have:

$$\binom{2\cdot N}{2}\cdot X^{2\cdot(2N-2)} = \binom{2\cdot N}{2} - \binom{2\cdot N}{1}\cdot\varphi_1 + \binom{2\cdot N}{0}\cdot\varphi_2 \tag{9}$$

and, with Eq.(8) we obtain ($X^{2.(2N-2)}$ ~1-2.K.2.(2N-2)). Then

$$-(2N-2)\cdot\lambda_1 = -2\cdot N\cdot\lambda_1 + \varphi_2$$

i.e.
$\varphi_2 = 2\cdot\varphi_1 = 2.\lambda_1$ for every N.

Additionally, with the definition $\varphi_2 =(1/2)\cdot(\lambda_1^2 + \lambda_2)$ we have
$\lambda_2' = 4\cdot\lambda_1 - \lambda_1^2$.

Now, for our truncation of the order N (the degree of the Polynomial = $2\cdot N$!) , for i=N, all φ'_s of index from i = 0 to i =N appear: the Equation of interest for i= N - from above- is given by:

$$\sum_{K=0}^{N}\binom{2\cdot N}{N-K}\cdot(-1)^K\cdot\varphi_k = X^{N\cdot N}\cdot\binom{2\cdot N}{N} \tag{10}$$

and for small K we have:

$$\sum_{K=0}^{N}\binom{2\cdot N}{N-K}\cdot(-1)^K\cdot\varphi_K=\binom{2\cdot N}{N}\cdot(1-N^2\cdot 2\cdot K)$$

Then:

$$\sum_{K=1}^{N-1}\binom{2\cdot N}{N-K}\cdot(-1)^K\cdot\varphi_K+(-1)^N\cdot\varphi_N=-\binom{2\cdot N}{N}\cdot(N^2\cdot 2\cdot K) \quad (11)$$

using Eq.(8) for K in Eq.(12) we obtain:

$$\sum_{k=1}^{N-1}\binom{2\cdot N}{N-k}\cdot(-1)^k\cdot\varphi_k+(-1)^N\cdot\varphi_N=$$

$$=-\binom{2\cdot N}{N}\cdot\left(\frac{N}{(2\cdot N-1)\cdot 2}\right)\cdot\lambda_1 \quad (12)$$

The equality $\varphi_n = n\cdot\lambda_1$ was checked for n=2,3, 4,5 in [3]. We now show that $\varphi_n = n\cdot\lambda_1$ holds for all n by induction.

Mathematical Induction

From above, $\varphi_1 = 1\cdot\lambda_1$.

We now suppose that $\varphi_k = k.\ \lambda_1$ for k=1..N-1.then from Eq.(12) :

$$\sum_{k=1}^{N-1}\binom{2\cdot N}{N-k}\cdot(-1)^k\cdot k\cdot\lambda_1+(-1)^N\cdot\varphi_N=-\binom{2\cdot N}{N}\cdot\left(\frac{N}{(2\cdot N-1)\cdot 2}\right)\cdot\lambda_1 \quad (13)$$

Now indicating with A(N-1, λ_1) the first term in Eq.(13) we have:

$$A(N-1,\lambda_1)+(-1)^N\cdot\varphi_N=-\binom{2\cdot N}{N}\cdot\left(\frac{N}{(2\cdot N-1)\cdot 2}\right)\cdot\lambda_1 \quad (14)$$

$$\text{If N is even,}\quad \varphi_N=-\binom{2\cdot N}{N}\cdot\left(\frac{N}{(2\cdot N-1)\cdot 2}\right)\cdot\lambda_1-A(N-1,\lambda_1) \quad (15)$$

$$\text{If N is odd}\quad \varphi_N=-\left(-\binom{2\cdot N}{N}\cdot\left(\frac{N}{(2\cdot N-1)\cdot 2}\right)\cdot\lambda_1+A(N-1,\lambda_1)\right) \quad (16)$$

Below, the plots of the right hand side of Equations (15) and (16) together with the plots of the functions y = n and y = - n (we have divided the terms of the Equations Eq.(15) and (16) by λ_1).

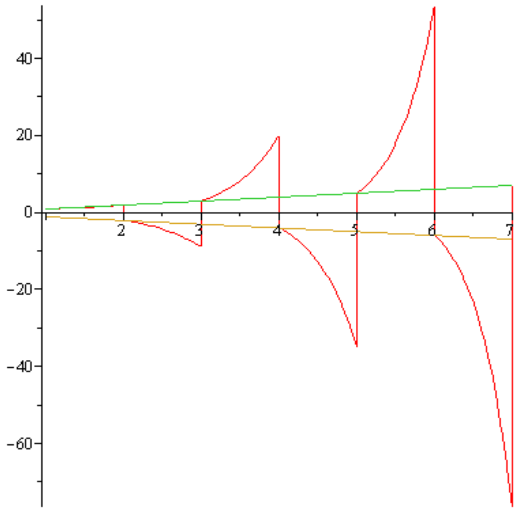

Fig. 1. Plots of the functions which give φ_n/λ_1 (in red),y=n (in green) and y=-n (in maroon).

Moreover we have:

$$\sum_{k=1}^{N} \binom{2\cdot N}{N-k}\cdot(-1)^k\cdot k\cdot\lambda_1 = -\binom{2\cdot N}{N}\cdot\left(\frac{N}{(2\cdot N-1)\cdot 2}\right)\cdot\lambda_1$$

(17)

We have thus proven that in the small K limit where K is the reciprocal of the temperature in the spin model i.e. $X = e^{(-2\beta\cdot J/k\cdot T)} = e^{(-2\cdot K)} \sim 1-2\cdot K$ we have:

$$\varphi_n = n\cdot\lambda_1 \quad \text{for all positive n.} \tag{18}$$

Since for n=0 we have $\varphi_0 = 1$ then :

$$e^{\left(\sum_{n=1}^{\infty}\frac{\lambda_n\cdot z^n}{n}\right)} = \sum_{n=0}^{\infty}\varphi_n\cdot z^n = 1+\lambda_1\cdot(1z+2z^2+3z^3+\cdots) =$$

$$= 1+\lambda_1\cdot\left(\sum_{n=0}^{\infty} n\cdot z^n\right) = 1+\frac{\lambda_1\cdot z}{(1-z)^2} = 1+\lambda_1\cdot K(z)$$

(19)

where now $K(z) = z/(1-z)^2$ is the Koebe function of argument z [5].
We notice that a perturbation around the K function entered in one of our recent work in another approach to the study of the tiny fluctuations in the Li-Keiper coefficients [4].In the above limit we have that

$$\sum_{n=1}^{\infty}\frac{\lambda_n\cdot z^n}{n} = log(1+\lambda_1\cdot K(z)) = log\left(\frac{(1-z)^2+\lambda_1\cdot z}{(1-z)^2}\right) =$$
$$= f(z) := log\left(\frac{(z-z_1)\cdot(z-z_2)}{(1-z)^2}\right) \tag{20}$$

where z_1 and z_2 are the solutions of the Equation $z^2 - z\cdot(2-\lambda_1) + 1 = 0$, i.e. $z_1 = e^{i\varphi}$ and $z_2 = e^{-i\varphi}$.
It should be recalled that the partition function of the smallest spin½ system on the circle with an even number of spin sites 2·N, i.e. N=1,that is that of two spin variables, $Z_2(X) = Z_2(e^{-2K}, e^{-2h})$, with $X = e^{-2K}$ and with the magnetic spin variable $z = e^{-2h}$, is given by: $Z_2 = z^2 + 2\cdot X\cdot z + 1$ which, after the change $z\to -z$ as described in [3], i.e. $z^2 - 2\cdot X\cdot z + 1$, gives $X = e^{-2\cdot K} = (1-(\lambda_1)/2)$ and $2\cdot\lambda_1 = (½)\cdot(\lambda_2 + + \lambda_1^2)$ i.e. $\lambda_2 = 4\cdot\lambda_1 - \lambda_1^{\ 2}$.
Then, the derivative of f(z) is:

$$f'(z) = d/dz(f(z)) = 2/(1-z) - (1/z_1)\cdot(1/(1-z/z_1)) - (1/z_2).(1/(1-z/z_2)) =$$
$$= \sum_{n=0}^{\infty}\left(2-\frac{1}{z_1^{(n+1)}}-\frac{1}{z_2^{(n+1)}}\right)\cdot z^n = \sum_{n=0}^{\infty}\left(2-2\cdot cos(\varphi\cdot(n+1))\right)\cdot z^n$$

and finally

$$f'(z) = \sum_{n=0}^{\infty}\lambda_n\cdot z^{(n-1)} = \sum_{n=1}^{\infty} 4\cdot sin^2\left(\frac{\varphi\cdot(n)}{2}\right) z^{(n-1)} \tag{21}$$

A possible lower "bound" (we have verified its validity for low values of n) of the Li-Keiper coefficients, for all n greater than zero, would be:

$$\lambda_n \geq 4\cdot\sin^2(\varphi.(n)/2). \quad n > 0 \tag{22}$$

Below we give the plot of the proposed lower "bound" (periodic function) in the range n =[0..4] with the first four true values λ's.

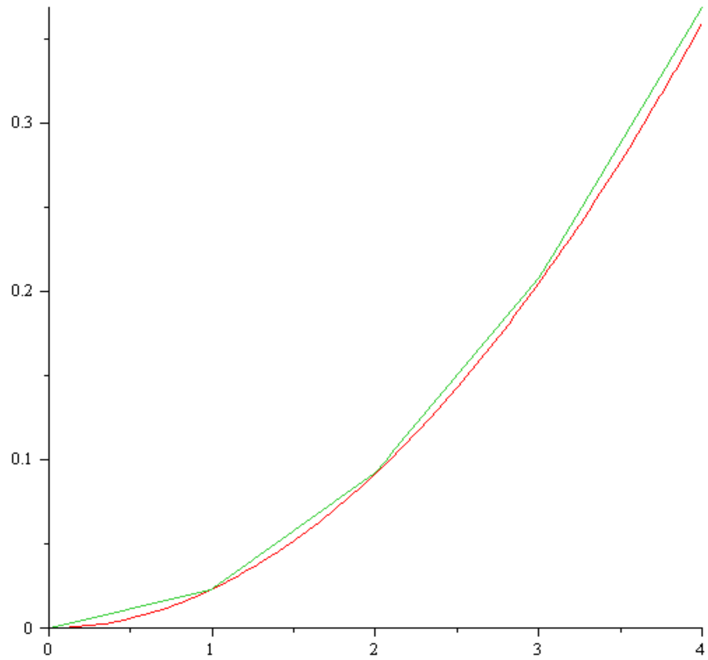

Fig. 2. In red the periodic function as lower bound; in green the polygonal of the first 4 values λ's.

Below on the Table, we give our first fifteen values of Eq.(22) (lower bounds) and the corresponding true values of Ref [10].

n	lower bound	true value
1	0.0230957	0.0230957
2	0.0918494	0.0923457
3	0.2046732	0.2076389
4	0.3589613	0.3687904
5	0.5511504	0.5755427
6	0.7768017	0.8275660
7	1.0307037	1.1244601
8	1.3069922	1.4657556
9	1.5992862	1.8509160
10	1.9008350	2.2793393
11	2.2046741	2.7503608
12	2.5037861	3.2632553
13	2.7912628	3.8172400
14	3.0604647	4.4114776
15	3.3051744	5.0450793

Table

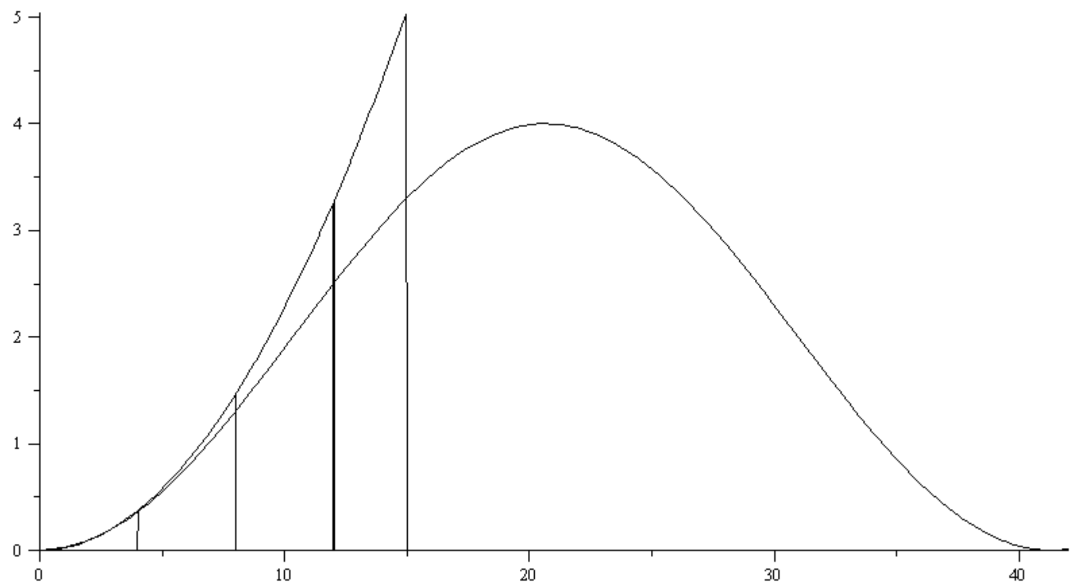

Fig. 3. The lower periodic "bound" (the periodic function up to the first maximum around n=20) and the first 15 true values, taken from [10], up to the true value $\lambda_{15} = 5.04$..

Remark
We now investigate about the possibility that the above inequality has the chance to be correct.
For this we take the infinite temperature limit ($K \rightarrow 0$) in Eq.(8))and we have - instead of Eq. (22) - Eq.(23) given by (since now $\varphi_k = 0$ all k!)

$$\lambda_n \geq 4 \cdot \sin^2(0 \cdot (n+1)) = 0 \text{ for all } n>0. \tag{23}$$

If the above inequality holds, then Eq.(23) coincides with the Li-Keiper Equivalent for the truth of the RH i.e. that all λ_n should all be non negative,for every n . We note that in the high temperature region (K small) it emerges our periodic function which is greater than the Li-Keiper Equivalent given by the above Equation. Analyzing the high temperature region we have thus remarked that the coefficients of f '(z) increase, a manifestation of the possible positiveness of all the Li-Keiper coefficients. (See Appendix 1 and Appendix 2 for additional completations).

IV. Inhomogeneous interactions

We now look at a spin model with inhomogeneous interactions between two spin variable i.e. $X_1 = \exp(-2K_1)$ between nearest neighbors, and so on … $X_{N-1} = \exp(-2K_{N-1})$ and $X_N = \exp(-2K_N)$ for two spin variable sitting on the opposite sites (diameter) of the circle. We restrict us to the second Equation (i=2) above. The two Equations are given by [3]:

$$2N \cdot [(\prod_{i=1}^{N-1} X_i^2) \cdot X_N] = 2N - \lambda_1 = 2N - \varphi_1 \tag{24}$$

$$2N \cdot [(\prod_{i=1}^{N-1} X_i^2) \cdot X_N]^2 \cdot \left[2N \cdot \left(\sum_{i=1}^{N-1} \left(\frac{1}{X_i^2}\right) + N \cdot \left(\frac{1}{X_N^2}\right)\right) \right] \tag{25}$$

As above, in the high temperature limit ($e^{(-2X)} \sim 1 - 2 \cdot X$)the first Equation gives

$$2 \cdot N \cdot (\sum_{i=1}^{N-1} 4 \cdot K_i + 2 \cdot K_N) = \lambda_1 \tag{26}$$

In the same way, the second Equation gives:

$$N^2 - N - 2N(N-1) \cdot \left(4 \cdot K_N + \sum_{i=1}^{N-1} K_i\right) + 8N \cdot \sum_{i=1}^{N-1} K_i - 8N \cdot \sum_{i=1}^{N-1} K_i \tag{27}$$

Finally, substituting λ_1 from the above Equation in the last Equation (27), we obtain:

$$\binom{2N}{2} - 2N \cdot (N-1) \cdot 2 \cdot \frac{\lambda_1}{2N} = \binom{2N}{2} - 2N \cdot \varphi_1 + \varphi_2$$

$$\varphi_2 = 2\cdot\varphi_1 = 2\cdot\lambda_1 \tag{28}$$

$$\text{Since} \quad \varphi_2 = (½)\cdot(\lambda_1^2 + \lambda_2) \ \rightarrow \lambda_2 = 4\cdot\lambda_1 - \lambda_1^2 \tag{29}$$

as for the homogeneous case i.e. where $K_i = K$, for all i.

Remark: Notice that $4\cdot\lambda_1 - \lambda_1^2 = 0.091849...$ while the true value is $\lambda_2 = 0.0923457...$ The value $\lambda_2' = 4\cdot\lambda_1 + - \lambda_1^2$ appears as a lower bound to the true value λ_2.
We now show, using the density of the zeros, that $\lambda_2 > \lambda_2' = 4\cdot\lambda_1 - \lambda_1^2$.
From the definition we have in fact:

$$\lambda_2 = \sum_\rho \left(1 - \left(1 - \frac{1}{\rho}\right)^2\right) = 2 \cdot \sum_\rho \frac{1}{\rho} - \sum_\rho \frac{1}{\rho^2} = 2 \cdot \lambda_1 - \sum_\rho \frac{1}{\rho^2}$$

and the above inequality is true if

$\sum 1/\rho^2 < \lambda_1^2 - 2\cdot\lambda_1$ that is with $\rho = \sigma + i\cdot t$

if $\lambda_2 = 2\cdot\lambda_1 - \sum (2\cdot\sigma^2 - 2\cdot t^2)/(\sigma^2 + t^2)^2 = 4\cdot\lambda_1 - \sum 4\cdot\sigma^2/(\sigma^2 + t^2)^2 > 4\cdot\lambda_1 - \lambda_1^2$
that is

$$\text{if} \ \lambda_1^2 > \sum_{i=1}^{\rho} \frac{(4\cdot\sigma_i^2)}{(\sigma_i^2 + t_i^2)^2} \quad \text{i.e.}$$

$$\text{if} \quad \lambda_1^2 > \sum_{i=1}^{\infty} 4 \cdot \left(\frac{1}{t_i^4}\right) \tag{30}$$

Using the density of the nontrivial zeros $dN = (1/(2\pi))\cdot\log(t/(2\pi))\cdot dt$ and integrating from $t_1 = 14.134725..$ (the first t value)to infinity we obtain that $0.000576... > 0.00002149301199\cdot 4 = 0.000086...$
Thus $\lambda_2' = 4\cdot\lambda_1 - \lambda_1^2$ is a lower bound to the true value λ_2 .

V. The case 2N= 4

We now treat in details the case 2N=4, i.e. a spin system with 4 particles (4 spins ½) and the corresponding truncated ξ function, i.e. a Polynomial in z = 1-1/s of degree 4: this because in this manner one see in detail the computations which leads to the possible lower bound to the Li-Keiper coefficients for all N.
For this small spin system we have, with $X = e^{(-2K)}$ and $z = e^{(-2h)}$:

$$Z_4(X, z) = 1 + 4\cdot X^3\cdot(z + z^3) + 6\cdot X^4\cdot z^2 + z^4 \tag{31}$$

for $0<X<1$ and we know that the zeros in z of Z_4 are on the unit circle (Lee and Yang Theorem, and others) [3].
The corresponding truncated ξ function reads:

$$Z_4(\lambda_1, \lambda_2) = 1 + (4 - \lambda_1)\cdot(z + z^3) + (6 - 4.\varphi_1 + \varphi_2)\cdot(z^2 + z^4) =$$
$$1 + (4 - \lambda_1)\cdot(z + z^3) + (6 - 4.\lambda_1 + (½)\cdot(\lambda_1^2 + \lambda_2))\cdot z^2 + z^4 \tag{32}$$

and with λ_1 and λ_2 solutions of the system of the 2 Equations below for values $0 < X < 1$ i.e.

$$4\cdot X^3 = 4 - \lambda_1 \tag{33}$$
$$6\cdot X^4 = 6 - 4\cdot\lambda_1 + (½)\cdot(\lambda_1^2 + \lambda_2) \tag{34}$$

the zeros in z of $Z_4(\lambda_1, \lambda_2)$ are on the unit circle. With the change of variable $w = z + 1/z$ we obtain:

$$w^2 + 4\cdot X^3\cdot w + 6\cdot X^4 - 2 = 0. \tag{35}$$

with the two real solutions given and represented below as a function of X, $0 < X < 1$.

$$w = \frac{1}{2}\left\{-4 \cdot X^3 \pm \left[\sqrt{(X^2-1)^2 \quad \cdot (2 \cdot X^2 + 1)}\right]\right\} \quad (36)$$

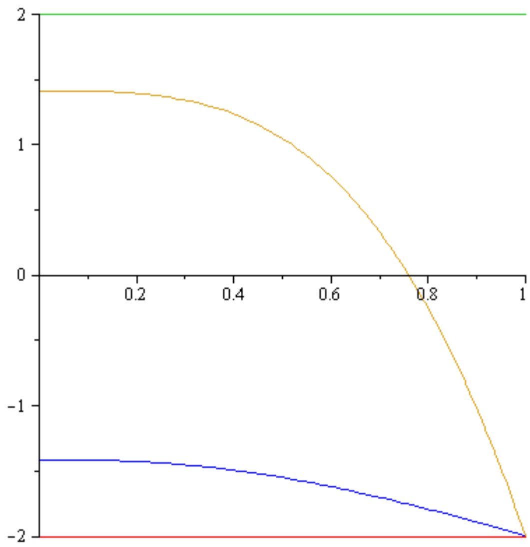

Fig. 4. w_1 and w_2 as a function of $X = e^{-2.K}$ in the ferromagnetic region 0<X<1.

For the Equation of the truncated ξ function we have:

$$w^2 + (4-\lambda_1)\cdot w + 4 - 4\cdot\lambda_1 + \varphi_2 = 0 \quad (37)$$

with the two real solutions given by:

$$w = \frac{1}{2}\left\{-(4-\lambda_1) \pm \left[\sqrt{\lambda_1^2 \quad + 8\cdot\lambda_1 - 4\cdot\varphi_2}\right]\right\} \quad (38)$$

if $\lambda_2 \leq 4\cdot\lambda_1 - (½)\cdot\lambda_1^2$.
The zeros in z are on the unit circle if $|w| <= 2$, i.e. if $\lambda_2 >= 4.\lambda_1 - \lambda_1^2$. (See Eq.(29)). Below we represent, as a function of X, 0<X<1,

$$\lambda_1(X) = 4\cdot(1-X^3) \quad \text{and} \quad (39)$$
$$\lambda_2(X) = 12\cdot(X^4-1) + 32\cdot(1-X^3) - 16\cdot(1-X^3)^2 \quad (40)$$

We notice that for the true value $\lambda_1 = 0.0230957089661$, we obtain X=0.9980716414, argument that inserted in λ_2 (X) gives λ_2 (0.9980716414) = 0.09193843822 to be compared with the true value $\lambda_2 = 0.0923457$...(See the above remark).

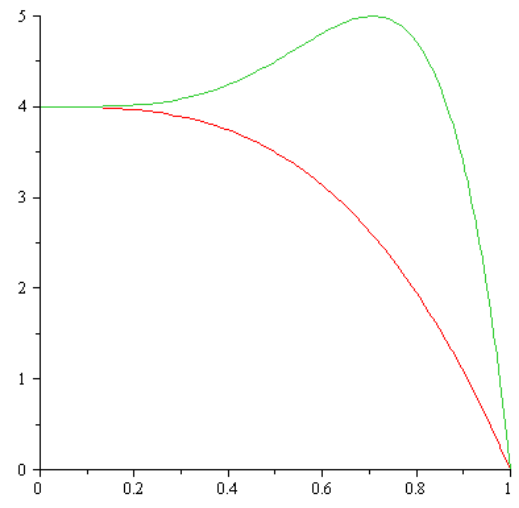

Fig. 5. In red λ_1 (X), in green λ_2 (X) for 0 <X<1.

High temperature limit for the spin model.
In Z_4 (X, z) we set $K \to 0$ i.e. $X^n \sim 1 - 2K \cdot n$, here $n \leq 4$ and we obtain Z_4'. Then

$$w^2 + 4\cdot(1\text{-}6K)\cdot w + 6\cdot(1\text{-}8K) - 2 = 0. \tag{41}$$

with the solutions $w_1 = -2$ and $w_2 = -2 + 24\cdot K$ which gives 4 zeros in z of Z'_4 on the unit circle. From above, $24\cdot K = \lambda_1$ and for the truncated ξ function $Z_4'(\lambda_1, \lambda_2)$ we have

$$w^2 + (4-\lambda_1)\cdot w + 4 - 2\cdot\lambda_1 = 0. \tag{42}$$

Then with the second Equation i.e.,
$6\cdot(1\text{-}8K) = 6 - 4\cdot \lambda_1 + \varphi_2$ we have $\varphi_2 = 2\cdot\lambda_1 \to \lambda_2 = 4\cdot\lambda_1 - \lambda_1^2$.

Then, $w_1 = -2$ and $w_2 = -2 + 24\cdot K = -2 + \lambda_1$.
The solutions in z of $Z_4'(\lambda_1,\lambda_2) = 0$ are thus given by:
$w_1 = z + 1/z = -2 \rightarrow (z+1)^2 = 0$ and, $z_1 = z_2 = -1$.
$w_2 = z + 1/z = -2 + \lambda_1$
and

$$z_{3,4} = \frac{1}{2}\left[-(2-\lambda_1) \pm \sqrt{\lambda_1^2 \quad - 4\cdot\lambda_1}\right] \tag{43}$$

In order that all 4 zeros be on the unit circle we should have $\lambda_1^2 - 4\cdot\lambda_1 = -\lambda_2 \leq 0$ i.e. $\lambda_2 \geq 0$.The above high temperature limit gives $\varphi_2 = 2\cdot\lambda_1$ and $\lambda_2 > 0$ ensure that all zeros in z for this limit are on the unit circle.

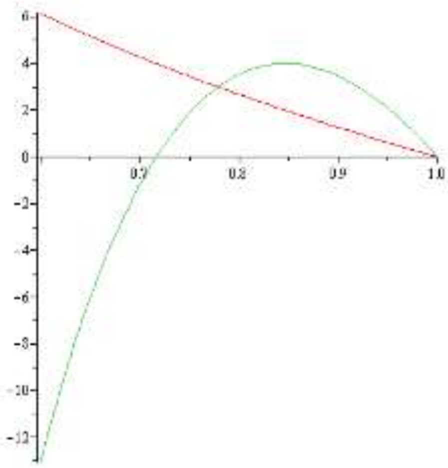

Fig. 6. $\lambda_1'(X) = -12\cdot\log(X)$ (in red) and $\lambda_2'(X) = -12\cdot\log(X)\cdot(4+12\cdot\log(X))$ (in green), for $0<X<1$.

Notice that in this limit -for $\lambda_1 = 0.0230957089661$ -we have X= 0.9980772085 which gives $\lambda_2 = 0.09184942519$ equal to $\lambda_2 = 4\cdot\lambda_1 - \lambda_1^2$.
Finally since in this limit (for 2N=4!),

$$Z_4'(\lambda_1, \lambda_2) = Z_4'(\lambda_1, \lambda_2 = 4\cdot\lambda_1 - \lambda_1^2) = (1+z)^2\cdot(z^2 + (2-\lambda_1)\cdot z + 1) =$$

$$= (1+z)^4\cdot(1 - \lambda_1\cdot z/(1+z)^2) \tag{44}$$

changing back from $z \to -z$ and remembering the factor $(1-z)^{2N} = (1-z)^4$ of multiplication for the truncation of ξ [3], we see that the factor $(1 + \lambda_1\cdot z/(1-z)^2) = (1 + \lambda_1\cdot K(z))$ involves here too the Koebe function K(z) [5], for every N as discussed above.
In fact, the Taylor expansion of $\log(1+\lambda_1\cdot K(z))$ around z=0 is given (defining f as below), by:

$$f(z) = \log(1 + \lambda_1\cdot z/(1-z)^2) \sim \lambda_1\cdot z/ + (4\cdot\lambda_1 - \lambda_1^{\,2})\cdot z^2/2 +$$
$$+ \quad (9\cdot\lambda_1 - 6\cdot\lambda_{11} + \lambda_1^{\,3})\cdot z^3/3 + \ldots. \tag{45}$$

i.e., by introducing the two zeros in z of $(1-z)^2 + \lambda_1\cdot z = z^2 - (2-\lambda_1)\cdot z + 1$

$z_1 = e^{i\cdot\phi}$ and $z_2 = e^{-i\cdot\phi}$ we have
$d/dz\,(f(z)) = 2/(1-z) - (1/z_1)\cdot(1/(1-z/z_1)) - (1/z_2)\cdot(1/(1-z/z_2)) =$
$2-2\cdot\cos(\phi) + (2-2\cdot\cos(2\cdot\phi)\cdot z + (2-2\cdot\cos(3\cdot\phi))\cdot z^2 + \ldots =$

$$= \sum_{n=1}^{\infty}(2-2\cos(n\phi)\cdot z^{n-1} = \sum_{n=1}^{\infty} 4\cdot sin^2\left(\frac{n\phi}{2}\right)\cdot z^{n-1} =$$

$$= \sum_{n=1}^{\infty} \lambda_n' \cdot z^{n-1}$$

Thus:

$$\lambda_n' = 4\cdot sin^2\left(\frac{n\phi}{2}\right) \qquad n=1, 2, \ldots. \tag{46}$$

Notice that it may be shown that for an hexagon (2N=6 spins variable and the corresponding truncation up to z^6 of ξ), $Z_6'(\lambda_1) = (1+z)^6\cdot(1-\lambda_1\cdot z/(1+z)^2)$.
Similarly for an "Octagon",

$$Z_8'(\lambda_1) = (1+z)^8\cdot(1-\lambda_1\cdot z/(1+z)^2). \tag{47}$$

From our analysis we have established that our high temperature limit for the truncated ξ function i.e. Eq.(45) holds for- and is the same -for all N - (apart the factor $(1-z)^{2N}$ which is recovered and which have been dropped out) and is given by

$$\log(1+\lambda_1\cdot z/(1-z)^2) = \log[(z^2-(2-\lambda_1)\cdot z+1)/(z-1)^2] \tag{48}$$

where the numerator in the argument of the log is the expression which gives the zeros in the magnetic field variable $z=e^{-2h}$ ($h=H/\beta.T$)of the truncation of smaller order of the ξ function corresponding to the thermodynamic reduced partition function of the ferromagnetic model with two spins (two particles) of Statistical Mechanics. Our periodic function appears as a lower bound to the Li-Keiper coefficients for N=2 (system with 2N= 4 spins, that is λ_1 and $\lambda_2 = 4.\,\lambda_1 - \lambda_1^2$) and the periodic function above is the same for all N, thus concluding: for a non negative possible lower bound to the Li-Keiper coefficients which would ensure the truth of the RH.

VI. Concluding remark

In this work, starting with a comparison between the partition functions of a spin ½ lattice system on a circle with two-body long range ferromagnetic interaction and those corresponding to a truncation of the Riemann's ξ function started in [3],we extended analytical computations in a high temperature region – thus-obtaining and proposing also in our high temperature region a new possible non negative lower "bound" on the Li-Keiper coefficients in the form of a periodic function containing the Koebe function and the first Li-Keiper coefficient λ_1 – which- if it is equal to the reciprocals[11] of all the zeros on the critical line ensures the truth of the Riemann Hypothesis: for the infinite temperature limit, such a lower "bound" coincides with the Li-Keiper condition for the truth of the RH i.e. the non negativity property [9, 13] of all the Li-Keiper coefficients: in our height but "finite" temperature region , such a lower "bound" increases from 0 to a positive discrete periodic function of a maximum value which is equal to 4.

The high temperature property of the coefficients of z^N (advanced and controlled in [3] up to N=5) is proven here for all N.

In the Appendix we give a proof (for a system with 2N = 6 spin variable, i.e. for the corresponding truncation of the ξ function to order 6) that the λ_i, i=1, 2, 3 of the small system have as lower bounds λ_i', i =1, 2, 3, the values emerging from the high temperature limit we have constructed.

Finally, extensions of the approach with more general Lee-Yang measures for models with long range interactions are expected to yield possibly more information about the values of the Li-Keiper coefficients.

Our (positive) periodic function (a background Riemann wave depending only on γ (the Euler-Mascheroni constant) and π) i.e. on the first Li-Keiper coefficient λ_1 and the Li-Keiper constant function (constant zero) are represented on the Figure below.

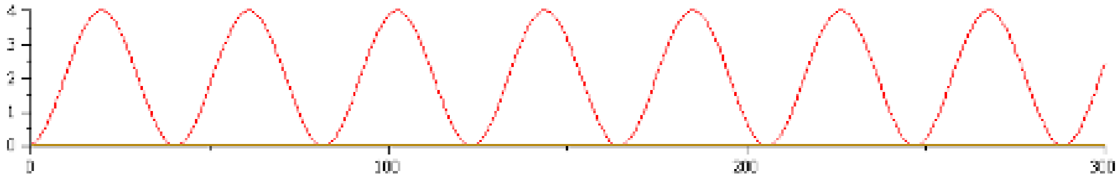

Fig. 7. The Li-Keiper constant (in black) and the background Riemann Wave (in red). Notice that the minima of the periodic function are positive (the first one in the interval n=[40-41]).

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Appendix 1

We know that for 2N=6, the truncated ξ function (a polynomial of order 6 in z) or the spin model (6 spins with the magnetic field variable z) have all their 6 zeros on the unit circle [3, 6, 7].Here we construct a proof for this small system ,that the λ's are bounded below from the λ's emerging from our high temperature limit $X \rightarrow 1$ in some manner for this system.

The three Equations obtained from the general set Eq.(1) and Eq.(5) are given by:

$$6-\lambda_1 = 6 \cdot X^5 \tag{a1}$$

$$15 - 6 \cdot \lambda_1 + \varphi_2 = 15 \cdot X^8 \tag{a2}$$

$$20 - 15 \cdot \lambda_1 + 6 \cdot \varphi_2 - \varphi_3 = 20 \cdot X^9 \tag{a3}$$

where $\varphi_1 = \lambda_1$ is a positive free parameter , $\varphi_2 = (½) \cdot (\lambda_2 + \lambda_1^2)$ and $\varphi_3 = (1/3) \cdot (\lambda_3 + +(3/2) \cdot \lambda_1 \cdot \lambda_2 + \lambda_1^3/2)$.
From these relations we compute φ_2 and φ_3 as a function of X i.e. of λ_1 and the same for λ_2 and λ_3 .
We verify that $\varphi_2 > 2 \cdot \lambda_1$ and that $\varphi_3 > 3.\lambda_1$, also that $\lambda_2 > 0.091938..$ (the true value is 0.0923457..) and also that $\lambda_3 > 0.205...$ (the true value is 0.20763...).The plots of λ_2 and of λ_3 as a function of X, i.e. of λ_1 with Eq.(a1) is given below in the interval of λ_1 from 0.022 to the highest value 0.0230957, value which gives the two values greater then the two lower bounds i.e.
$\lambda_2 = 0.0920628 > \lambda_2' = 0.091849$ and $\lambda_3 = 0.205951 > \lambda_3' = 0.204673$
(λ_2' is obtained from $\varphi_2 = 2 \cdot \lambda_1$ and λ_3' is obtained from $\varphi_3 = 3 \cdot \lambda_1$).

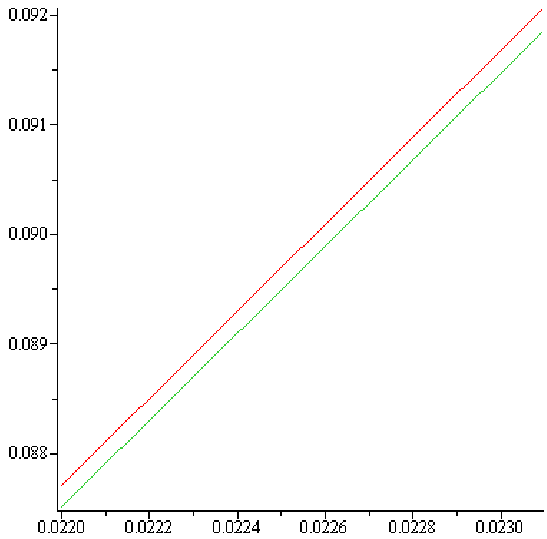

Fig. (a1). λ_2 (in red) and λ_2' (in green) as a function λ_1 in the region [0.022..0.0230957]

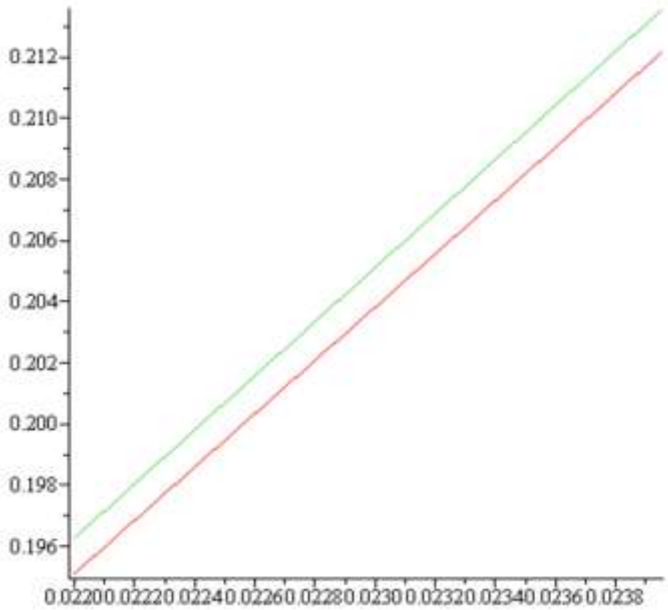

Fig. (a2). λ_3 (green) and λ_3' (red) as a function λ_1 in the region [0.022..0.0230957]

With φ_2 and φ_3 the formulas are given by:
$\lambda_2' = 4\cdot\lambda_1 - \lambda_1^2$ and $\lambda_3' = 9\cdot\lambda_1 - 6\cdot\lambda_1^2 + \lambda_1^3$ ¨.
For X=1 in Eq. (a1) we obtain $\lambda_1 = 0$; from Eq.(a2), $\lambda_2 = 0$ and from Eq.(a3) , $\lambda_3 = 0$;these values may be seen as lower bounds for the first three values of λ in the spirit of the Li-Keiper condition for the truth of the Riemann Hypothesis.(Non negativity of all the Li-Keiper coefficients).
Our high temperature limit $X\rightarrow 1$provides a discrete periodic function which should constitute in the same spirit a condition for the truth of the Riemann Hypothesis. Concerning Equivalents of the Riemann Hypothesis [8] we can consider the Li-Keiper Equivalent of the RH and affirm the following:
"If our emerging periodic function is correct in the sense of Statistical Mechanics, then all Li-Keiper coefficients are non negative.
If RH is true, then the Li-Keiper coefficients are surely greater than the periodic bound, thus if our periodic function is correct true for all N, then we have a new "Equivalent" of the RH".

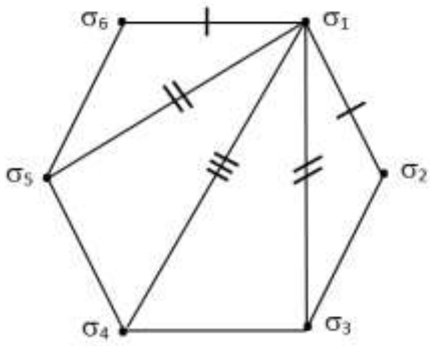

Fig. (a3). Model with six spin variables nearest neighbors interactions $\sigma_1\sigma_2$ and $\sigma_1\sigma_6$.Spin variables with nearest, next nearest and the two body interaction$\sigma_1\sigma_4$ of the two opposite spins on the diameter.

Appendix 2

It may be interesting to prove that $\varphi_2 = (½)\cdot(\lambda+\lambda_1^2 > 2\cdot\lambda_1$using the formula for the structure of the zeros i.e.

$$\lambda_2 = \sum_{\rho}\left(1-\left(1-\frac{1}{\rho}\right)^2\right)$$

(where the sum is over all nontrivial zeros of ζ) , without a comparison with a spin model and without assuming RH is true.

Proof

If $\rho = \sigma + i\cdot t$ is a zero then $(1-\rho) = (1-\sigma - i\cdot t)$ is a zero and also their conjugates i.e. σ - i.t and $1-\sigma + i\cdot t$, are nontrivial zeros, thus:

$$\lambda_2 = \sum (2/\rho) - (1/\rho^2) = 2.\lambda_1 - \sum (1/\rho^2)$$

Notice that $\lambda_1 = \sum_\rho (1-(1-1/\rho)^1) = \sum_\rho (1/\rho) =$

$$= \sum[\,(2\sigma/(\sigma^2+t^2) + 2.(1-\sigma)/((1-\sigma)^2+t^2))\,] \ (1)$$

(the summation is always on all nontrivial zeros of ξ)

We have:

$$\sum -(1/\rho^2) = -\sum 2(\sigma^2 - t^2)/(\sigma^2+t^2)^2 - 2\cdot(1-\sigma)^2 - t^2)/((1-\sigma)^2+t^2)^2 =$$
$$= -\sum 2(2\cdot\sigma^2 - \sigma^2 - t^2)/(\sigma^2+t^2)^2 - 2\cdot(2\cdot(1-\sigma)^2 - (1-\sigma)^2 - t^2)/((1-\sigma)^2+t^2)^2 =$$

$$= \sum [\,2/(\sigma^2+t^2) + 2/((1-\sigma)^2+t^2)] - \sum[4\cdot\sigma^2/(\sigma^2+t^2)^2 + 4((1-\sigma)^2/((1-\sigma)^2+t^2)^2\,]$$

The second term is of the Form $-\sum X_n^2$, i.e. a sum of squares of positive numbers and since $-\sum X_n^2 \geq -(\sum X_n)^2 = -\lambda_1^2$ from Eq.(1), we obtain

$$\lambda_2 \geq 2\cdot\lambda_1 - \lambda_1^2 + \sum [\,2/(\sigma^2+t^2) + 2/((1-\sigma)^2+t^2)] =$$

$$2\cdot\lambda_1 - \lambda_1^2 + \sum [\,2\cdot\sigma/(\sigma^2+t^2) + 2\cdot(1-\sigma)/((1-\sigma)^2+t^2)] + R(\sigma,t) \geq 4.\lambda_1 - \lambda_1^2 +$$

$$+ R(\sigma, t)$$

where

$$R(\sigma, t) = 4\cdot(\sigma - 1/2)\cdot[\,1/((1-\sigma)^2+t^2) - 1/((\sigma^2+t^2)]\ (2)$$

Notice here the appearance of the Riemann symmetry for the function R,- for all t- given by R(σ, t) = R(1- σ, t) which has a minimum at σ=1/2, i.e. R(1/2,t) =0 for all t. As an illustration we give below the plot of R for three value of t.

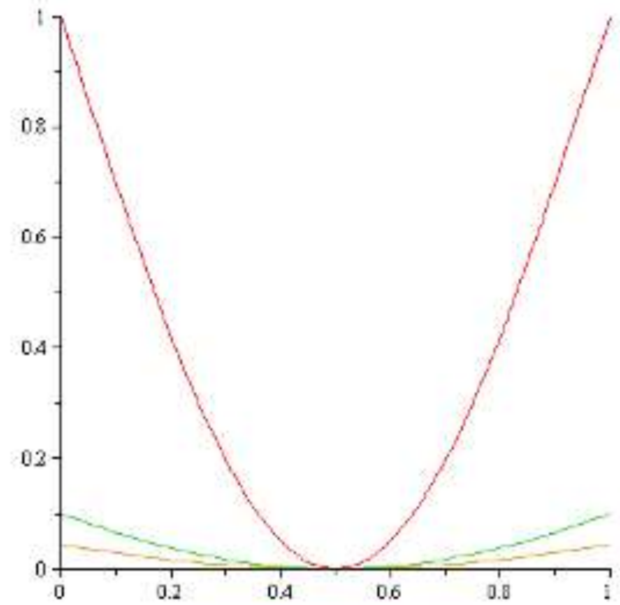

Fig. (a4). Plots of the function R for t=1.5(maroon, t=2(green) and t=2.5(red) in the range [0, 1] of σ.

In conclusion: $\lambda_2 \geq 4\cdot\lambda_1 - \lambda_1^2$ i.e. $\varphi_2 \geq 2\cdot\lambda_1$ a result obtained without assuming the RH is true.
Notice that the lower bound is given by $4\cdot\lambda_1 - \lambda_1^2 = 0.09184938..$
We now perform a numerical experiment to calculate the increment from λ_1 to λ_2 i.e., $\Delta_2 = \lambda_2 - \lambda_1$. From the Definition of λ_n, in general we have:

$$\Delta_n = \lambda_n - \lambda_{n-1} = \sum_{\rho}\left(1-\left(1-\frac{1}{\rho}\right)^n\right) - \sum_{\rho}\left(1-\left(1-\frac{1}{\rho}\right)^{n-1}\right) =$$

$$= \sum_{\rho}\left(\left(1-\frac{1}{\rho}\right)^{n-1}\right)\cdot\left(\frac{1}{\rho}\right) = \sum_{\rho}\left(\left(\frac{\rho-1}{\rho}\right)^n\cdot\left(\frac{1}{\rho-1}\right)\right)$$

Remark: if there is a zero off the critical line, then if $\rho = \sigma + i\cdot t$ is such a zero with $\sigma > 1/2$ we have the two contributions:
$((\sigma-1-i\cdot t)/(\sigma+i\cdot t))^n.(1/(\sigma-1-i\cdot t))$ very small for n big in absolute value and the amount $((-\sigma-i\cdot t)/((1-\sigma-i\cdot t))^n .(1/(-\sigma-i\cdot t))$ exploding as n is big in absolute value since
$(\sigma/(1-\sigma)) > 1$.For the numerical experiment on $\Delta_2 = \lambda_2 - \lambda_1$ we assume here the RH; then
$\rho = ½ \pm i\cdot t$ and we obtain:

$$\Delta_n = \lambda_n - \lambda_{n-1} =$$
$$= \sum -(-1)^n\cdot (\cos(2\cdot n\cdot \arctan(2\cdot t))+2\cdot t\cdot \sin(2\cdot n\cdot \arctan(2\cdot t)) \cdot(1/(1/4+t^2)).$$

For n=2, we take the first 20 zeros from the Tables and form the 21-ten zero we integrate with the weight $((1/(2\cdot\pi)) \cdot\log(t/(2\cdot\pi))\cdot dt$ up to infinity. We obtain :

$\Delta_2 = 0.0478413 + 0.0212776 = 0.0691189$ to be compared

with the exact value $\Delta_{2=}$ 0.0923457 – 0,0230957 = 0.0692500..
With the lower bound we obtain instead $\Delta_2' = 0.091849-0.0230957=0.0687536$.
We may also consider the equivalent formula

$$\Delta_2(\sigma) = \sum[(2\cdot\sigma)/(\sigma^2 + t^2) + 2\cdot(1-\sigma)/((1-\sigma)^2+t^2) +$$

$- (2\cdot \sigma^2 -2\cdot t^2)/(\sigma^2 +t^2)^2 - (2\cdot(1-\sigma)^2 -2\cdot t^2)/((1-\sigma)^2+t^2)^2] =$

$= \sum R'(\sigma, t)$

where R' is the contribution of the four zeros ($\sigma\pm i\cdot t$ and $(1-\sigma)\pm i\cdot t$). Notice that here R' has the Riemann symmetry i.e. $R'(\sigma, t) = R'(1-\sigma, t)$ for all t but, contrary to the function $R(\sigma, t)$ (which has a minimum) R' has a maximum at $\sigma =1/2$, (See Figure below) and thus $R'(\sigma, t) > R'(0,t) = R'(1,t) =2/t^2 +4t^2 /(1+t^2)^2$.

A lower bound to Δ_2 is given by $\Delta_2= \sum R'(0, t) =\sum(2/t^2+ 4t^2 /(1+t^2)^2)$where the sum is on the heights $t_k > 0$ of all the nontrivial zeros.

This proves here too that $\Delta_2> 0$. For any n,

$$\Delta_n(\sigma) = \sum\nolimits_{t_k} R'_n\,(n, \sigma, t_k\,)$$

where $R'(n, \sigma, t_k)$ has the Riemann symmetry and have a maximum for $\sigma=1/2$, but for big n there are contributions of negative amounts in a corresponding region of the t values . The truth of the RH is equivalent to the Δ_n's having the maximum possible value for each n, and every height t (maximum increment). Below the plot of $R'(\sigma, t)$ for n=2 and n=10 at the value of t_1 =14.134725 i.e. the height of the first nontrivial zero.

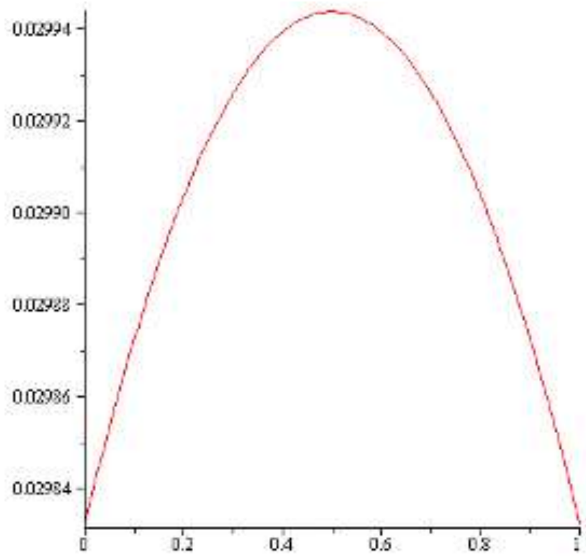

Fig. (a5). R'(σ, t=14.134725.., n=2)

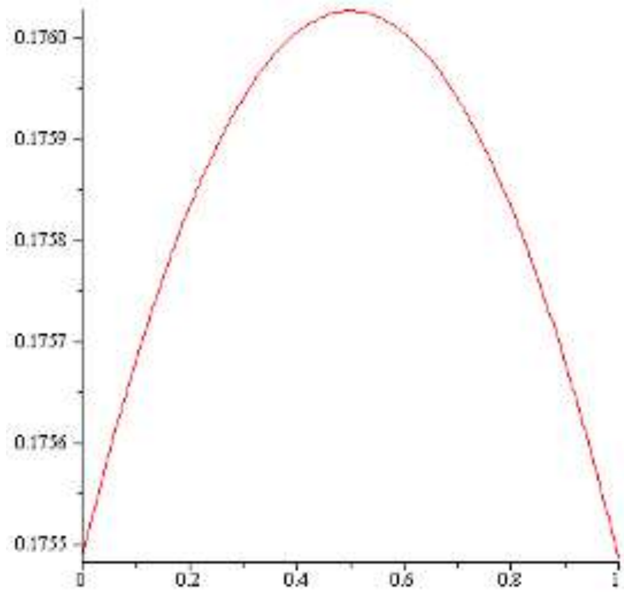

Fig. (a6). R'(σ, t=14.134725.., n=10)

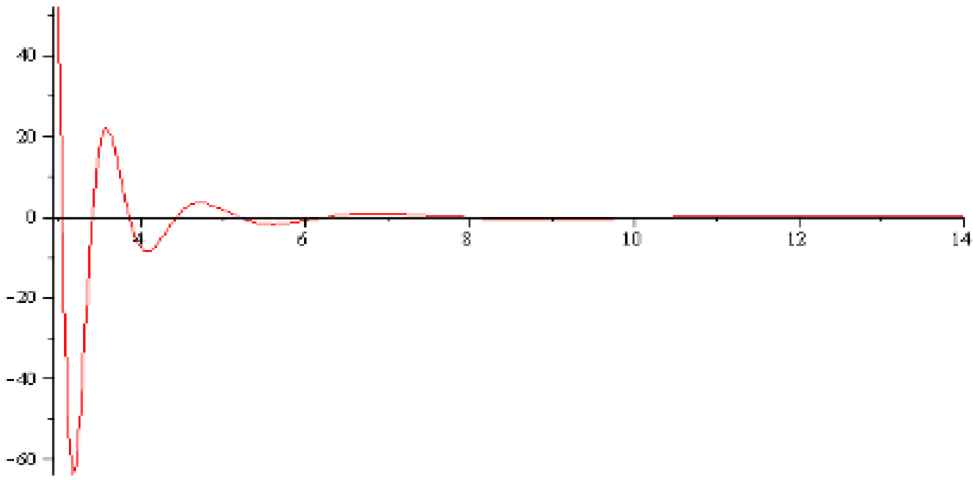

Fig. (a7). R'(σ =0, n=100, t) in the range 3..14 to illustrate the on set of negative values of R' at "large" n.

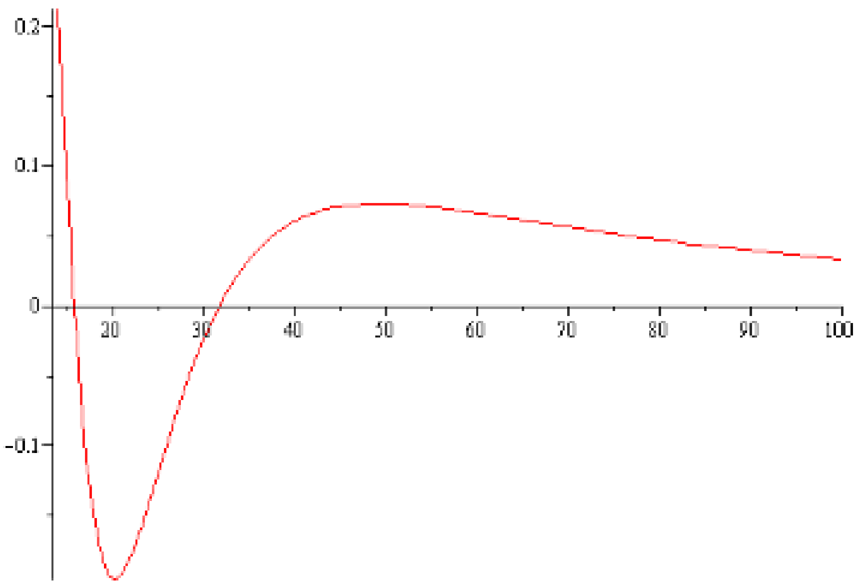

Fig. (a8). R'(σ =0, n=100, t) in the range [14..100]of t, to illustrate the onset of negative values of R' at "large" n.

We also add the analysis for n=3 as follows: we study

$\varphi_3 = (1/3)\cdot(\lambda_3 + (3/2)\cdot \lambda_1\cdot\lambda_2 + (1/2)\cdot\lambda_1^3) \geq 3\cdot \lambda_1$

i.e.

$\lambda_3 \geq 9\cdot\lambda_1 - (3/2)\cdot \lambda_1\cdot\lambda_2 - (1/2)\cdot\lambda_1^3$

The right hand side with λ_1 = 0.0230957089661 and with λ_2 = 0.0923457352280 [10] gives 0.20465603289 smaller than λ_3= 0.207638920554 [10]. We have thus verified that $\varphi_3 \geq 3\cdot \lambda_1$.

In conclusion, the relation between spin models and truncations of the ξ function offered by Eq.(19) for general n as a "lower bound" and here, without assuming RH is true, the proof has been given for n=2 and verified for n=3. We think, it would be difficult to disprove such "stability bound" given by Eq.(18) and connected with theorems concerning models of Statistical Mechanics.

Danilo Merlini. " A Possible Non Negative Lower Bound on the Li-Keiper Coefficients (A high temperature limit for the Riemann ξ Function)." IOSR Journal of Mathematics (IOSR-JM) 15.6 (2019): 01-16.