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The embodiment of analogy and transformation in complex analysis

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Abstract –*Analogy is an effective thinking method to explore problems, solve problems and discover problems, and in mathematics, it is an important means to discover concepts, methods, theorems and formulas. Transformation is an important method to explore and prove new proposition and to make mathematical creations. Pursuing analogy and transformation in depth is very beneficial to the study of mathematics. Complex analysis is based on real analysis, so, many concepts, properties, definitions and methods for dealing with problems are similar. Based on the complex functions, we discuss the embodiment of analogy and transformation in complex analysis from several aspects, in this paper.*

Keywords: *analogy, transformation, complex variable functions.*

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I. Introduction

Mathematical thought is people's rational understanding and basic views on the regularity of mathematical knowledge and methods, which refers to the spatial form and quantitative relationship of the real world reflected in human consciousness and the result generated after thinking activities. It is the essential understanding of mathematical facts and theories[1]. As an important mathematical thought, analogical thought has been paid more and more attention by educators[2]. The so-called analogy is a form of inference that some identical or similar properties of two objects can be inferred to be identical or similar in other properties[3]. As early as in the 1930s, American mathematician G Polya reappeared the origin and process of some major discoveries, and proposed that the most basic method to put forward mathematical conjectures is analogy and induction through research and analysis of major mathematical conjectures in history [4]. The discovery of the Pythagoras theorem and the Right triangle altitude theorem, the establishment of polynomial theory are all successful applications of analogy. The basis of analogy method is the universal relation existing between objective reality and objects. In the study of mathematics, the idea of analogy is embodied in the comprehensive use of logical deduction, calculation and other methods, reasoning according to the nature of things. This analogy requires that two objects are not always mathematical isomorphism, such as linear and non-linear. The idea of transformation of mathematics is to transform the problem to be solved from one form to another to make it easier to solve. It is a commonly used method to transform unfamiliar and complex mathematical problems into known and simple mathematical problems in mathematics learning. In the seventeenth century, the French mathematician Rene Descartes created the theory of analytic geometry which used the coordinate system to transform an algebraic equation with two unknowns into a curve on a plane to solve a geometric problem algebraically. Many unknown mathematical problems are solved by using transformation ideas to simplify complex problems, familiarize unfamiliar problems, and concretize abstract problems, so as to make full use of existing knowledge, experience and methods to solve problems [5, 6].

In complex analysis, the use of analogy and transformation thought is inseparable, and analogy mainly reflected in formation of important concept and construction of knowledge system. The establishment and perfection of complex function based on real function, which is the expansion of real number field to complex number field, and therefore, many of its concepts and properties are similar to real functions. But some of the properties of complex function are also obviously different from real functions. It can make us understand the difference between complex analysis and real analysis clearly in some theorems, which make us have a deeper understanding of complex analysis. This is helpful for us to construct the knowledge system of complex variable function, which also brings great convenience to the knowledge transfer of the system. Transformation is embodied in the solutions of the specific problem in complex analysis, and the main body of real analysis and complex analysis is function. The main line of analysis research is the consistent, which is variable, function, limit, derivative, series and integral, so some complex analysis and real analysis problems have similar research methods. Therefore, analogy and transformation ideas run through many parts of the concept formation and problem solving system of complex analysis.

II. Application of analogy and transformation in limit

The limit reflects the tendency property of variables in the process of change. Continuity, differentiability and derivability of functions are all based on the limit. For a real function $f(x)$ defined on D , A real number A is a limit of $f(x)$ as $x \to x_0$, if for any $\varepsilon > 0$, there exists positive number $\delta > 0$ such that for all x with $|x - x_0| < \delta$, $|f(x) - A| < \varepsilon$. It's analogous to the above concept of limit of real function, for a complex function $w=f(z)$ ($z \in E$), the following definition is obviously obtained. $\lim\limits_{z \to z_0} f(z) = w_0$. If for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all z with $|z - z_0| < \delta$, $|f(z) - w_0| < \varepsilon$, where z_0 is a limit point of E . We can also obtain the conclusion that the limit must be unique if it exists, by analogy with the limit of a real function of one variable.

However, we find that the limit of a complex variable function is much more demanding than that of a real function with one variable. As for the limit of real function $f(x)$ of one variable, x approaches x_0 only from the left and right directions along x axis. It is why complex analysis is complicated and different from real analysis. And by analogy with the limit of a real variable function and the limit of a complex variable function, we can get a better understanding of the limit of a complex function. However, to find the limit of complex variable function, we can transfer it into the limit of real part and imaginary part functions, the double limit theory in mathematical analysis can be applied to solve the limit.

For instance, the limit $\lim\limits_{z \to 0} \frac{1}{2i}\left(\frac{z}{\bar{z}} - \frac{\bar{z}}{z}\right)$ does not exist because

$$\lim_{z \to 0} \frac{1}{2i}\left(\frac{z}{\bar{z}} - \frac{\bar{z}}{z}\right) = \lim_{z \to 0} \frac{1}{2i}\left(\frac{z^2}{z\bar{z}} - \frac{\bar{z}^2}{z\bar{z}}\right) == \lim_{\substack{x \to 0 \\ y \to 0}} \frac{1}{2i}\left(\frac{x^2 + y^2 + 2xyi}{z\bar{z}} - \frac{x^2 + y^2 - 2xyi}{z\bar{z}}\right) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{2xy}{x^2 + y^2},$$

and the last real function with two variables is divergent.

Moreover, we can also apply L'hopital's rule to find the limit of a complex variable function, by analogy to the method of solving the limit of real function.

Example 1. Find the limit $\lim\limits_{z \to 0} \left(\frac{\sin z}{z}\right)^{\frac{1}{1-\cos z}}$.

Solution. Let $y = \left(\frac{\sin z}{z}\right)^{\frac{1}{1-\cos z}}$, then $\ln y = \frac{1}{1-\cos z} \ln\left(\frac{\sin z}{z}\right)$. And

$$\lim_{z \to 0} \ln y = \lim_{z \to 0} \frac{1}{1-\cos z} \ln\left(\frac{\sin z}{z}\right) = \lim_{z \to 0} \frac{2\ln\left(\frac{\sin z}{z}\right)}{z^2} = \lim_{z \to 0} \frac{\frac{z}{\sin z} \cdot \frac{z\cos z - \sin z}{z^2}}{z}$$

$$= \lim_{z \to 0} \frac{z\cos z - \sin z}{z^2 \cos z} = \lim_{z \to 0} \frac{z\cos z - \sin z}{z^3} = \lim_{z \to 0} \frac{-z\sin z}{3z^2} = -\frac{1}{3} \quad .$$

Therefore, we have $\lim\limits_{z \to 0} \left(\frac{\sin z}{z}\right)^{\frac{1}{1-\cos z}} = e^{\frac{-1}{3}}$.

III. Iii. Application Of Analogy And Transformation In Complex Integral

Complex integral is the core of complex analysis theory. In general, by analogy with the definition of line integral, we can understand the complex integral. In real variable analysis, vector function $f(x,y)=(p, q)$ be defined on the simple curve of the two-dimensional plane, where $p=p(x,y)$ and $q=q(x,y)$. The line integral is defined as: $\int_\tau f(x,y) \cdot dr = \lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} f(\xi_k) \Delta z_k$. The line integral is the integral against the coordinate, which has two dimensions. In topology, a two-dimensional plane is the same as a complex plane. The line integral involves two real functions of two variables, and we mentioned earlier that the study of complex functions can be transformed into the study of two real functions of two variables. Therefore, by analogy to the line integral, in the complex variable function $f(x,y)= u(x,y)+\mathrm{i}v(x,y)$ is defined on the simple curve of the complex plane, where $u=u(x,y)$, $v=v(x,y)$, then the complex integral is defined as $I = \int_\tau f(z) \cdot dz = \int_\tau udx - vdy + \mathrm{i}\int_\tau vdx - udy$. The complex integral can be transformed into a sum of two line integrals of $I_1 = \int_\tau udx - vdy$ and $I_2 = \int_\tau vdx - udy$. For example, let C be the parabola $y = x^2 (0 \le x \le 1)$), then we can calculate that

$$\int_C (xy+y\mathrm{i})dz=\int_0^1\left[x\cdot x^2-x^2\cdot 2x\right]dx+\mathrm{i}\int_0^1\left[x\cdot x^2\cdot 2x+x^2\right]dx=-\int_0^1 x^3dx+\mathrm{i}\int_0^1\left[2x^4+x^2\right]dx=\frac{-1}{4}+\frac{11}{15}.$$

For complex line integrals, Green's formula can be used to convert them into double integrals. With the transformation idea, we can apply Green's formula and other methods of calculating curvilinear integrals (such as parametric equation method) to complex integrals, which opens up ideas for solving the integration problem of complex variable function.

Example 2. Let *C* be a simple closed curve that can be measured, and *A* the area enclosed by the curve *C*. Prove that $\oint_C \bar{z}dz=2\mathrm{i}A$.

Proof. Let $z=x+iy$, then $\oint_C \bar{z}dz=2\mathrm{i}A$. $\oint_C \bar{z}dz=\oint_C xdx+ydy+\mathrm{i}\oint_C xdy-ydx$. By Green's formula, we have that $\oint_C xdx+ydy=0$, $\oint_C xdy-ydx=2A$.

Transformation thought is reflected in the number of isolated singularity too complicated, since

$$\sum_{k=1}^{n}\mathrm{Res}\left[f(z),a_k\right]+\mathrm{Res}\left[f(z),\infty\right]=0,$$

we can convert the calculation of the residue of high-order singularity into that of low-order or less singularity, and the calculation of the residue at the infinite point can be transformed into the residue at the origin through inversion, and then calculated by using the residue formula of finite point.

Example 3. Calculate $\oint_C \frac{1}{(z+\mathrm{i})^{10}(z-1)(z-3)}dz$, $C:|z|=2$.

Solution. Let $f(z)=\frac{1}{(z+\mathrm{i})^{10}(z-1)(z-3)}$. The three isolated singularities are –i, 1, 3. There are two isolated singularities are –i, 1 in *C*. since z= –i is a tenth order singularity, the calculation is tedious. And

$$\mathrm{Res}\left[f(z),3\right]=\lim_{Z\to 3}(z-3)f(z)=\frac{1}{2(3+\mathrm{i})^{10}},$$

$$\mathrm{Res}\left[f(z),\infty\right]=-\mathrm{Res}\left[f(\frac{1}{z})\frac{1}{z^2},0\right]=-\mathrm{Res}\left[\frac{z^{10}}{(\mathrm{i}z+1)^{10}(1-z)(1-3z)},0\right]=0,$$

so $\oint_C f(z)dz=2\pi\mathrm{i}\left\{\mathrm{Re}\,s\left[f(z),-\mathrm{i}\right]+\mathrm{Re}\,s\left[f(z),1\right]\right\}=-2\pi\mathrm{i}\left\{\mathrm{Re}\,s\left[f(z),3\right]+\mathrm{Re}\,s\left[f(z),\infty\right]\right\}=\frac{-\pi\mathrm{i}}{(3+\mathrm{i})^{10}}$.

IV. The application of analogy and transformation in complex series

Analytic functions are special contents of complex analysis and play an important role in the theory of complex variable functions. The real functions (real part and imaginary part) of analytic functions are not independent, which satisfy Cauchy-Riemann equation. Obviously, analytic functions are special contents of complex functions, and series is an important tool to study analytic functions. Analogous to the representation of real series, for a complex series as $\sum_{n=1}^{\infty}\alpha_n=\alpha_1+\alpha_2+...+\alpha_n+...$, according to $\alpha_n=a_n+\mathrm{i}b_n$ $(n=1,2,...)$, to discriminate the convergence and divergence of complex series, one has: complex infinite series $\sum_{n=1}^{\infty}\alpha_n$ converges to $s=a+\mathrm{i}b$ (a,b are real numbers) if and only if the real series $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ converge to a and b respectively.

Example 4. Investigate the convergence and divergence of complex series $\sum_{n=2}^{\infty}(\frac{n^{\ln n}}{(\ln n)^n}+\frac{n!}{n^{\sqrt{n}}}\mathrm{i})$.

Solution. Let $a_n=\frac{n^{\ln n}}{(\ln n)^n}=\frac{e^{(\ln n)^2}}{e^{nln(\ln n)}}=e^{-\left[nln(\ln n)-(\ln n)^2\right]}$, and $b_n=\frac{n!}{n^{\sqrt{n}}}$. Since $\lim_{n\to\infty}\frac{nln(\ln n)-(\ln n)^2}{n}=\infty$, there exist n_0 with $n\ge n_0$, $n\ln(\ln n)-(\ln n)^2\ge A_n$ (*A*>0). Therefore, $\frac{a_n}{\frac{1}{n^2}}=\frac{n^2}{e^{nln(\ln n)-(\ln n)^2}}\le\frac{n^2}{e^{A_n}}\to 0$, and $\sum_{n=1}^{\infty}a_n$ is convergence.

Moreover, Since $n! > \left(\frac{e}{n}\right)^n$, let $\frac{n!}{n^{\sqrt{n}}} > \frac{1}{e^n} n^{n-\sqrt{n}} = c_n$, then $\sqrt[n]{c_n} = \frac{1}{e} n^{1-\frac{1}{\sqrt{n}}} \to \infty \ (n \to \infty)$, and $\sum_{n=1}^{\infty} c_n$ is divergent. Therefore $\sum_{n=1}^{\infty} b_n$ is divergent, and so the complex series is divergent.

In real analysis, to determine the convergence and divergence of infinite series whose general formulas is product form, there is Abel Test[7]: if i) $\{a_n\}$ is a monotone bounded sequence, and ii) real series $\sum_{n=1}^{\infty} b_n$ is convergent, the series is convergent. When the analogy is extended to the complex number domain, the corresponding conditions should be changed: if i) complex series $\sum_{n=1}^{\infty} u_n$ is convergent, ii) $v_n = a_n + \mathrm{i} b_n$ is a complex sequence, where $\{a_n\}$ and $\{b_n\}$ are monotone bounded complex sequences, and $|a_n| \le M$, $|b_n| \le M \ (n = 1, 2, ...)$. Then complex series $\sum_{n=1}^{\infty} u_n v_n$ is convergent. It provides a new idea for us to judge the convergence and divergence of complex series.

For instance, consider the complex series $\sum_{n=1}^{\infty} \frac{(n+\mathrm{i})(3+5\mathrm{i})^n}{n \times n!}$. There is $\sum_{n=1}^{\infty} \frac{(n+\mathrm{i})(3+5\mathrm{i})^n}{n \times n!} = \sum_{n=1}^{\infty} \frac{(3+5\mathrm{i})^n}{n!} \frac{(n+\mathrm{i})}{n}$. It is able to judge that $\sum_{n=1}^{\infty} \frac{(3+5\mathrm{i})^n}{n!}$ is convergent by using D' Alembert's test, and $\left\{\frac{n+\mathrm{i}}{n}\right\}$ is a monotone bounded complex sequence. Hence $\sum_{n=1}^{\infty} \frac{(n+\mathrm{i})(3+5\mathrm{i})^n}{n \times n!}$ is convergent.

V. Conclusion

Complex analysis is a logical knowledge system. Every concept, theorem and formula in the complex analysis is developed according to a strict and orderly logical system. The complex analysis of the knowledge structure is holistic and systematic. Learning the complex analysis and mastering the logical relationship of each part of the complex analysis requires us to have a strong analogy and transformation consciousness. On the one hand, the theory of continuity, differential calculus, integral and series can be extended to functions of complex by applying analogy and transformation ideas, so as to help us build a knowledge system of derivative, analytic function, integral and series knowledge in complex analysis. On the other hand, it helps to find the difference between real analysis and complex analysis and prevent negative transfer. Therefore, analogy and transformation are inseparable in the learning of complex variable functions and run through the whole complex analysis learning system. Analogical is also one of the sources of mathematical knowledge discovery. By applying transformation, the known method theory can be applied to solve unknown problems, which can not only consolidate analytical theory, but also easily accept theories of complex variable functions. Using real integral to solve complex integral problems, complex integrals can be transformed into curvilinear integral; the conclusion of the real series is used to solve the questions about converge and expansion of the complex series; the concept of residue is given based on complex variable functions, but applying variable substitution or satisfying certain conditions can be used to calculate some real integrals and improper integrals Solving some forms of real and generalized integrals, which are the manifestations of analogy and transformation in complex variable functions [8].

When applying analogy and transformation in complex analysis, we can actively guide ourselves to conduct creative analysis and solve problems. When learning complex analysis, we can actively use analogy and transformation for reasoning, arguing, and gradually digest and understand complex analysis, which is beneficial to improving our exploration and learning ability. Using analogy and transformation to learn complex analysis, thinking from different directions and observing the same problem from multiple angles can exercise divergent thinking. In the process of analogy, we should pay attention to the strict demonstrations of various states of analogy, and make use of the advantage of analogy to transfer the knowledge between the two systems, so as to prove the rationality of the result of analogy [9]. When applying transformation ideas, we should also pay attention to that the transformation is not arbitrary, but targeted. We should mainly pay attention to the internal and connection between knowledge and problem-solving methods, turn unfamiliar problems into familiar ones, and turn complex problems into simple ones. Therefore, when applying analogy and transformation ideas, we should not only pay attention to the commonness, but also pay attention to the differences, pay attention to the new situations and problems, and discuss the reasons for the new problems.

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