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Some Properties on State Residuated Lattices

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Abstract *In the paper, we give characterizations of semi-states on residuated lattices, and discuss relations between Rl -morphism and semi-states on residuated lattices. Then we discuss the relations between maps defined by a frame. Finally, Using conanical frame on residuated lattices, we discuss the relations between state operator and the binary relation of Rl-morphisms.*

Keywords *residuated lattice · state operator · frame*

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I. Introduction

It is well known that non-classical logic has become a formal and useful tool for computer science to deal with uncertain information and fuzzy information. Var- ious logical algebras have been proposed as the semanti- cal systems of non-classical logic systems. Among these logical algebras, residuated lattices are very basic and important algebraic structures. Residuated lattices (cf. Ward M. (1939)) constitute the semantics of Höhle's MonoidalLogic (*ML*) (cf.Hohle U. 1995), which are the basis for the majority of formal fuzzy logics. Apart from their logical interest, residuated lattices have interesting algebraic properties and include two important classes of algebras: *BL*-algebras (introduced by Hâjek (1998) as the algebraic counterpart of his Basic Logic) and *MV*-algebras (introduced by Chang (1958) to prove the completeness theorem for Łukasiewicz calculus).

This paper is organized as follows: In Section 2, we review some basic definitions and results about residu- ated lattices, dyadic numbers and MV-terms and frame and Galois connections. In Section 3, we introduce the notion of semi-states on residuated lattices and inves- tigate some related properties of semi-states. Also, we characterize arbitrary meets of *Rl*-morphisms as semi- states. In Section 4, we focus on a conanical frame of state residuated lattice (L, τ).

II. Preliminaries

2.1 Residuated lattices and states

In this section, we summarize some definitions and re- sults about residuated lattices and lattices, which will be used in the following sections of the paper.

First, we recall some definitions and properties about residuated lattices.

Deftnition 2.1 (Zhou XN, Li QG, Wang GJ 2007)(Ad- joint pair) *Let P be a poset. Then the two binary operations $\otimes$ and $\rightarrow$ on P are adjoint each other with the following properties:*

(1) $\otimes : P \times P \rightarrow P$ *is isotone.*
(2) $\rightarrow: P \times P \rightarrow P$ *is antitone in the first variable and isotone in the second variable.*
(3) $x \otimes y \leq z$ *if and only if* $x \leq y \rightarrow z$,

for all $x,y,z \in P$. *And we call* $(\otimes,\rightarrow)$ *an adjoint pair on P.*

Deftnition 2.2(Zhou XN, Li QG, Wang GJ 2007)(Resid- uated lattice) *A structure* $(L; \leq, \otimes, \rightarrow, 0, 1)$ *is called a* residuated lattice *if the following conditions are satis- fied:*

(1) $(L,\leq)$ *is a bounded lattice,* 0 *is the smallest element and* 1 *is the greatest of* L, *respectively.*
(2) $(L, \otimes, 1)$ *is a commutative monoid;*

(3) $(\otimes, \to, 1)$ *is anjoint pair on L.*
Morphisms of residuated lattices(shortly *RL*-morphisms) are defined as usual, ie., they are functions which preserve the binary operations $\wedge$, $\vee$, $\otimes$ and $\to$, the unary operation $\neg$ and the constants 0 and 1.

Deftnition 2.3 (Pei DW, 2004)(Involution residu- ated lattice) *Let* $L = (L; \leq, \otimes, \to, 0, 1)$ *be a residuated lattice. Define on L the unary operation,* $\neg : L \to L$, *such that* $\neg x = x \to 0$, *for each* $x \in L$. *The we call L an involution residuated lattice if* $\neg\neg x = x$, *for each* $x \in L$.

Proposition 2.4 (Hajek, 1998; Turunen, 1999) *In any residuated lattice* $(L, \wedge, \vee, \otimes, \to, 0, 1)$, *the following properties hold:*
(1) $1 \to x = x$, $x \to 1 = 1$,
(2) $x \leq y$ *if and only if* $x \to y = 1$,
(3) $x \otimes \neg x = 0$, $x \otimes y = 0$ *if and only if* $x \leq \neg y$,
(4) *If* $x \leq y$, *then* $y \to z \leq x \to z$, $z \to x \leq z \to y$ *and*
$x \otimes z \leq y \otimes z$, *(5)* $x \otimes (x \to y) \leq y$,
(6) $x \otimes y \leq x \wedge y$, $x \leq y \to x$,
(7) $x \to (y \to z) = (x \otimes y) \to z = y \to (x \to z)$, *(8)* $\neg 0 = 1$, $\neg 1 = 0$, $x \leq \neg\neg x$, $\neg\neg\neg x = \neg x$,
(9) $x \otimes (y \to z) \leq y \to (x \otimes z) \leq (x \otimes y) \to (x \otimes z)$,
(10) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$,
(11) $x \vee (y \otimes z) \geq (x \vee y) \otimes (x \vee z)$, *hence* $x \vee y^n \geq (x \vee y)^n$ *and* $x^m \vee y^n \geq (x \vee y)^{mn}$ *for any natural numbers m, n,*
(12) $x \to (x \wedge y) = x \to y$,
(13) $x \otimes y = x \otimes (x \to x \otimes y)$,
(14) $x \leq (y \to x \otimes y)$.
for any $x, y, z \in L$.

Proposition 2.5(Pei DW, 2004) *Let* $(L, \wedge, \vee, \otimes, \to, 0, 1)$ *be a involution residuated lattice, then the following conditions hold:*
(1) $a \to \neg b = b \to \neg a$, $\neg a \to b = \neg b \to a$; *(2)* $a \otimes b = \neg(a \to \neg b)$;
(3) $a \to b = \neg(a \otimes \neg b)$;
(4) $\neg a \to (a \to b) = 1$;
(5) $\neg \wedge_{i \in I} a_i = \vee_{i \in I} \neg a_i$; *where* $a, b, a_i \in L (\forall i \in I)$.
Next, we recall the notions of states on residuated lattices. For more details about these concepts, we refer the readers to (Ciungu, 2008).

In a residuated lattice L, we say that two elements $x, y \in L$ are said to be *orthogonal* and we write $x \perp y$, if $\neg\neg x \leq \neg y$. It is easy to check that $x \perp y$ iff $x \leq \neg y$

and iff $x \otimes y = 0$. It is clear that $x \perp y$ iff $y \perp x$, and $x \perp 0$ for each $x \in L$.
For two orthogonal elements x,y in L, we define the partial addition $x + y := \neg y \to \neg\neg x (= \neg x \to \neg\neg y)$.

Deftnition 2.7 (Ciungu L.C., 2008) *Let* $(L, \wedge, \vee, \otimes, \to, 0, 1)$ *be a residuated lattice. A* Bosbach state *on L is a function* $s : L \longrightarrow [0, 1]$ *such that the following con- ditions hold:*
(1) $s(0) = 0$, $s(1) = 1$,
(2) $s(x) + s(x \to y) = s(y) + s(y \to x)$ *for all* $x, y \in L$.

Deftnition 2.8 (Ciungu L.C. 2008) *Let* $(L, \wedge, \vee, \otimes, \to, 0, 1)$ *be a residuated lattice. A* Riečan state *on L is a function* $s : L \longrightarrow [0, 1]$ *such that the following condi- tions hold:*
(1) $s(1) = 1$,
(2) $s(x + y) = s(x) + s(y)$ *whenever* $x \perp y$.

Deftnition 2.9(He P.F., 2015) *Let* $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ *be a residuated lattice. A mapping* $\tau : L \rightarrow L$ *is called a* state operator *on L if it satisfies the following conditions:*
(L1) $\tau(0) = 0$;
(L2) $x \rightarrow y = 1$ *implies* $\tau(x) \rightarrow \tau(y) = 1$; (L3) $\tau(x \rightarrow y) = \tau(x) \rightarrow \tau(x \wedge y)$;
(L4) $\tau(x \otimes y) = \tau(x) \otimes \tau(x \rightarrow (x \otimes y))$;
(L5) $\tau(\tau(x) \otimes \tau(y)) = \tau(x) \otimes \tau(y)$;
(L6) $\tau(\tau(x) \rightarrow \tau(y)) = \tau(x) \rightarrow \tau(y)$;
(L7) $\tau(\tau(x) \vee \tau(y)) = \tau(x) \vee \tau(y)$;
(L8) $\tau(\tau(x) \wedge \tau(y)) = \tau(x) \wedge \tau(y)$, *for any* $x, y \in L$.
The pair (L, τ) *is said to be a* state residuated lat- tice, *or more precisely, a residuated lattice with inter- nal state. We say that a state operator* τ *is contractive (transitive) if* $\tau(x) \leq x (\tau(x) \leq \tau(\tau(x)))$ *for all* $x \in L$. *A state operator* τ *that is both contractive and transi- tive is called a conucleus.*

2.2 Dyadic numbers and MV-terms

The content of this part summarizes the basic results about certain $MV-$ terms form (Teheux B. 2009) in the setting of residuated lattices.

The set $\mathbb{D}$ of dyadic numbers is the set of the rational numbers that can be written as a finite sum of powers of 2. We denote by $f_0(x)$ and $f_1(x)$ the terms $x \rightarrow x$ and $x \otimes x$ respectively, and by $T_{\mathbb{D}}$ the clone generated by $f_0(x)$ and $f_1(x)$.

An example of residuated lattice is the real unit interval $L = [0, 1]$ equipped with the operations. For all

$$x, y \in L, x \otimes y = min\{x, y\} \text{ and } x \rightarrow y = \begin{cases} 1, & x \leq y \\ y, & otherwise \end{cases}.$$

We refer to it as the Gödel structure.

Corollary 2.10(Teheux B. 2009)*Let L be the Gödel structure,* $x \in [0, 1]$ *and* $r \in (0, 1) \cap \mathrm{D}$. *Then there is a term* t_r *in* T_{D} *such that* $t_r(x) = 1$ *if and only if* $r \leq x$.

Lemma 2.11 *Let L be a linearly ordered residuated lattice,* $s : L \longrightarrow [0, 1]$ *be a RL− morphism,* $x \in L$ *such that* $s(x) = 0$. *Then* $x \otimes x = 0$.

Proof Assume that $x \geq \neg x$. Then $0 = \neg s(x) \otimes s(x) = s(\neg x \otimes x) \geq s(\neg x \otimes \neg x) = \neg s(x) \otimes \neg s(x) = 1$ which is absurd. Therefore $x < \neg x$, then $x \otimes x = 0$.

Proposition 2.12 *Let L be a linearly ordered resid- uated lattice,* $s : L \longrightarrow [0, 1]$ *be a RL− morphism,* $x \in L$. *Then* $s(x) = 1$ *if and only if* $t_r(x) = 0$ *for all* $r \in (0, 1) \cap \mathrm{D}$.

Proof Assume that $x f= 0$ since $s(1) = 1$ and $t_r(1) = 1$for all $r \in (0, 1) \cap \mathrm{D}$. Note that $s(t_r(x)) = t_r(s(x))$ since s is an $RL-$ morphism. Then $s(x) = 0$ iff $r > s(x)$ for all $r \in (0, 1) \cap \mathrm{D}$ iff $t_r(s(x)) = 0$ for all $r \in (0, 1) \cap \mathrm{D}$ if $s(t_r(x)) = 0$ for all $r \in (0, 1) \cap \mathrm{D}$.

Now suppose that $t_r(x) = 0$ for all $r \in (0, 1) \cap \mathrm{D}$. Then $s(t_r(0)) = 0$ for all $r \in (0, 1) \cap \mathrm{D}$ and by above considerations we have that $s(x) = 0$.

Conversely, let $s(x) = 0$ and $r \in (0, 1) \cap \mathrm{D}$. Then $t_r(x) = t(x) \otimes t(x)$ such that $t(x)$ is some term from clone T_{D} constructed entirely from the operations $(-) \otimes (-)$ and $(-) \rightarrow (-)$. Therefore $s(t(x)) = t(s(x)) = t(0) = 0$. By Lemma 2.11 we get $t_r(x) = t(x) \otimes t(x) = 0$.

2.3 Frame and Galois connections

In what follows, we review some notions and results about frames which will be necessary in the following.

By a frame it is meant a triple (S, T, R) where S, T are non-void sets and $R \subseteq S \times T$. If $S = T$, we will write briefly (T, R) for the frame (T, T, R) and we say that (T, R) is a time frame. The relation sRt expresses a relationship "s to be before t" and "t to be after s". Having an involution residuated lattice $(L, \leq, \otimes, \rightarrow, 0, 1)$ and a non-void set T, we can produce the direct power L^T, ie. the base set of L^T is the set of all functions from T to L and $\otimes$, $\neg$ are defined pointwise.
Let $(A, \leq)$ and $(B, \leq)$ be two ordered sets. A map- ping $f : A \rightarrow B$ is called residuated if there exists a mapping $g : B \rightarrow A$ such that $f(a) \leq b$ if and only

if $a \leq g(b)$ for all $a \in A$, $b \in B$. In this situation, we say that f and g form a residuated pair or that the pair (f, g) is called a (monotone)*Galois connection*. In a Galois connection (f, g), f is called the *left adjoint* of g or a lower adjoint of g. Dually, g is called the *right adjoint* of f or an upper adjoint of f , see (cf. Gierz G. (2003)).
We note in particular that if f has a right adjoint then its right adjoint is unique. Similarly, if g has a left adjoint then its left adjoint is unique.

Lemma 2.13 *Let $(A; \leq)$ and $(B; \leq)$ be ordered sets. Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be mappings. The following conditions are equivalent:*
(1) *(f, g) is a Galois connection.*
(2) *f and g are monotone, $id_A \leq g \circ f$ and $f \circ g \leq id_A$.*
(3) *$g(b) = sup\{x \in A \mid f(x) \leq b\}$ and $f(a) = inf\{y \in B \mid a \leq g(y)\}$ for all $a \in A$ and $b \in B$.*
In the above case, g is determined uniquely by f and, similarly, f is determined uniquely by g. Moreover, f preserves all existing joins in $(A; \leq)$ and g preserves all existing joins in $(B; \leq)$. If, in addition, both $(A; \leq)$ and $(B; \leq)$ are complete ordered sets we have the converse, i.e. if f preserves all joins in $(A; \leq)$ then f has an upper adjoint g given by the condition $g(b) = sup\{x \in A \mid f(x) \leq b\}$, for all $b \in B$. Similarly, if g preserves all meets in $(A; \leq)$ then g has a lower adjoint f given by the condition $f(a) = inf\{y \in B \mid a \leq g(y)\}$, for all $b \in B$.

III. Semi-states on residuated lattices

Deftnition 3.1 *Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a residuated lattice. A map $s : L \longrightarrow [0, 1]$ is called*
(1) *a* semi-state *on L if (i)* $s(1) = 1$,
(ii) *$x \leq y$ implies $s(x) \leq s(y)$,*
(iii) $s(x) = 1$ *and* $s(y) = 1$ *implies* $s(x \otimes y) = 1$,
(iv) $s(x) \otimes s(x) = s(x \otimes x)$,
(v) $s(x) \rightarrow s(x) = s(x \rightarrow x)$.
(2) *a strong* semi-state *on L if it is a semi-state such that*
(vi) $s(x) \otimes s(y) \leq s(x \otimes y)$,
(vii) $s(x) \rightarrow s(y) \leq s(x \rightarrow y)$,
(viii) $s(x \wedge y) = s(x) \wedge s(y)$,
(ix) $s(x^n) = s(x)^n$ *for all $n \in$* N, *wherever* $x^n = x^{n-}1 \otimes x$ *for* $n \geq 1$.
Note that any RL-morphism into a unit interval is a strong semi-state.

Lemma 3.2 *Let L be a residuated lattice, $s : L \longrightarrow [0,1]$ is a semi-state on L. Then for all $x, y \in L$,* $s(x) =$

1 *and* $s(y) = 1$ *implies there is* $z \le x$ *and* $z \le y$ *such that* $s(z) = 1$.

Proof Let $x, y \in L$ such that $s(x) = 1$ and $s(y) = 1$. Since $x, y \le 1$, so from Proposition 2.4(6) we get that $z = x \otimes y \le x, y$, by Definition 3.1(iii) we have $s(z) = 1$.

Proposition 3.3 *Let L be a residuated lattice, S a non-empty set of semi-states(strong semi-states) on L. Then the point-wise meet* $t = \bigwedge S : L \to [0, 1]$ *is a semi-states(strong semi-states) on L.*

Proof The proof is a straightforward checking of condi- tions $(i) - (v)((i) - (ix))$.

Proposition 3.4 *Let L be a residuated lattice, s, t be semi-states on L. Then* $t \le s$ *iff* $t(x) = 1$ *implies* $s(x) = 1$ *for all* $x \in L$.

Proof Assume that $t(x) = 1$ implies $s(x) = 1$ for all $x \in L$ is valid and that there is $y \in L$ such that $s(y) < t(y)$. Thus there is a dyadic number $r \in (0, 1) \cap \mathrm{D}$ such that $s(y) < r < t(y)$. By Corollary 2.10, there is a term t_r in T_{D} such that $t_r(s(y)) < 1$ and $t_r(t(y)) = 1$. It follows that $s(t_r(y)) = t_r(s(y)) < 1$ and $t(t_r(y)) = t_r(t(y)) = 1$. So by the last condition, we have $s(t_r(y)) = 1$, this contradict with $s(t_r(y)) < 1$. Therefore $t \le s$.

Deftnition 3.5(Botur M. 2015). Let P, Q be bounded posets and let S be a set of order-preserving maps from P to Q. Then

(i) S is called order determining if $((\forall s \in S) s(a) \le s(b)) \Longrightarrow a \le b$ for any elements $a, b \in P$;

(ii) S is called strongly order determining if $((\forall s \in S) s(a) = 1 \Longrightarrow s(b) = 1) \Longrightarrow a \le b$ for any elements $a, b \in P$.

Proposition 3.6 *Let L be a residuated lattice, S be a semi-state on L and* $S^{\mathrm{D}} = \{s \circ t_r \mid s \in S, r \in (0,1) \cap \mathrm{D}\}$. *Then the following conditions are equivalent.*

(1) S is strongly order determining,

(2) $((\forall s \in S, r \in (0,1) \cap \mathrm{D}) s(a) \ge r \Longrightarrow s(b) \ge r) \Longrightarrow a \le b$ *for any elements* $a, b \in L$,

(3) S is order determining.

Proof (1) $\Longrightarrow$ (2) Suppose that S^{D} is strongly order determining, so for all $s \in S$, $r \in (0, 1) \cap \mathrm{D}$, $t_r(s(a)) = s(t_r(a)) = 1 \Longrightarrow t_r(s(b)) = s(t_r(b)) = 1$, it follows that $t_r(a) \le t_r(b)$. And by Corollary 2.10, it follows that $(s(a) \ge r \Longrightarrow s(b) \ge r) \Longrightarrow a \le b$.

(2) $\Longrightarrow$ (3) Suppose that (2) holds. Choose $a, b \in L$ such that, for all $s \in S$, $s(a) \le s(b)$. Let $r \in (0, 1) \cap \mathrm{D}$, $s(a) \ge r$. Then $s(b) \ge r$. This yields by (2) that $a \le b$.

(3) $\Longrightarrow$ (1) Suppose that S is order determining and that there are $a, b \in L$, $a \not\le b$ such that for all $s \in S$, $r \in (0, 1) \cap \mathrm{D}$, $s(t_r(a)) = 1 \Longrightarrow s(t_r(b)) = 1$. Since S is order determining, so that there is $t \in S$ such that $t(a) > t(b)$. Thus there is a dyadic number $r \in (0, 1) \cap \mathrm{D}$ such that $t(b) < r < t(a)$. By Corollary 2.10, there is a term t_r in T_D such that $t_r(t(a)) = t(t_r(a)) = 1$ and $t_r(t(b)) < 1$. By $t, s \in S$ and $t(t_r(a)) = 1$, we have $1 = t(t_r(b)) = t_r(t(b)) < 1$. This a contradiction. Thus $a \le b$.

Proposition 3.7 *Let L be a residuated lattice,* $s : L \to [0,1]$ *a RL-morphism on L. If* $s(0) = 0$, *then s is a Rieč state.*

Proof If $s(0) = 0$, and s is a RL-morphism, then $s(1) = s(0 \to x) = s(0) \to s(x) = 0 \to s(x) = 1$. $s(x + y) = s(\neg x \to \neg\neg y) = s(\neg x) \to s(\neg\neg y) = \neg s(x) \to \neg\neg s(y) = s(x) + s(y)$.

IV. The representation of state residuated lattice

In this section, we introduce the frame of state opera- tors in a state residuated lattice and investigate some related properties of such frame. Also, we give the rep- resentation theorem of state residuated lattice.

Deftnition 4.1 *Let* (L, τ) *be a state residuated lat- tice, and denote by T the set of all RL− morphisms from L to the Gödel structure. We define a frame* (T, R_τ) *by* $sR_\tau t$ *if and only if* $s(\tau(x)) \le t(x)$ *for all* $x \in L$, *We call* (T, R_τ) *a conanical frame.*

Lemma 4.2 *Let (L, τ) be a state residuated lattice, $s, t \in T$. Then $sR_\tau t$ if and only if $s(\tau(x)) = 1 \Longrightarrow t(x) = 1$ for all $x \in L$.*

Proof Assume first $sR_\tau t$. Then $1 = s(\tau(x)) \leq t(x)$. Now, assume that $s(\tau(x)) = 1 \Longrightarrow t(x) = 1$ forall $x \in L$ and there is $x \in L$ such that $t(x) < s(\tau(x))$. It follows that there is a dyadic number $r \in (0, 1) \cap D$ such that $t(x) < r < s(\tau(x))$. By Corollary 2.10, we obtain that $t(t_r(x)) = t_r(t(x)) < 1 = t_r(s(\tau(x))) = s(t_r(\tau(x))) \leq s(\tau(t_r(x)))$. Therefore $s(\tau(t_r(x))) = 1$ yields $t(t_r(x)) =$ 1, a contradiction.

Lemma 4.3 *Let (L, τ) be a state residuated lattice. Then*

(i) *If τ is contractive then R_τ is reflexive; (ii)If τ is transitive then R_τ is transitive.*

Proof (i) If $\tau(x) \leq x$, then $s(\tau(x)) \leq s(x)$ for all $x \in L$ and all $s \in T$. Hence $sR_\tau s$.

(ii) Let $s, t, u \in T$, $sR_\tau t$ and $tR_\tau u$. Let $x \in L$, then $s(\tau(x)) \leq s(\tau(\tau(x)) \leq t(\tau(x)) \leq u(x)$, hence $sR_\tau u$.

Theorem 4.4 Let $(L1, \tau 1)$ and $(L2, \tau 2)$ be state residuated lattices, $f : L1 \to L2$ and $g : L2 \to L1$ be mappings such that (f, g) is a Galois $\tau-$connection. Let T, S be sets of all order determining semi-states on L_2 and L_1, respectively.

Further, let (S, T, R_g) be a frame such that the relation $R_g \subseteq S \times T$ is defined by

$sR_g t$ if and only if $s(g(x)) \leq t(x)$ from and $x \in L_2$.

Then g is order determining semi-state via the canonical Galois connection (P^*, G^*) between complete residuated lattices G^S and G^T induced by the frame (S, T, R_g) and the Gödel structure G, $G^* : [0, 1]^T \to [0, 1]^S$.

Proof Assume that $x \in L_2$ and $s \in S$. Then $i^S_{L_1}(g(x))(s) = s(g(x)) \leq t(x)$ for all $t \in T$ such that $(s, t) \in R_g$. It follows that $i^S_{L_1}(g(x)) \leq G^*(i^S_{L_1}(x))$. Note that $s \circ g$ is a semi-state on L_1. Clearly, $s \circ g(1) = s(g(1)) = s(1) = 1$, $s \circ g$ is order determining semi-state, so by Lemma 3.2, assume that $s(g(x)) = s(g(y))$. Then there is an element $z \in L_1$, $z \leq g(x)$, $z \leq g(y)$. Since g is a right adjoint to the map $f : L_1 \to L_2$, then $f(z) \leq x$, $f(z) \leq y$. It follows that $s(f(z)) = 1$. By Proposition 3.3 we get that

$s \circ g = \bigwedge\{t : L_2 \to [0, 1] \mid t \ is \ a \ semi-state, t \geq s \circ g\}$
$= \bigwedge\{t \in T \mid (s, t) \in R_g\}$. This means that $i^S_{L_1}(g(x)) = G^*(i^S_{L_1}(x))$.

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