IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 6 Ser. I (Nov – Dec 2019), PP 01-03
www.iosrjournals.org

Completeness and Compactness of Function Spaces on Uniform Groups

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Abstract: *This paper deals with a necessary and sufficient condition for the completeness of a uniform group of continuous functions from a topological spaces to a Housdorff complete uniform group. We also defined on the completion of abelian or non- abelianHousdorff uniform group of functions.*
Key-words: *Topological space, Housdorff space, Uniform group, uniformity*

Date of Submission: 17-10-2019 Date of acceptance: 01-11-2019

I. Introduction

Compactness is phrased in terms of the existence of finite families of open sets that “cover” the space in the sense that each point of the space lies in some set contained in the family. This more subtle notion, introduced by PalvenAlexandrov and PavelUrysohn in 1929 exhibits compact spaces as generalization of finite sets.

Completeness the state of being complete and entire; having everything that is neededcompleteness an attribute of a logical system that is so constituted that a contradiction arises if any proposition is introduced that cannot be derived from the axious of the system.

Definition:A uniform group is a group with a uniform structure having a translation invariant base for the uniformity of the uniform structure.

II. Mathematical Approach:

Theorem: 1

Let E be a set and S a family of subsets of E satisfying the conditions:

(i) A, B $\in$ S $\Rightarrow \exists$ C $\in$ S Such that $A \cup B \subseteq C$

(ii) U A = E
A $\in$ S

Let F be an (additive) uniform group. Let m (E, F) be the set of all maps of E into F then the following results hold:

(a) There exists an (additive) group structure on m (E,F) and a translation- invariant base for a unique uniformity J_s for m (E,F) such that m (E, F) is an (additive) uniform group.

(b) m (E,F) is a Hansdorff if F is Hansdorff.

Proof: We first note by usual definition of addition of map given by (f + g) (x) = f(x) + g(x) for all x $\in$ E the set m(E, F) can be given an (additive) group structure. If o be identity element of F then the map $f_\circ(x)$ = x for all x $\in$ E serves as the identity element of m (E, F).

Let μ be the uniformity for F and B be the translation invariant base of μ for each A $\in$ and U $\in$ B.

We define,

M (A, U) = {(f, g) : f(x), g(x)} $\in$ U for all x $\in$ A }.

We claim that the family {M(A, U)} A $\in$ S forms a base for a U $\in$ B uniformity $J_s$for m (E, F) such that M (A,U) is translation invariant in the (additive) group m (E, F). To verify our claim, we first prove that Δ(m(E,F)) = { (f, f): f $\in$ m (E, F) } $\subseteq$ M(A, U) for all A $\in$ S and U $\in$ B. Let M (B, V) is arbitrary in {M (A, U)} .Let (g, g) be any element of Δ(m (E, f)) then (g (x), g(x)) $\longleftarrow \Delta$ F for all x $\in$ E $\Rightarrow$ (g(x) , g(x)) $\in$ V for all x $\in$ B, since Δ (F) $\subseteq$V and B $\subseteq$ E $\Rightarrow$ (g, g) $\in$ M(B, V) and also M (B, V) is arbitrary . Therefore, Δ (m (E, F)) $\subseteq$ M (A, U) for all A $\in$ S and U $\in$ B

Next, since U $\in$ B so $\exists$ V $\in$ β such that $V \in U^{-1}$. Let M (A, U) $\in$ {M(A, U)} we claim that there exists
M (A, V) $\in$ { M (A, U) } such that

$M(A, V) \subseteq M(A, U)^{-1}$

Let $(f, g) \in M(A, V) \Rightarrow (f(x), g(x)) \in V$ for all $x \in V$.

$\Rightarrow (f(x), g(x)) \in 1\, U^{-1} x \in A$

$\Rightarrow (f, g) \in M\left(A, U^{-1}\right)$

Hence $M(A, V) \subseteq M\left(A, U^{-1}\right) = M(A, U)^{-1}$

Next, Let M(B, V) be any member of the family { M (A, U)}. Then since $V \in B$, there exists $W \in B$ such that $m^2 \subseteq v$. Now we claim that $M(B, W)^2 \subseteq M(B, V)$.

To verify this, let $(f, g) \in M(B, V)^2$. Then there exists $h \in m(E, F)$ such that $(f, g) \in M(B, W)$ and $(h, g) \in M(B,W)$.

This implies that $(f(x), h(x)) \in W$ and $(h(x), g(x)) \in W$ for all $x \in B$.

$\Rightarrow (f(x), g(x)) \in \mu^2$ for all $x \in B$

$\Rightarrow (f(x), g(x)) \in V$ for all $x \in B$

$\Rightarrow (f, g) \in M(B, V)$

This verifies our claim that $M(B, W)^2 \subseteq M(B, V)$. Next, M (A, U) and M (B, V) be any two members of the family {M (A, U) } for $A, B \in S$ and $U, V \in B$. Since $A, B \in S$ there exists $C \in S$ such that $A \cup B \subseteq C$. Since $U, V \in B$ there exists $W \in B$ such that $W \subseteq U \cap V$. Now we claim that $M(C, W) \subseteq M(A,U) \cap M(B, V)$.

To verify this, Let $(f, g) \in M(C, W) \Rightarrow (f(x), g(x) \in W)$ for all $x \in C$; $\Rightarrow (f(x), g(x) \in U$ for all $x \in A$ and $(f(x), g(x)) \in V$ for all $x \in B$; $\Rightarrow (f, g) \in M(A, U) \cap M(B, V)$. This implies that $M(C, W) \subseteq M(A, U) \cap M(B, V)$. Therefore, the family {M (A, U)} with $A \in S$ and $U \in B$ is a base for a uniformity J_s for, m (E, F).

Next, we prove that each M(A, U) in {M (A, U)} $A \in S$ $U \in B$ is translation-invariant in the group m(E, F).

For this, Let $(f, g) \in M(A, V)$, then $(f(x), g(x) \in U$ for all $x \in A$. Let h be any element of m (E, F). since U is translation-invariant, we get

$(f(x)+h(x), g(x) + h(x)) \in U$ for all $x \in A$

$\Rightarrow (f+h, g+h) \in M(A, U)$

Similarly, $(h+f, h+) \in M(A, U)$.

Hence, each M (A, U) in {M(A, U)} $A \in S$ is a translation-invariant. Thus the family {M (A, U)}$A \in S$ is a base for M(A, U) is translation-invariant in the (additive) group m (E, F) is an (additive) uniform group.

ii) Let F is Housdorff. Then we have to prove that m (E, F) in Housdorff. For this it is sufficient to show that

$$\bigcap_{\substack{B \in S \\ V \in B}} M(B, V) = \Delta(m(E, F))$$

Clearly, $\Delta\left(m(E,F)\right) \subseteq \bigcap_{\substack{B \in S \\ V \in B}} M(B,V)$

We now prove that $\bigcap_{\substack{B \in S \\ V \in B}} M(B, V) \subseteq \Delta(m, (E, F))$

Let, $x \in E$ be arbitrary. Then since $\bigcup_{A \in S} A = E$, then exists $B \in S$ such that $x \in B$

Now, let, $(f, g) \in \bigcap_{\substack{B \in S \\ V \in S}} M(B, V)$

$\Rightarrow (f, g) \in M(B, V)$ for all $B \in S$ and $V \in B$

$\Rightarrow (f(x), g(x)) \in V$ for all $V \in B$

$\Rightarrow (f(x), g(x)) \bigcap_{V \in B} V$

$\Rightarrow (f(x), g(x) \in \Delta(f)$ (since F is Housdorff)

$\Rightarrow f(x) = g(x)$

Since, $x \in E$ is arbitrary, it follows that f(x) = g(x) for all $x \in E$. Hence, f = g.
Therefore, (f, g) $\in \Delta$(m (E, F)). Hence m (E, F0 is Housdorff.
This completes the proof.

Theorem: Every uniform group is a topological going. Translation-invariant base for the uniformity then the mapping: (x, y) $\rightarrow$ x + y and x $\rightarrow$ -x are continuous for let U [x + y] and U [-x] be any basic neighbourhoods of x + y and -x respectively.
Hence, E is topological group.
This completes the proof.

Theorem: If E is an (additive) abelian group, then E is uniform group if and only if E is topological group.

Proof: It remains to prove that E is an (additive) abelian topological group then E is uniform group. We know that if {U} is a system of all neighbourhoods of O, and if for each $U \in$ {U} we define

$$L(U) = \{(xn\ y) \in E \times E: -x + y \in U\}$$

and

$$R(U) = \{(x, y) \in E \times E: y - x \in U\}$$

then {L(U)} and similarly {R(U)} where U $\in$ {U) forms a base for a uniformity and L (U) = R (U) if E is abelian. So E is uniform space. Again, since E is abelian, it will follow that each L (U) is translation-invariant. Hence, E is an uniform group.

Conclusion:

We conclude that each M (A, U) in $\{M(A, U)\}_{\substack{A \in S \\ U \in B}}$ is translation-invariant in the uniform group m (E, F), also $\bigcap_{\substack{B \in S \\ V \in \mathbf{B}}} M(B, V) \subseteq \Delta$(m (E, F)) then m (E, F) is Hansdorff.

References

[1]. Coth, E. : 'Completeness and Compactness in linear topological spaces', Frans, Amer, MathSoz., 79 (1955) 256-280.
[2]. James, I. M. : Topological and uniform spaces; springer verlag. Berlin 1987
[3]. Isbll, J. R. : Uniform spaces, Amer Math Soz. Survey providence, 1964
[4]. Kelley, J. L. : General Topology, Van Nostrand, New-York, 1955.
[5]. Jha, K. K. : Advanced general topological, Nav Bharat Prakashan 1977.

Ranjankumarsingh " Completeness and Compactness of Function Spaces on Uniform Groups." IOSR Journal of Mathematics (IOSR-JM) 15.6 (2019): 01-03.