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The Convergence of Homotopy Analysis Method for Solving Onchocerciasis (Riverblindness)

Sotonwa O.E[1], Obabiyi O.S[2]

Department of Mathematics, University of Ibadan, Oyo State, Nigeria.

Abstract: *In this paper, a mathematical modelling of Onchocerciasis also known as Riverblindness epidemics is discussed. Approximate analytical solutions for the non-linear equations in Onchocerciasis epidemics are obtained by using the Homotopy analysis method (HAM). For this purpose, a theorem is proved to show the convergence of the series solution obtained from the proposed method. Analytical expressions pertaining to the number of susceptible and infected individuals are derived for all possible values of parameters. Furthermore, we present a condition enabling the homotopy analysis method (HAM) to converge to the exact solution of the nonlinear differential equations.*

Keyword*: Onchocerciasis epidemics, Mathematical modeling, Homotopy analysis method (HAM), Convergence, Nonlinear differential equations*

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I. Introduction

Since the nonlinear ordinary differential equations or systems for initial and boundary value problems are most challenging in finding their exact solutions, besides the classical perturbation methods, some new perturbation or analytical methods have been introduced and developed by researchers in the literature. Among these methods proposed is to find analytic approximate solutions of a given nonlinear mathematical model, the most recent popular and powerful technique is the homotopy analysis method (HAM). In this method, which requires neither a small parameter nor a linear term, a homotopy with an embedding parameter $p \in [0,1]$ is constructed [1]. The solution is considered as the sum of an infinite series converging rapidly and accurately to the exact solutions by means of enjoying the convergence control parameter. Fundamental characteristics and advantages of the Homotopy Analysis Method over the existing analytical techniques were clearly laid out by S. J. Liao in the recent book [2]. In addition to its early success in several nonlinear problems as summarized in the book [3]. Furthermore, numerous nonlinear problems in science, finance and engineering were successfully treated by the method [4]. Particularly, a few new solutions of some nonlinear problems were discovered by means of the method [5], which were unfortunately neglected by other analytic methods and even by numerical techniques. The method was successively applied recently to some series of strongly nonlinear problems, such as the Blasius equation for the flow over a permeable stretching plate [6] reported by S.J. Liao, the system of differential equations concerning the flow over a rotating cone [7] , the system of differential equations related to the rotating disk [8] , the series solution of non linear two-point singularly perturbed boundary layer problems [9], the singularly perturbed boundary layer problems [10], the uniform solutions of undamped and dumped Duffing oscillators [11,12], the limit cycle of duffing-van der Pol equation [13], the rotational approximation to the Thomas Fermi equation [14] and the strongly nonlinear differential equations [15]. An analytic shooting approach combined with the HAM was also proposed in [16]. The use of the HAM is more safe now since an optimal parameter controlling the fast convergence can always be picked from the squared residual error ensuring to gain the most accurate results, as also implemented in the above citations. Despite the fact that all these demonstrate the validity and high potential of the homotopy analysis method for strongly nonlinear problems of real life, apart from some general approaches as presented in [2], the question of convergence of the method is yet to be answered. The present chapter is devoted to the investigation of the homotopy analysis technique from a mathematical point of view to serve to its convergence issue. The aim is thus to analyze the method and to show that under a given constraint the HAM converges to the exact solution desired. The convergence control parameter and optimal value for the convergence control parameter, a new conceptual definition is offered, which makes use of the ratios of the homotopy series based on a properly chosen norm. It is shown through examples that both yield approximately the same values regarding the convergence control parameter, though the newly introduced scheme seems more advantageous in some aspects at least in terms of computational efforts.The given convergence criterion is justified by basic examples from non-linear algebraic, differential-difference, integro-differential, the fractional differential, ordinary and partial differential equations and also systems often studied in the literature. The convergence of the HAM for the considered problems is not only guaranteed, but the interval of convergence and further the optimum value for the convergence can also be determined by the presented theory.

II. Statement of the Problem

Consider the standard model under the assumption of constant population size. Susceptible individuals become infected either by contact with infected individuals or through contact with infected black flies. Infected individuals thus generates secondary infections in two ways; by shedding the pathogen into the susceptible vector (blackflies) which susceptible individuals subsequently come into contact with it. In the model to be consider below, we have the following variables and parameters which are: H_S is the Human Susceptible, H_I is the Human Infected, V_S is the Vector Susceptible, V_I is the Vector Infected, Ψ is the Recruitment rate of Humans (Immigration rate, Birth rate), N_h is the total human population size , $\varphi$is the Recruitment rate of Vectors, N_v is the total vector population size, ξ is the Movement rate from H_S to H_I, α is the Recovery rate from H_I to H_S (proportion of ivermectin / Mectizan treatment successfully cure the patients), ρ is the Transmission rate parameters for H_I to V_S, δ is the Transmission rate parameters for V_I to H_S, b is the bitting rate, m is the Movement rate from V_S to V_I, μ is the Natural death rate, β is the death rate for vector, σ is the Artificial death rate (cause as a result of chronic river blindness diseases), u_i for $1 \leq i \leq 4$, is the control parameters, a is Ivermectin, reducing the probability of successful re-infection, b is Moxidectin and doxycycline, kills or sterilizes the adult worm, c for insecticides and d for insecticides and others preventive measures. The corresponding model equations to the compartments above are

$$\dot{H}_S = \Psi N_h + \alpha H_I - b\delta H_S V_I - \mu H_S + au_1 H_S$$

$$\dot{H}_I = b\delta H_S V_I - \alpha H_I - \sigma H_I - \mu H_I + bu_2 H_I \qquad (1)$$

$$\dot{V}_S = \varphi N_v - b\rho H_I V_S - mV_S - \beta V_S + cu_3 V_S$$

$$\dot{V}I = b\rho HIVS - \beta VI + du4VI$$

where $H_S(0) = H_{S0}$, $H_I(0) = H_{I0}$, $V_S(0) = V_{S0}$, $V_I(0) = V_{I0}$, transforming the above equations into matrix notation, we have

$$\dot{x}(t) = x(t)A + x(t)Bu(t) + f(x(t))$$

$$\dot{x}(t) = x(t)(A + Bu(t)) + f(x(t)) \qquad (2)$$

$$x(t_o) = x_o, x(T) = x_T$$

where

$$A = \begin{bmatrix} -(\square) & \alpha & 0 & 0 \\ 0 & -(\alpha + \sigma + \square) & 0 & 0 \\ 0 & 0 & -(\beta + m) & 0 \\ 0 & 0 & 0 & -\beta \end{bmatrix}$$

$$B = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} \dot{H}_S \\ \dot{H}_I \\ \dot{V}_S \\ \dot{V}_I \end{bmatrix}$$

$$u(t) = \begin{bmatrix} u1 & 0 & 0 & 0 \\ 0 & u2 & 0 & 0 \\ 0 & 0 & u3 & 0 \\ 0 & 0 & 0 & u4 \end{bmatrix}$$

$f^T(x(t)) = (-b\delta H_S V_I, b\delta H_S V_I, -b\rho H_I V_S, b\rho H_I V_S)$

where A and B are real constant matrices of appropriate dimensions.

$x \in \mathbb{R}^4$ is the state vector

u is the control matrix

f is a non linear polynomial vector function

where $f(0) = 0$, $x_0 \in \mathbb{R}^4$, $x_t \in \mathbb{R}^4$ are the initial and final states, respectively.

III. A description of the homotopy analysis method

A systematic description of the homotopy analysis method is outlined in this section.Within this purpose, let us consider the following general nonlinear equation below

$$N[x(t)] = 0 \qquad (3)$$

where N is a nonlinear operator, t denotes an independent variable, $x(t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions in the case of a differential equation (or a system), which can be treated in the similar way. We construct the zeroth-order deformation of equation

$$(1 - p)L[\phi(t;p) - x_0(t)] + phJ(t)N[\phi(t;p)] = 0 \qquad (4)$$

and we define the operator

$$N[x(t;p)] = \dot{x}(t;p) - x(t;p)(A + Bu(t)) - f(x(t;p)) \qquad (5)$$

where $p \in [0,1]$ is called the homotopy embedding parameter, h is a nonzero auxiliary parameter which is called the convergence control parameter, L is an auxiliary linear operator, $x_0(t)$ is an initial guess for $x(t)$,and $J(t)$ is an auxiliary function and $\phi(t;p)$ is an unknown function which satisfies equation (4).Obviously, when $p = 0$ and $p = 1$, we have

$$\phi(t;0) = x_0(t), \qquad \phi(t;1) = x(t) \qquad (6)$$

Thus, as p increases from 0 to 1, the solution $\phi(t;p)$ varies from the initial guess $x_0(t)$ to the final solution $x(t)$ of the original nonlinear equation (1). Expanding $\phi(t;p)$ about p, it reads

$$(7) \qquad \phi(t;p) = x_0(t) + \sum_{k=1}^{\infty} x_k(t)p^k$$

where the series coefficients x_k are defined by

$$(8) \qquad x_k(t) = \frac{1}{k!}\frac{\partial^k \phi(t;p)}{\partial p^k}|_{p=0}$$

Here, the series (7) is called the homotopy series and the expression (8) is called the kth-order homotopy-derivative of ϕ. If the auxiliary linear operator L, the initial guess $x_0(t)$, the convergence control parameter h and the auxiliary function $J(t)$ are so properly chosen in equation (4), the homotopy-series (7) converges at $p = 1$, then using the relationship $\phi(t;1) = x(t)$, one has the so-called homotopy series solution

$$) \qquad (9) \qquad x(t) = x_0(t) + \sum_{k=1}^{\infty} x_k(t$$

which must be one of the solutions of original nonlinear equation (1). Based on the equation (8), the governing equation for the homotopy series (9) can be deducted from the zeroth-order deformation equation (4). Differentiating the zeroth-order deformation equation (4)

$$(1 - p)L[\phi(t;p) - x_0(t)] + phJ(t)N[\phi(t;p)] = 0$$

$$L[\phi(t;p) - x_0(t)] - pL[\phi(t;p) - x_0(t)] + phJ(t)N[\phi(t;p)] = 0$$

differentiating with respect to *p* we have

$$L\frac{\partial[\phi(t;p)]}{\partial p} - L[\phi(t;p)] - pL\frac{\partial[\phi(t;p)]}{\partial p} + L[x_0(t)] + hJ(t)N[\phi(t;p)] + phJ(t)\frac{\partial[\phi(t;p)]}{\partial p} = 0$$

thus, setting $p = 0$ we have

$$L\frac{\partial[\phi(t;p)]}{\partial p}\mid_{p=0} - L[\phi(t;0)] = hJ(t)N[\phi(t;0)]$$

$$L\left[\frac{\partial[\phi(t;p)]}{\partial p}\mid_{p=0} - \phi(t;0)\right] = hJ(t)N[\phi(t;0)]$$

for the second derivatives with respect to *p*

$$L\frac{\partial^2[\phi(t;p)]}{\partial p^2} - L\frac{\partial[\phi(t;p)]}{\partial p} - L\frac{\partial[\phi(t;p)]}{\partial p} - pL\frac{\partial^2[\phi(t;p)]}{\partial p^2} + hJ(t)\frac{\partial N[\phi(t;p)]}{\partial p} + hJ(t)\frac{\partial N[\phi(t;p)]}{\partial p} + phJ(t)\frac{\partial^2 N[\phi(t;p)]}{\partial p^2} = 0$$

setting $p = 0$, we have

$$L\frac{\partial^2[\phi(t;p)]}{\partial p^2}\mid_{p=0} - 2L\frac{\partial[\phi(t;p)]}{\partial p}\mid_{p=0} = 2hJ(t)\frac{\partial N[\phi(t;p)]}{\partial p}\mid_{p=0}$$

$$L\frac{\partial^2[\phi(t;p)]}{\partial p^2}\mid_{p=0} - 2L\frac{\partial[\phi(t;p)]}{\partial p}\mid_{p=0} = 2hJ(t)\frac{\partial N[\phi(t;p)]}{\partial p}\mid_{p=0}$$

dividing by 2!

$$L\left[\frac{\partial^2[\phi(t;p)]}{2!\partial p^2}\mid_{p=0} - \frac{\partial[\phi(t;p)]}{\partial p}\mid_{p=0}\right] = hJ(t)\frac{\partial N[\phi(t;p)]}{\partial p}\mid_{p=0}$$

for the third derivatives with respect to *p*

$$L\frac{\partial^3[\phi(t;p)]}{\partial p^3} - L\frac{\partial^2[\phi(t;p)]}{\partial p^2} - L\frac{\partial^2[\phi(t;p)]}{\partial p^2} - L\frac{\partial^2[\phi(t;p)]}{\partial p^2} - pL\frac{\partial^3[\phi(t;p)]}{\partial p^3} + hJ(t)\frac{\partial^2 N[\phi(t;p)]}{\partial p^2} + hJ(t)\frac{\partial^2 N[\phi(t,p)]}{\partial p^2} + hJ(t)\frac{\partial^2 N[\phi(t;p)]}{\partial p^2} + phJ(t)\frac{\partial^3[\phi(t;p)]}{\partial p^3} = 0$$

setting $p = 0$, we have

$$L\frac{\partial^3[\phi(t;p)]}{\partial p^3}\mid_{p=0} - L\frac{\partial^2[\phi(t;p)]}{\partial p^2}\mid_{p=0} - L\frac{\partial^2[\phi(t;p)]}{\partial p^2}\mid_{p=0} - L\frac{\partial^2[\phi(t;p)]}{\partial p^2}\mid_{p=0} + hJ(t)\frac{\partial^2 N[\phi(t;p)]}{\partial p^2}\mid_{p=0} + hJ(t)\frac{\partial^2 N[\phi(t;p)]}{\partial p^2}\mid_{p=0} + hJ(t)\frac{\partial^2 N[\phi(t;p)]}{\partial p^2}\mid_{p=0} = 0$$

$$L\frac{\partial^3[\phi(t;p)]}{\partial p^3}\mid_{p=0} - 3L\frac{\partial^2[\phi(t;p)]}{\partial p^2}\mid_{p=0} = 3hJ(t)\frac{\partial^2 N[\phi(t;p)]}{\partial p^2}\mid_{p=0}$$

dividing by 3!

$$L\left[\frac{\partial^3[\phi(t;p)]}{3!\partial p^3}\mid_{p=0} - \frac{\partial^2[\phi(t;p)]}{2!\partial p^2}\mid_{p=0}\right] = hJ(t)\frac{\partial^2 N[\phi(t;p)]}{2!\partial p^2}\mid_{p=0}$$

differentiating *k* times with respect to the homotopy parameter *p*, setting $p = 0$ and finally dividing them by *k*!, we have the so-called *k*th-order deformation equation

$L[x_k(t) - \chi_k x_{k-1}(t)] = hJ(t)R_{k-1}[\phi(t;p)]$ (10)

we chose our auxiliary linear operator as

$L[\phi(t;p)] = \phi(t;p)$ (11)

the equations (10) and (11), now implies the

$$x_k(t) - \chi_k x_{k-1}(t) = hJ(t)R_{k-1}[\phi(t;p)]$$

$x_k(t) = \chi_k x_{k-1}(t) + hJ(t)R_{k-1}[\phi(t;p)]$ (12)

also by using equations (5)

$N[\phi(t;p)] = \dot{x}(t;p) - x(t;p)[A + Bu(t) - f(x(t;p))]$ (13)

where R_k is called kth-order homotopy derivative operator given by

$$R_k[\phi(t;p)] = \frac{1}{k!}\frac{\partial^k N[\phi(t;p)]}{\partial p^k}|_{p=0} \quad (14)$$

$$R_k[\phi(t;p)] = \frac{1}{k!}\frac{\partial^k \dot{x}(t;p) - x(t;p)\,[A + Bu(t) - f(x(t;p))]}{\partial p^k}|_{p=0}$$

$$R_k[\phi(t;p)] = \frac{1}{k!}\frac{\partial^k \dot{x}(t;p)}{\partial p^k}|_{p=0} - \frac{1}{k!}\frac{\partial^k x(t;p)[A + BU(t)]}{\partial p^k}|_{p=0} + \frac{1}{k!}\frac{\partial^k x(t;p) f(x(t;p))}{\partial p^k}|_{p=0}$$

$$R_k[\phi(t;p)] = -(A + Bu(t))x_k - f(x(t;p))x_k - f(x(t;p))\chi_k$$

$R_k[\phi(t;p)] = [-(A + Bu(t)) - f(x(t;p))[1 - \chi_k]]x_k$ (15)

where $f \in C^1$, finally using (12) we obtain

$x_k(t) = \chi_k x_{k-1}(t) + hJ(t)R_{k-1}[\phi(t;p)]$ (16)

substituting equation (15) into equation (16)

$x_k(t;p) = \chi_k x_{k-1}(t;p) + hJ(t)[-(A + Bu(t)) - f(x(t;p))[1 - \chi_k]]x_{k-1}(t;p)$ (17)

and

$$\chi_k = \begin{cases} 0, & k < 1 \\ 1, & k \geq 1 \end{cases} \quad (18)$$

Note that the right-hand side term $R_{k-1}[\phi(t;p)]$ of (10) is dependent only upon,

$$x_0(t), x_1(t), x_2(t), \cdots, x_{k-1}(t)$$

which are known for the kth-order deformation equation described above. Because of the fact that the higher-order deformation equation (10) is linear in nature, an appropriate linear operator L will easily generate the homotopy terms x_k in homotopy series (9) by means of computer algebra systems such as Mathematica, Maple and so on. Finally, an Mth-order approximate analytic solution is given by truncating the homotopy series (9)

$$x_M(t) = x_0(t) + \sum_{k=1}^{M} x_k(t) \tag{19}$$

and the exact solution is given by the limit

$$x(t) = \lim_{M\to\infty} x_M(t) \tag{20}$$

It should be reminded that the homotopy terms

$$x_1(t), x_2(t), \cdots, x_k(t)$$

strongly depend on both the physical variable t and the convergence control parameter h. In essence, h is an artificial parameter without physical meanings but it can adjust and control the convergence region of the homotopy series solution (19). In fact, the use of such an auxiliary parameter distinguishes the HAM from other perturbation-like analytical techniques. With the initial guess $x_0 = 0$, $h = -1$, we have

$$x_1 = x(t,p)f(x)J(t)$$

$$x_2 = x(t,p)f(x)J(t) + (A + Bu(t))x(t;p)f(x)J(t)$$

$$= J(t)x(t;p)f(x)[1 - (A + Bu(t))] \tag{21}$$

.
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.

$$x_n = x_{n-1}(t;p)J(t)f(x(t;p))[1 - (A + Bu(t))]^{n-1}$$

by substituting equation (21) into equation (9) we obtain

$$x = x_1 + x_2 + \cdots + x_n$$

$$x = \sum_{n=0}^{\infty} x_{n-1}(t;p)J(t)f(x(t;p))\left[1 - (A + Bu(t))\right]^{n-1} \tag{22}$$

Now we have to prove the convergence of equation (22)

Theorem 1. *: The sequence*

$$x^{[m]} = \left[\sum_{n=0}^{m}[I - (A + Bu(t))]^k\right] J(t)f(x(t))$$

is a Cauchy sequence if $\|I - (A + Bu(t))\| < 1$.

we have to show that

$$\lim_{m\to\infty} \|x^{[m+p]} - x^{[m]}\| = 0$$

Now considering

$$x^{[m+p]} - x^{[m]} = \left[\sum_{k=1}^{p}\left[I - (A + Bu(t))\right]^{m+k}\right] J(t)f(x(t))$$

then

$$\|x^{[m+p]} - x^{[m]}\| \le \|(A + Bu(t))f(x(t))\| \sum_{k=1}^{p} \|I - (A + Bu(t))\|^{m+k}$$

let $\beta = \|I - (A + Bu(t))f(x(t))\|$, then

$\|x^{[m+p]} - x^{[m]}\| \leq$

$$\| J(t)f(x(t)) \| \beta^m \sum_{k=1}^{p} \beta_k \leq \| J(t)f(x(t)) \| \beta^m \left(\frac{\beta^p - 1}{\beta - 1}\right)$$

so we have

$$\lim_{m\to\infty} \|x^{[m+p]} - x^{[m]}\| \leq \left(\frac{\beta^p - 1}{\beta - 1}\right) \|J(t)f(x(t))\| lim_{m\to\infty}\beta^m$$

since $\beta < 1$, then we obtain

$$\lim_{m\to\infty} \|x^{[m+p]} - x^{[m]}\| = 0$$

which completes the proof.

IV. A Convergence Theorem

Existing theorems about the convergence of the resulting homotopy series of a given nonlinear problem unfortunately have general meanings [2,3]. The convergence of solution series is believed to take place since the HAM logically contains the famous Euler Theorem as proved by Liao [2]. However, a firm answer to the very basic question of why the series (9) obtained by setting $p = 1$ in (4) should be convergent with respect to the equation (17) remains. To make sure of the convergence, the analyticity of solutions is generally presumed, otherwise, a Maclaurin series of a function may not necessarily converge to that function. This may limit the homotopy method leading to divergent homotopy series solutions especially for nonlinear problems with strong nonlinearity. Moreover, although it is fortunate to know that the convergence control parameter h can greatly modify the convergence of the homotopy series solution, the guarantee of convergence still needs a mathematical explanation. We provide the subsequent theorems. It should be noted that even though the proofs require prescription of convergence control parameter h, how to find a proper value, or even better, to get a fastest convergent one. Since the homotopy analysis methodology as described in section 3 is used for solving complicated highly nonlinear problems, the convergence criterion to be given should also be easy-to-use beyond the generality and in the absence of exact solution to the nonlinear equation under consideration. This is essential in deeper understanding of whether the HAM performed for a specific problem will converge to the true exact solution or not. Such a convergence criterion was made use of in several physical problems. In what follows we state the criterion, that is based on the fixed point theorem well known in the functional analysis.

Theorem 2. *Suppose that Q is a subset of R be a Banach space donated with a suitable norm $\|.\|$, over which the functional sequence $x_k(t)$ of (7) is defined for a prescribed value of h. Assume also that the initial approximation $x_0(t)$ remains inside the solution $x(t)$ of (1). Taking $r \in R$ be a constant, the following statements hold true:*
For a prescribed convergence control parameter h, if $\|x_{k+1}(t)\| \leq r \|x_k(t)\|$ for all k, provided that $0 < r < 1$, then the series solution $\phi(t,p)$ defined in (7) converges absolutely at $p = 1$ to $x(t)$ given by (9).

Proof. If $S_n(t)$ denote the sequence of partial sum of the series (9),it is demanded that $S_n(t)$ be a Cauchy sequence in Q. For this purpose, the subsequent inequalities are constructed

$$\|S_{n+1}(t) - S_n(t)\| = \|x_{n+1}(t)\| \leq r \|x_n(t)\| \leq r^2 \|x_{n-1}(t)\| \leq \cdots \leq r^{n+1} \|x_0(t)\| \quad (23)$$

It is remarked that owing to (23), all the approximations produced by the homotopy analysis method by (4) in section 3 will lie within the solution $x(t)$. For every $m,n \in N$, such that $n \geq m$, the following results in making use of (23) and thetriangle inequality successively,

$$\| S_n(t) - S_m(t) \| = \| (S_n(t) - S_{n-1}(t)) + \cdots + (S_{m+1}(t) - S_m(t)) \| \leq \left(\frac{1 - r^{n-m}}{1 - r}\right) r^{m+1} \| x_0(t) \| \quad (24)$$

since by the hypothesis $0 < r < 1$, we get from (24)

$$\lim_{n,m\to\infty} \| S_n(t) - S_m(t) \| = 0 \qquad (25)$$

Therefore, $S_n(t)$ is a Cauchy sequence in the Banach space A, which implies that the series solution (9) is indeed convergent. This completes the proof.

Theorem 3. *If the series solution defined in (7) is convergent at p = 1, then the resulting series (9) converges to an exact solution of the nonlinear problem given in (1).*

Theorem 4. *Assume that the series solution defined in (9) is convergent to the solution $x(t)$ for a prescribed value of h. If the truncated series $x_M(t)$ expressed in equation (19) is used as an approximation to the solution $x(t)$ of problem (1), then an upper bound for the error, that is, $E_M(t)$, is estimated as*

$$E_M(t) \leq \frac{r^{M+1}}{1-r} \| x_0(t) \| \qquad (26)$$

Proof. Making use of the inequality (24) of Theorem 2, we immediately obtain

$$\| x(t) - S_M(t) \| \leq \left(\frac{1 - r^{n-M}}{1-r} \right) r^{M+1} \| x_0(t) \| \qquad (27)$$

and taking into account the constraint $(1 - r^{n-M}) < 1$, (26) leads to the desired formula (25). This completes the proof

V. Convergence on Onchocerciasis

In this Section, we prove the convergence of the series solution obtained from the homotopy analysis method to the exact solution of the Equation (1)

Theorem 5. *If the series solution*

$$HS = HS0 + HS1 + HS2 + \cdots$$

$$H_I = H_{I0} + H_{I1} + H_{I2} + \cdots \qquad (28)$$

$$VS = VS0 + VS1 + VS2 + \cdots$$

$$VI = VI0 + VI1 + VI2 + \cdots$$

from the HAM are convergent, and the series

$$\sum_{m=0}^{\infty} \frac{\partial H_{S_m}}{\partial t}, \quad \sum_{m=0}^{\infty} \frac{\partial H_{I_m}}{\partial t}, \quad \sum_{m=0}^{\infty} \frac{\partial V_{S_m}}{\partial t}, \quad \sum_{m=0}^{\infty} \frac{\partial V_{I_m}}{\partial t},$$

are convergent, then the series

$$\sum_{m=0}^{\infty} H_{S_m}(t), \quad \sum_{m=0}^{\infty} H_{I_m}(t), \quad \sum_{m=0}^{\infty} V_{S_m}(t), \quad \sum_{m=0}^{\infty} V_{I_m}(t)$$

converge to the exact solution of Equation (1)

Proof. By hypothesis, the series is convergent, it holds

$$H_S(t;p) = \sum_{m=0}^{\infty} H_{S_m}(t;p) \qquad (29)$$

In this case, the necessary condition for the convergence of the series is valid, i.e

$$lim_{m\to\infty}H_{Sm}(t;p) = 0 \quad (30)$$

so using equations (10) and (29), we have

$$\sum_{m=1}^{\infty}\left[H_{S_m}(t;p) - \chi_m H_{S_{m-1}}(t;p)\right] = hJ(t)\sum_{m=1}^{\infty}[R_{m-1}(H_{S_m}(t;p))] \quad (31)$$

$$\sum_{m=1}^{\infty}[H_{S_m}(t;p) - \chi_m H_{S_{m-1}}(t;p)] = lim_{m\to\infty}H_{S_m}(t;p) = 0 \quad (32)$$

$$L[HSm - \chi mHSm-1] = hJ(t)Rm-1(HSm) \quad (33)$$

Where

$$R_{m-1}(H_{S_m}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[H_{S_m}(t;p)]}{\partial p^{m-1}} \quad (34)$$

and

$$\chi_m = \begin{cases} 0, & m < 1 \\ 1, & m \geq 1 \end{cases} \quad (35)$$

since the operator L is linear, from equation (33) we have:

$$\sum_{m=1}^{\infty}L\left[H_{S_m}(t;p) - \chi_m H_{S_{m-1}}(t;p)\right] = L\left(\sum_{m=1}^{\infty}[H_{S_m}(t;p) - \chi_m H_{S_{m-1}}(t;p)]\right) = 0 \quad (36)$$

By applying

$$L\left[H_{S_m}(t;p) - \chi_m H_{S_{m-1}}(t;p)\right] = hJ(t)R_{m-1}(H_{S_m}(t;p)) \quad (37)$$

we get

$$\sum_{m=1}^{\infty}L[H_{S_m}(t;p) - \chi_m H_{S_{m-1}}(t;p)] = hJ(t)\sum_{m=1}^{\infty}R_{m-1}(H_{S_m}(t;p)) \quad (38)$$

since $h,J(t) \neq 0$ then

$$\sum_{m=1}^{\infty}R_{m-1}(H_{S_m}) = 0 \quad (39)$$

$$\sum_{m=1}^{\infty}R_{m-1}(H_{S_m}) = \sum_{m=1}^{\infty}\left[\frac{\partial H_{S_m}(t;p)}{\partial t} - \Psi N_h - \alpha H_{I_m}(t;p) + b\delta H_{S_m}(t;p)V_{I_m}(t;p) + \mu H_{S_m}(t;p) - au_1 H_{S_m}(t;p)\right.$$

$$\sum_{m=1}^{\infty}[R_m(H_{S_{m-1}})] = \sum_{m=1}^{\infty}\left[\frac{\partial H_{S_m}(t;p)}{\partial t}\right] - \sum_{m=1}^{\infty}\Psi N_{h_m} - \alpha\sum_{m=1}^{\infty}H_{I_m}(t;p) + b\sum_{m=1}^{\infty}\delta H_{S_m}(t;p)V_{I_m}(t;p) + \mu\sum_{m=1}^{\infty}H_{S_m}(t;p) - au_1\sum_{m=1}^{\infty}H_{S_m}(t;p) \quad (40)$$

$$\sum_{m=1}^{\infty}[R_{m-1}(H_{S_m})] = \sum_{m=1}^{\infty}\left[\frac{\partial H_{S_m}(t;p)}{\partial t}\right] - \Psi\sum_{m=1}^{\infty}N_{h_m} - \alpha\sum_{m=1}^{\infty}H_{I_m}(t;p) + b\delta\sum_{m=1}^{\infty}H_{S_m}(t;p)\sum_{m=1}^{\infty}V_{I_m}(t;p) + \mu\sum_{m=1}^{\infty}H_{S_m}(t;p) - au_1\sum_{m=1}^{\infty}H_{S_m}(t;p)$$

$$\frac{\partial H_S(t;p)}{\partial t} - \Psi N_h - \alpha H_I(t;p) + b\delta H_S(t;p)V_I(t;p) + \mu H_S(t;p) - au_1 H_S(t;p) = 0$$

so, from equation (38), we obtain

$$\dot{H}_S(t;p) - \Psi N_h - \alpha H_I(t;p) + b\delta H_S(t;p)V_I(t;p) + \mu H_S(t;p) - au_1 H_S(t;p) = 0$$

$$\dot{H}_S(t;p) = \Psi N_h + \alpha H_I(t;p) - b\delta H_S(t;p)V_I(t;p) - \mu H_S(t;p) + au_1 H_S(t;p) \quad (41)$$

Also from initial conditions (10) and (24), the following holds

$$H_S(0) = \sum_{m=0}^{\infty} Hsm(0) = Hso \quad (42)$$

since H_S satisfies equation (40) and (41), we conclude that it is an exact solution of(1). This completes the proof

Proof. By hypothesis, the series is convergent, it holds

$$H_I(t;p) = \sum_{m=0}^{\infty} H_{I_m}(t;p) \quad (43)$$

In this case,the necessary condition for the convergence of the series is valid, i.e

$$lim_{m\to\infty} H_{Im}(t;p) = 0 \quad (44)$$

using equation (10) and equation (44), we have

$$\sum_{m=1}^{\infty}[H_{I_m}(t;p) - \chi_m H_{I_{m-1}}(t;p)] = hJ(t)\sum_{m=1}^{\infty}[R_{m-1}(H_{I_m}(t;p))] \quad (45)$$

by the left side of equation (45)

$$\sum_{m=1}^{\infty}[H_{I_m}(t;p) - \chi_m H_{I_{m-1}}(t;p)] = lim_{m\to\infty} H_{I_m}](t;p) = 0 \quad (46)$$

$$L[HIm - \chi mHIm-1] = hJ(t)Rm-1(HIm) \quad (47)$$

where

$$R_{m-1}(H_{I_m}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[H_I(t;p)]}{\partial p^{m-1}} \quad (48)$$

and

$$\chi_m = \begin{cases} 0, & m < 1 \\ 1, & m \geq 1 \end{cases} \quad (49)$$

and since the operator L is linear, from equation (46) we have:

$$\sum_{m=1}^{\infty} L[H_{I_m}(t;p) - \chi_m H_{I_{m-1}}(t;p)] = L\left(\sum_{m=1}^{\infty}[H_{I_m}(t;p) - \chi_m H_{I_{m-1}}(t;p)]\right) = 0 \quad (50)$$

By applying

$$L[H_{Im}(t;p) - \chi_m H_{Im-1}(t;p)] = hJ(t)R_{m-1}(H_{Im}(t;p)) \quad (51)$$

we get

$$\sum_{m=1}^{\infty} L[H_{Im}(t;p) - \chi_m H_{Im-1}(t;p)] = hJ(t)\sum_{m=1}^{\infty} LR_{m-1}(H_{Im}(t;p)) \quad (52)$$

since $h,J(t) \neq 0$ then

$$\sum_{m=1}^{\infty}[R_{m-1}(H_{I_m})(t;p)] = 0 \quad (53)$$

$$\sum_{m=1}^{\infty}[R_{m-1}(H_{I_m})] = \sum_{m=1}^{\infty}\left[\frac{\partial H_{I_m}(t;p)}{\partial t} - b\delta H_{S_m}(t;p)V_{I_m}(t;p)\right.$$

$$+ \alpha H_{Im}(t;p) + \sigma H_{Im}(t;p) + \mu H_{Im}(t;p) - bu_2 H_{Im}(t;p)$$

$$\sum_{m=1}^{\infty}[R_{m-1}(H_{I_m})] = \sum_{m=1}^{\infty}\frac{\partial H_{I_m}(t;p)}{\partial t} - b\delta\sum_{m=1}^{\infty}H_{S_m}(t;p)V_{I_m}(t;p) + (\alpha+\sigma+\mu)\sum_{m=1}^{\infty}H_{I_m}(t;p) - bu_2\sum_{m=1}^{\infty}H_{I_m}(t;p) \quad (54)$$

$$\sum_{m=1}^{\infty}[R_{m-1}(H_{I_m})] = \sum_{m=1}^{\infty}\frac{\partial H_{I_m}(t;p)}{\partial t} - b\delta\sum_{m=1}^{\infty}H_{S_m}(t;p)\sum_{m=1}^{\infty}V_{I_m}(t;p) + (\alpha+\sigma+\mu)\sum_{m=1}^{\infty}H_{I_m}(t;p) - bu_2\sum_{m=1}^{\infty}H_{I_m}(t;p)$$

$$\frac{\partial H_{I_m}(t;p)}{\partial t} - b\delta H_{S_m}(t;p)V_{I_m}(t;p) + \alpha H_{I_m}(t;p) + \sigma H_{I_m}(t;p) + \mu H_{I_m}(t;p) - bu_2 H_{I_m}(t;p) = 0$$

so, from equation (52), we obtain

$$\dot{H}_I(t;p) - b\delta H_S(t;p)V_I(t;p) + \alpha H_I(t;p) + \sigma H_I(t;p) + \mu H_I(t;p) - bu_2 H_I(t;p) = 0 \quad (55)$$

$$\dot{H}_I(t;p) = b\delta H_S(t;p)V_I(t;p) + \alpha H_I(t;p) + \sigma H_I(t;p) + \mu H_I(t;p) - bu_2 H_I(t;p) \quad (56)$$

Also from initial conditions (10) and (25), the following holds

$$H_I(0) = \sum_{m=0}^{\infty} H_{I_m}(0) = H_{I_0} \quad (57)$$

since H_I satisfies equation (55) and (56), we conclude that it is an exact solution of (1). This completes the proof.

Proof. By hypothesis, the series is convergent, it holds

$$V_S(t;p) = \sum_{m=0}^{\infty} V_{S_m}(t;p) \quad (58)$$

In this case,the necessary condition for the convergence of the series is valid, i.e

$$lim_{m\to\infty} V_{Sm}(t;p) = 0 \quad (59)$$

so using equation (10) and equation (59), we have

$$\sum_{m=1}^{\infty}[V_{S_m}(t;p) - \chi_m V_{S_{m-1}}(t;p)] = hJ(t)\sum_{m=1}^{\infty}[R_{m-1}(V_{S_m}(t;p))] \quad (60)$$

where

thus,

$$\sum_{m=1}^{\infty}[V_{S_m}(t;p) - \chi_m V_{S_{m-1}}(t;p)] = lim_{m\to\infty} V_{S_m}](t;p) = 0 \quad (61)$$

$$L[V_{Sm} - \chi_m V_{Sm-1}] = hJ(t)R_{m-1}(V_{Sm}) \quad (62)$$

(63)

and

(64)

and since the operator L is linear, from equation (61) we have:

$$\sum_{m=1}^{\infty} L[V_{S_m}(t;p) - \chi_m V_{S_{m-1}}(t;p)] = L\left(\sum_{m=1}^{\infty}[V_{S_m}(t;p) - \chi_m V_{S_{m-1}}(t;p)]\right) = 0 \quad (65)$$

By applying

$$L[V_{Sm}(t;p) - \chi_m V_{Sm-1}] = hJ(t)R_{m-1}(V_{Sm}(t;p)) \quad (66)$$

from Equation (64) we get

$$\sum_{m=1}^{\infty} L[V_{S_m}(t;p) - \chi_m V_{S_{m-1}}(t;p)] = hJ(t)\sum_{m=1}^{\infty} R_{m-1}(V_{S_m}(t;p)) \quad (67)$$

since $h, J(t) \neq 0$ then

$$\sum_{m=1}^{\infty} R_{m-1}(V_{Sm}) = 0 \quad (68)$$

$$\sum_{m=1}^{\infty}[R_{m-1}(V_{S_m}(t;p))] = \sum_{m=1}^{\infty}\left[\frac{\partial V_{S_m}(t;p)}{\partial t} - \varphi N_{v_m} - b\rho H_{I_m}(t;p)V_{S_m}(t;p) + mV_{Im}(t;p) + \beta V_{Sm}(t;p) - cu_3 V_{Sm}(t;p)\right]$$

$$\sum^{\infty}[R^{\infty}_{m-1}(V^{\infty}_{S_m})] = \sum^{\infty} \frac{\partial V_{S_m}(t;p)}{\partial t} - \varphi \sum N_{v_m} - b\varphi) \sum \rho H_{I_m}(t;p) V_{S_m}(t$$

m=1m=1 m=1 $m=1$

$$+ m \sum_{m=1}^{\infty} V_{I_m}(t;p) + \beta \sum_{m=1}^{\infty} V_{S_m}(t;p) - cu_3 \sum_{m=1}^{\infty} V_{S_m}(t;p) \quad (69)$$

$$\sum_{m=1}^{\infty}[R_{m-1}(V_{S_m})] = \sum_{m=1}^{\infty} \frac{\partial V_{S_m}(t;p)}{\partial t} - \varphi \sum_{m=1}^{\infty} N_{v_m} - b \sum_{m=1}^{\infty} \rho H_{I_m}(t;p) \sum_{m=1}^{\infty} V_{S_m}(t;p)$$

$$+ m \sum_{m=1}^{\infty} V_{I_m}(t;p) + \beta \sum_{m=1}^{\infty} V_{S_m}(t;p) - cu_3 \sum_{m=1}^{\infty} V_{S_m}(t;p)$$

$\frac{\partial Vs(t;p)}{\partial t} - \varphi N_v - b\rho H_I(t;p)V_S(t;p) + mV_I(t;p) + \beta V_S(t;p) - cu_3 V_S(t;p) = 0$

so, from equation (67), we obtain

$$\dot{V}_S(t;p) - \varphi N_v - b\rho H_I(t;p)V_S(t;p) + mV_I(t;p) + \beta V_S(t;p) - cu_3 V_S(t;p) = 0$$

$$\dot{V}_S(t;p) = \varphi N_v + b\rho H_I(t;p)V_S(t;p) - mV_I(t;p) - \beta V_S(t;p) + cu_3 V_S(t;p) \quad (70)$$

Also from initial conditions (10), the following holds

$$V_S(0) = \sum_{m=0}^{\infty} V_{S_m}(0) = V_{S_0} \quad (71)$$

since V_S satisfies equation (69) and (70), we conclude that it is an exact solution of (1). This completes the proof

Proof. By hypothesis, the series is convergent, it holds

$$V_I(t;p) = \sum_{m=0}^{\infty} V_{I_m}(t;p) \quad (72)$$

In this case,the necessary condition for the convergence of the series is valid, i.e

$$lim_{m\to\infty} V_{Im}(t;p) = 0 \quad (73)$$

so using equation (10) and equation (73), we have

$$\sum_{m=1}^{\infty}[V_{I_m}(t;p)-\chi_m V_{I_{m-1}}(t;p)] = hJ(t)\sum_{m=1}^{\infty}[R_{m-1}(V_{I_m}(t;p))] \quad (74)$$

then,

$$\sum_{m=1}^{\infty}[V_{Im}(t;p)-\chi_m V_{Im-1}(t;p)] = lim_{m\to\infty}V_{Im}(t;p) = 0 \quad (75)$$

$$L[VIm-\chi mVIm-1] = hJ(t)Rm-1(VIm) \quad (76)$$

Where

$$R_{m-1}(V_{I_m}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[V_I(t;p)]}{\partial p^{m-1}} \quad (77)$$

and

$$\chi_m = \begin{cases} 0, & m<1 \\ 1, & m\geq 1 \end{cases} \quad (78)$$

and since the operator L is linear, from equation (75) we have:

$$\sum_{m=1}^{\infty}L[V_{I_m}(t;p)-\chi_m V_{I_{m-1}}(t;p)] = L\left(\sum_{m=1}^{\infty}[V_{I_m}(t;p)-\chi_m V_{I_{m-1}}(t;p)]\right) = 0 \quad (79)$$

By applying

$$L[V_{Im}(t;p)-\chi_m V_{Im-1}] = hJ(t)R_{m-1}(V_{Im}(t;p)) \quad (80)$$

we get

$$\sum_{m=1}^{\infty}[L[V_{Im}(t;p)-\chi_m V_{Im-1}(t;p)] = hJ(t)\sum_{m=1}^{\infty}[R_{m-1}(V_{Im}(t;p))] \quad (81)$$

since $h,J(t)\neq 0$ then

$$\sum_{m=1}^{\infty}R_{m-1}(V_{I_m}) = 0 \quad (82)$$

$$\sum_{m=1}^{\infty}[R_{m-1}(V_{I_m})] = \sum_{m=1}^{\infty}\left[\frac{\partial V_{I_m}(t;p)}{\partial t} - b\rho H_{I_m}(t;p)V_{S_m}(t;p) + \beta V_{I_m}(t;p) - du_4 V_{I_m}(t;p)\right.$$

$$\sum_{m=1}^{\infty}[R_{m-1}(V_{I_m})] = \sum_{m=1}^{\infty}\frac{\partial V_{I_m}(t;p)}{\partial t} - b\rho\sum_{m=1}^{\infty}H_{I_m}(t;p)V_{S_m}(t;p) + \beta\sum_{m=1}^{\infty}V_{I_m}(t;p) - du_4\sum_{m=1}^{\infty}V_{I_m}(t;p) \quad (83)$$

$$\sum_{m=1}^{\infty}[R_{m-1}(V_{I_m})] = \sum_{m=1}^{\infty}\frac{\partial V_{I_m}(t;p)}{\partial t} - b\rho\sum_{m=1}^{\infty}H_{I_m}(t;p)\sum_{m=1}^{\infty}V_{S_m}(t;p) + \beta\sum_{m=1}^{\infty}V_{I_m}(t;p) - du_4\sum_{m=1}^{\infty}V_{I_m}(t;p)$$

so,we have

$$\frac{\partial V_I(t;p)}{\partial t} - b\rho H_I(t;p)V_S(t;p) + \beta V_I(t;p) - du_4 V_I(t;p) = 0 \quad (84)$$

so, from equation (81), we obtain

$\dot{V}_I(t;p)\partial t - b\rho H_I(t;p)V_S(t;p) + \beta V_I(t;p) - du_4 V_I(t;p) = 0$

$\dot{V}_I(t;p) = b\rho H_I(t;p)V_S(t;p) - \beta V_I(t;p) + du_4 V_I(t;p)$ (85)

Also from initial conditions (10) and (24), the following holds

$$V_I(0) = \sum_{m=0}^{\infty} V_{I_m}(0) = V_{I_0} \quad (86)$$

since V_I satisfies equations (84) and (85), we conclude that it is an exact solution of (1). This completes the proof.

Hence from equation (84) we conclude that $V_I(t;p) = \sum_{m=0}^{\infty} V_{I_m}(t;p)$ is the exact solution of equation (28) and the proof is completed. This convergence theorem is important. It is because of this theorem that we can focus on ensuring that the approximation series converge.

VI. Conclusion

In this paper, the homotopy analysis method was applied to solve Onchocerciasis (riverblindness) and the theorem of convergence of homotopy analysis method was proved. Therefore, Homotopy Analysis Method can be a reliable and powerful method to obtain the analytical solution of which is a non-linear differential equation with complicated nonlinearity. The ability of the Homotopy Analysis Method is mainly due to the fact that the method provides a way to ensure the convergence of the series solution.

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