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Applications for Non expansive and monotone sequence of mappings with Viscosity approximation

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Abstract: *In this article we establish the application for viscosity approximation methods for Nonexpansive sequence of mappings. we associate converges strongly to a common element of the set of fixed points of sequence of mappings and also the set of solutions of the Variational inequality for an inverse strongly-monotone sequence of mappings which solves some Variational inequality.*

Keywords: *Viscosity approximation; fixed point; Inverse-strongly monotone mapping; Nonexpansive mapping; Variational inequalities*

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I. Introduction

In this work we give a generalized for Nonexpansive and monotone sequence of mappings with applications ,we supposed a closed convex subset of a real Hilbert space H denoted by C. and the metric projection of H onto C by P_C.Recall that a self-mapping $f: C \to C$ is a contraction on Cif there is a constant $0 < \epsilon < 1$ such that

$$\|f\ (u_m)\ - f\ (u_{m+1})\| \leq (1-\epsilon)\|u_m - u_{m+1}\|\ , \quad u_m, u_{m+1} \in C.$$

Π_C denotes the set of all contractions on C. Note that fhas aexclusive fixed point in C.

A mapping A of C into H is called monotone sequence if $\langle Au_m - Au_{m+1}, u_m - u_{m+1}\rangle \geq 0$, for all$u_m, u_{m+1} \in C$.The variational inequality problem is to find $u_m \in C$ such that $\langle Au_m, u_{m+1} - u_m\rangle \geq 0$ for all $u_{m+1} \in C$ (See [1,2]). The series of solutions of the variational inequality is denoted by $\mathrm{VI}(C, A)$. A mapping Aof CtoHis called inverse-strongly monotone of sequence if we have $\left(\frac{\lambda+\epsilon}{2}\right) \in \mathbb{R}^+$such that

$$\langle u_m - u_{m+1}, Au_m - Au_{m+1}\rangle \geq \frac{\lambda+\epsilon}{2}\|Au_m - Au_{m+1}\|^2$$

for all $u_m, u_{m+1} \in C$. For such a case, Ais $\frac{\lambda+\epsilon}{2}$-inverse-strongly monotone sequence .

A mapping Sof Cinto itself is called nonexpansive of sequence if $\|Su_m - Su_{m+1}\| \leq \|u_m - u_{m+1}\|$for all $u_m, u_{m+1} \in C$(Ref. [3]). We denoted by F(S)the set of fixed points of S.

The viscosity approximation methodology of choosing a selected fastened purpose of given Nonexpansive sequence of mapping was planned by Moudafi [4] established the subsequent sturdy convergence of each the implicit and specific method in Hilbert space.

Theorem 1.1. In a Hilbert space define$\{(u_m)_n\}$by implicit way

$$(u_m)_n = \frac{1}{1+\varepsilon_n}T(u_m)_n + \frac{\varepsilon_n}{1+\varepsilon_n}f\ ((u_m)_n),$$

where ε_n is a sequence in $(0,1)$ tending to zero. Then $\{(u_m)_n\}$ converges strongly to the exclusive solution $\widetilde{(u_m)} \in C$ of the variational inequality

$$\langle (I-f\ )\widetilde{(u_m)},\ \widetilde{(u_m)} - (u_m)\rangle \leq 0.$$

In other words, $\widetilde{(u_m)}$is the exclusive fixed point of $P_{Fix(T\ )}f$.

Theorem1.2. In a Hilbert space define$\{(u_m)_n\}$by $((u_m)_0 \in C \ is arbitrary)$

$$(u_m)_{n+1} = \frac{1}{1+\varepsilon_n} T(u_m)_n + \frac{\varepsilon_n}{1+\varepsilon_n} f\ ((u_m)_n),$$

Suppose that $\{\varepsilon_n\}$ satisfies the conditions

$$\lim_{n\to\infty} \varepsilon_n = 0, \qquad \sum_{n=1}^{\infty} \varepsilon_n = \infty; \ \lim_{n\to\infty} \left|\frac{1}{\varepsilon_n} - \frac{1}{\varepsilon_{n-1}}\right| = 0.$$

Then $\{(u_m)_n\}$converges strongly to the exclusive solution $\widetilde{(u_m)} \in C$ of the variational inequality $\langle (I - f)\widetilde{(u_m)},\ \widetilde{(u_m)} - (u_m)\rangle \le 0.$
In alternative words, $\widetilde{(u_m)}$ is the exclusive fixed point of $P_{Fix(T)}f$.

Theorem 1.3. (See [5].)Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive sequence of mappings with $F(T) \neq \emptyset$ and $f \in \Pi_C$. Let $\{(u_m)_t\}$be given by

$$(u_m)_t = tf\ ((u_m)_t) + (1-t)T\ (u_m)_t, \quad t \in (0,1).$$

Then:

$$\text{(i) } s - \lim_{t\to 0}(u_m)_t =: \widetilde{(u_m)} \text{ exists;}$$

$(ii)\widetilde{(u_m)} = P_S f\ \left(\widetilde{(u_m)}\right)$,or equivalently,$\widetilde{(u_m)}$is the exclusive solution in $F(T)$ to the variational inequality

$$\langle (I - f)\widetilde{(u_m)}, (u_m) - \widetilde{(u_m)}\rangle \ge 0, \qquad (u_m) \in S,$$

Where $S = F(T)$ and P_S is the metric projection from H to S.

Theorem 1.4. (See [5].)Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive sequence of mappings with $F(T) \neq \emptyset$, and$f : C \to C$ a contraction.Let $\{(u_m)_n\}$be given by

$$(u_m)_0 \in C, \qquad (u_m)_{n+1} = \left(\frac{\lambda+\epsilon}{2}\right)_n f((u_m)_n) + \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right) T(u_m)_n, \ n \ge 0.$$

Then under the following hypotheses

$$(H1)\ \left(\frac{\lambda+\epsilon}{2}\right)_n \to 0;$$

$$(H2) \sum_{n=0}^{\infty} \left(\frac{\lambda+\epsilon}{2}\right)_n = \infty;$$

$$(H3)\ either \sum_{n=0}^{\infty} \left|\left(\frac{\lambda+\epsilon}{2}\right)_{n+1} - \left(\frac{\lambda+\epsilon}{2}\right)_n\right| < \infty \ or \lim_{n\to\infty} \frac{\left(\frac{\lambda+\epsilon}{2}\right)_{n+1}}{\left(\frac{\lambda+\epsilon}{2}\right)_n} = 1,$$

$(u_m)_n \to \widetilde{(u_m)}$,where$\widetilde{(u_m)}$ is the exclusive solution of the variational inequality

$$\langle (I - f)\widetilde{(u_m)}, (u_m) - \widetilde{(u_m)}\rangle \ge 0, \qquad (u_m) \in S.$$

We verified the method of [11] by introducing the monotone sequence of mappings. With a little change.

II. Preliminaries

Let H be a real Hilbert space with inner product $\langle .\ ,\ .\rangle$and norm $\|\cdot\|$, and let$C$be a closed convex subset of H. We write $(u_m)_n \rightharpoonup (u_m)$to indicate that the sequence$\{(u_m)_n\}$converges weakly to (u_m). $(u_m)_n \to (u_m)$implies that $\{(u_m)_n\}$converges strongly to (u_m). For every point $(u_m) \in H$, there exists a exclusive nearest point in C, denoted by $P_C(u_m)$, such that

$$\|u_m - P_C u_m\| \le \|u_m - u_{m+1}\|$$

for all $u_{m+1} \in C$. $P_C$is called the metric projection of $H$to C. It is well known that $P_C$satisfies

$$\langle u_m - u_{m+1}, P_C u_m - P_C u_{m+1}\rangle \ge \|P_C u_m - P_C u_{m+1}\|^2 \qquad (1)$$

For every u_m , $u_{m+1} \in H$,and$P_C$is characterized by the following properties:

$$\langle u_m - P_C u_m, P_C u_m - u_{m+1} \rangle \geq 0, \tag{2}$$
$$\|u_m - u_{m+1}\|^2 \geq \|u_m - P_C u_m\|^2 + \|u_{m+1} - P_C u_m\|^2 \tag{3}$$

for all $u_m \in H, u_{m+1} \in C$. In the context of the variational inequality problem,
This implies

$$u_m \in VI(C,A) \Leftrightarrow u_m = P_C(u_m - \lambda A u_m), \ \forall \lambda > 0. \tag{4}$$

It is well known that Hsatisfies the Opial condition (Ref. [6]), i.e., for any sequence $\{(u_m)_n\}$ with $(u_m)_n \rightharpoonup (u_m)$ the inequality

$$\lim_{n\to\infty} inf \|(u_m)_n - (u_m)\| < \lim_{n\to\infty} inf \|(u_m)_n - u_{m+1}\|$$

holds for every $u_{m+1} \in H$with$u_{m+1} \neq u_m$. If$A$is an $\frac{\lambda+\epsilon}{2}$-inverse-strongly monotonesequence of mappings of $C$to H, then it is obvious that $A$is $\frac{2}{\lambda+\epsilon}$-Lipschitz continuous. We also have that for all $u_m, u_{m+1} \in C$and $\lambda > 0$,

$$\|(I - \lambda A)u_m - (I - \lambda A)u_{m+1}\|^2 = \|(u_m - u_{m+1}) - \lambda(Au_m - Au_{m+1})\|^2$$
$$= \|u_m - u_{m+1}\|^2 - 2\lambda\langle u_m - u_{m+1}, Au_m - Au_{m+1}\rangle + \lambda^2\|Au_m - Au_{m+1}\|^2$$
$$\leq \|u_m - u_{m+1}\|^2 - \epsilon\lambda\|Au_m - Au_{m+1}\|^2.$$

So, if $\frac{\lambda+\epsilon}{2}$ is given , then $I - \lambda A$ is a nonexpansive sequence of mappings of C into H.

A set-valued mapping $T : H \to 2^H$is called monotone sequence if for all $u_m, u_{m+1} \in H, f \in Tu_m$and $g \in Tu_{m+1}$imply $\langle u_m - u_{m+1}, f - g\rangle \geq 0$. A monotonesequence of mapping $T : H \to 2^H$is maximal if graph $G(T)$of Tisnot properly contained in the graph of any other monotone sequence of mapping. It is known that a monotone of sequence of mapping $T$is maximal if and only if for $(u_m, f) \in H \times H, \langle u_m - u_{m+1}, f - g\rangle \geq 0$ for every $(u_{m+1}, g) \in G(T)$implies$f \in Tu_m$. Let A is an inverse-strongly monotone sequence of mapping of C to H and $let\ N_C u_{m+1}$ be normal cone to C at $u_{m+1} \in C$, i.e., $N_C u_{m+1} = \{u_{m+2} \in H: \langle u_{m+1} - u_m, u_{m+2}\rangle \geq 0$, $\forall u_m \in C\}$, and define

$$Tu_{m+1} = \begin{cases} Au_{m+1} + N_C u_{m+1}, & u_{m+1} \in C \ , \\ \emptyset, & u_{m+1} \notin C \ , \end{cases}$$

then T is maximal monotone sequence and $0 \in Tu_{m+1}$if and only if $u_{m+1} \in VI(C,A)$(See [7]).

III. Main results

In this section, we show a strong convergence theorem (see [11]) for nonexpansive sequence of mappings and inverse strongly monotone sequence of mappings.

Lemma 1. (See [8].)Let C be a closed convex subset of a real Hilbert space$H$ and let $T : C \to C$ be a nonexpansive sequence of mapping such that $Fix(T) \neq \emptyset$. If a sequence $\{(u_m)_n\}$ in C is such that $(u_m)_n \rightharpoonup (u_{m+3})$ and $(u_m)_n - T(u_m)_n \to 0$, then$(u_{m+3}) = T(u_{m+3})$.

Lemma 2. (See [9].)Let $\{s_n\}$be a sequence of nonnegative sequence of real numbers such that:

$$s_{n+1} \leq (1 - \lambda_n)s_n + \beta_n, \quad n \geq 0,$$

where $\{\lambda_n\}, \{\beta_n\}$ satisfy the condition

$$(i)\ \{\lambda_n\} \subset (0,1)\ and \sum_{n=1}^{\infty} \lambda_n = \infty\ ,$$
$$(ii) \lim_{n\to\infty} sup \frac{\beta_n}{\lambda_n} \leq 0\ or \sum_{n=1}^{\infty} |\beta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

Proposition 3.1. Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction with coefficient $(1-\epsilon)$ $(0 < \epsilon < 1)$, A an $\frac{\lambda+\epsilon}{2}$-inverse-strongly monotone sequence of mapping of C to H and let S be a nonexpansive sequence of mapping of C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Suppose $\{(u_m)_n\}$be sequences generated by$(u_m)_0 \in C$,

$$(u_m)_{n+1} = \left(\frac{\lambda+\epsilon}{2}\right)_n f\,((u_m)_n) + (1 - \left(\frac{\lambda+\epsilon}{2}\right)_n) SP_C((u_m)_n - \lambda_n A(u_m)_n)$$

for every $n = 0,1,2,\ldots$, where $\{\lambda_n\} \subset [a,b]and\left\{\left(\frac{\lambda+\epsilon}{2}\right)_n\right\}$is a sequence in $(0,1)$. If $\left\{\left(\frac{\lambda+\epsilon}{2}\right)_n\right\}$ and $\{\lambda_n\}$are chosen so that $\lambda_n \in [a,b]$for some a,b with $0 < a < b < (\lambda+\epsilon)$,

$$\lim_{n\to\infty}\left(\frac{\lambda+\epsilon}{2}\right)_n = 0\,, \sum_{n=1}^{\infty}\left(\frac{\lambda+\epsilon}{2}\right)_n = \infty\,, \sum \left|\left(\frac{\lambda+\epsilon}{2}\right)_{n+1} - \left(\frac{\lambda+\epsilon}{2}\right)_n\right| < \infty\,, \sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| < \infty,$$

then$\{(u_m)_n\}$converges strongly to $q \in F(S) \cap VI(C,A)$, which is the exclusive solution in the $F(S) \cap VI(C,A)$ to the following variational inequality

$$\langle (I - f)q, \epsilon\rangle \le 0, \quad (q-\epsilon) \in F(S) \cap VI(C,A).$$

Proof. Put $(u_{m+1})_n = P_C((u_m)_n - \lambda_n A(u_m)_n)$for every $n = 0,1,2,\ldots$. Let $u_m \in F(S) \cap VI(C,A)$.We have

$$\begin{aligned}\|(u_{m+1})_n - u_m\| &= \|P_C((u_m)_n - \lambda_n A(u_m)_n) - P_C(u_m - \lambda_n A u_m)\|\\ &\le \|((u_m)_n - \lambda_n A(u_m)_n) - (u_m - \lambda_n A u_m)\|\\ &\le \|(u_m)_n - u_m\|\end{aligned}$$

for every $n = 1,2,3,\ldots$. Then we have

$$\begin{aligned}\|(u_m)_{n+1} - u_m\| &= \left\|\left(\frac{\lambda+\epsilon}{2}\right)_n f\,((u_m)_n) + \left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right) S(u_{m+1})_n - u_m\right\|\\ &\le \left(\frac{\lambda+\epsilon}{2}\right)_n \|f\,((u_m)_n) - u_m\| + \left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\|S(u_{m+1})_n - u_m\|\\ &\le \left(\frac{\lambda+\epsilon}{2}\right)_n \|f\,((u_m)_n) - f\,(u_m)\|\\ &+ \left(\frac{\lambda+\epsilon}{2}\right)_n \|f\,(u_m) - u_m\| + (1 - \left(\frac{\lambda+\epsilon}{2}\right)_n)\|(u_{m+1})_n - u_m\|\\ &\le \left(\frac{\lambda+\epsilon}{2}\right)_n (1-\epsilon)\|(u_m)_n - u_m\| + \left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\|(u_m)_n - u_m\|\\ &+ \left(\frac{\lambda+\epsilon}{2}\right)_n \|f\,(u_m) - u_m\|\\ &= \left(1 - \epsilon\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\|(u_m)_n - u_m\| + \left(\frac{\lambda+\epsilon}{2}\right)_n \|f\,(u_m) - u_m\|\\ &\le max\left\{\|(u_m)_n - u_m\|, \frac{1}{\epsilon}\|f\,(u_m) - u_m\|\right\}.\end{aligned}$$

By induction,

$$\|(u_m)_n - u_m\| \le max\left\{\|(u_m)_0 - u_m\|, \frac{1}{\epsilon}\|f\,(u_m) - u_m\|\right\}, \;\; n \ge 0.$$

Therefore, $\{(u_m)_n\}$is bounded,$\{(u_{m+1})_n\}$,$\{S(u_{m+1})_n\}$, $\{A(u_m)_n\}$, $\{f\,((u_m)_n)\}$are also bounded. Since $I - \lambda_n A$is nonexpansive of sequence and $u_m = P_C(u_m - \lambda_n A u_m)$, we also have

$$\begin{aligned}\|(u_{m+1})_{n+1} - (u_{m+1})_n\| &\le \|((u_m)_{n+1} - \lambda_{n+1}A(u_m)_{n+1}) - ((u_m)_n - \lambda_n A\,(u_m)_n)\|\\ &\le \|((u_m)_{n+1} - \lambda_{n+1}A(u_m)_{n+1}) - ((u_m)_n - \lambda_{n+1}A(u_m)_n)\|\\ &+ |\lambda_n - \lambda_{n+1}|\|A(u_m)_n\|\\ &\le \|(u_m)_{n+1} - (u_m)_n\| + |\lambda_n - \lambda_{n+1}|\|A(u_m)_n\|\end{aligned}$$

for every $n = 1,2,3,\ldots$. So we obtain

$$\begin{aligned}\|(u_m)_{n+1} - (u_m)_n\| &= \Big\|\left(\frac{\lambda+\epsilon}{2}\right)_n f\,((u_m)_n) + \left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right) S(u_{m+1})_n\\ &- \left(\frac{\lambda+\epsilon}{2}\right)_{n-1} f\,((u_m)_{n-1}) - \left(1-\left(\frac{\lambda+\epsilon}{2}\right)_{n-1}\right) S(u_{m+1})_{n-1}\Big\|\\ = \Big\|\left(\left(\frac{\lambda+\epsilon}{2}\right)_n - \left(\frac{\lambda+\epsilon}{2}\right)_{n-1}\right)&(f\,((u_m)_{n-1}) - S(u_{m+1})_{n-1}) + \left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)(S(u_{m+1})_n - S(u_{m+1})_{n-1})\\ &+ \left(\frac{\lambda+\epsilon}{2}\right)_n (f\,((u_m)_n) - f\,((u_m)_{n-1}))\Big\|\\ &\le \left|\left(\frac{\lambda+\epsilon}{2}\right)_n - \left(\frac{\lambda+\epsilon}{2}\right)_{n-1}\right| \|f\,((u_m)_{n-1}) - S(u_{m+1})_{n-1}\|\end{aligned}$$

$$+(1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n)\|(u_{m+1})_n - (u_{m+1})_{n-1}\|$$
$$+ \left(\tfrac{\lambda+\epsilon}{2}\right)_n (1-\epsilon)\|(u_m)_n - (u_m)_{n-1}\|$$

$$\leq \left(1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n\right)\left(\|(u_m)_n - (u_m)_{n-1}\| + |\lambda_{n-1} - \lambda_n|\|A(u_m)_{n-1}\|\right)$$
$$+ \left|\left(\tfrac{\lambda+\epsilon}{2}\right)_n - \left(\tfrac{\lambda+\epsilon}{2}\right)_{n-1}\right| \|f((u_m)_{n-1}) - S(u_{m+1})_{n-1}\|$$
$$+ \left(\tfrac{\lambda+\epsilon}{2}\right)_n (1-\epsilon)\|(u_m)_n - (u_m)_{n-1}\|$$
$$\leq \left(1 - \epsilon\left(\tfrac{\lambda+\epsilon}{2}\right)_n\right)\|(u_m)_n - (u_m)_{n-1}\|$$
$$+L|\lambda_n - \lambda_{n-1}| + M\left|\left(\tfrac{\lambda+\epsilon}{2}\right)_n - \left(\tfrac{\lambda+\epsilon}{2}\right)_{n-1}\right|$$

For every $n = 0,1,2,\ldots,$ where $L = sup\{\|f((u_m)_n) - S(u_{m+1})_{n-1}\|: n \in N\}$, $M = sup\{\|A(u_m)_n\|: n \in N\}$, since $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| < \infty, \sum_{n=1}^{\infty}\left|\left(\tfrac{\lambda+\epsilon}{2}\right)_n - \left(\tfrac{\lambda+\epsilon}{2}\right)_{n-1}\right| < \infty$ in view of Lemma 2, we have $\lim_{n\to\infty}\|(u_m)_{n+1} - (u_m)_n\| = 0$. then we also obtain $\|(u_{m+1})_{n+1} - (u_{m+1})_n\| \to 0$

$$\|(u_m)_n - S(u_{m+1})_n\| \leq \|(u_m)_n - S(u_{m+1})_{n-1}\| + \|S(u_{m+1})_{n-1} - S(u_{m+1})_n\|$$
$$\leq \left(\frac{\lambda+\epsilon}{2}\right)_{n-1}\|f((u_m)_{n-1}) - S(u_{m+1})_{n-1}\| + \|(u_{m+1})_{n-1} - (u_{m+1})_n\|,$$

we have $\|(u_m)_n - S(u_{m+1})_n\| \to 0. For u_m \in F(S) \cap VI(C,A)$,

$$\|(u_m)_{n+1} - u_m\|^2 = \left\|\left(\tfrac{\lambda+\epsilon}{2}\right)_n f((u_m)_n) + \left(1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n\right)S(u_{m+1})_n - u_m\right\|^2$$
$$\leq \left(\tfrac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2 + (1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n)\|(u_{m+1})_n - u_m\|^2$$
$$\leq \left(\tfrac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2$$
$$+\left(1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n\right)[\|(u_m)_n - u_m\|^2 + \lambda_n(\lambda_n - (\lambda+\epsilon))\|A(u_m)_n - Au_m\|^2]$$
$$\leq \left(\tfrac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2 + \|(u_m)_n - u_m\|^2$$
$$+\left(1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n\right)a\big(b - (\lambda+\epsilon)\big)\|A(u_m)_n - Au_m\|^2.$$

So, we obtain

$$-(1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n)a(b - (\lambda+\epsilon))\|A(u_m)_n - Au_m\|^2$$
$$\leq \left(\tfrac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2$$
$$+(\|(u_m)_n - u_m\| + \|(u_m)_{n+1} - u_m\|)(\|(u_m)_n - u_m\| - \|(u_m)_{n+1} - u_m\|)$$
$$\leq \left(\tfrac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2$$
$$+\left(\|(u_m)_n - u_m\| + \|(u_m)_{n+1} - u_m\|\right)\|(u_m)_n - (u_m)_{n+1}\|.$$

Since $\left(\tfrac{\lambda+\epsilon}{2}\right)_n \to 0$ and $\|(u_m)_n - (u_m)_{n+1}\| \to 0$, then $\|A(u_m)_n - Au_m\| \to 0, n \to \infty$. Further, from (1), we obtain

$$\|(u_{m+1})_n - u_m\|^2 = \|P_C((u_m)_n - \lambda_n A(u_m)_n) - P_C(u_m - \lambda_n Au_m)\|^2$$
$$\leq \langle (u_m)_n - \lambda_n A(u_m)_n - (u_m - \lambda_n Au_m), (u_{m+1})_n - u_m\rangle$$
$$= \tfrac{1}{2}\Big\{\left\|\left((u_m)_n - \lambda_n A(u_m)_n\right) - (u_m - \lambda_n Au_m)\right\|^2$$
$$+\|(u_{m+1})_n - u_m\|^2$$
$$- \|((u_m)_n - \lambda_n A(u_m)_n) - (u_m - \lambda_n Au_m) - ((u_{m+1})_n - u_m)\|^2\Big\}$$
$$\leq \tfrac{1}{2}\{\|(u_m)_n - u_m\|^2 + \|(u_{m+1})_n - u_m\|^2 - \|(u_m)_n - (u_{m+1})_n\|^2$$
$$+2\lambda_n\langle (u_m)_n - (u_{m+1})_n, A(u_m)_n - Au_m\rangle - {\lambda_n}^2\|A(u_m)_n - Au_m\|^2\}.$$

So, we obtain

$$\|(u_{m+1})_n - u_m\|^2 \leq \|(u_m)_n - u_m\|^2 - \|(u_m)_n - (u_{m+1})_n\|^2$$
$$+2\lambda_n\langle (u_m)_n - (u_{m+1})_n, A(u_m)_n - Au_m\rangle - {\lambda_n}^2\|A(u_m)_n - Au_m\|^2.$$

And hence

$$\|(u_m)_{n+1} - u_m\|^2 \le \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2 + \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right) \left\|S(u_{m+1})_n - u_m\right\|^2$$
$$\le \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2 + (1 - \left(\frac{\lambda+\epsilon}{2}\right)_n)\|(u_{m+1})_n - u_m\|^2$$
$$\le \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - u_m\|^2 + \|(u_m)_n - u_m\|^2$$
$$-(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n)\left\|(u_m)_n - (u_{m+1})_n\right\|^2$$
$$+2\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\lambda_n\langle (u_m)_n - (u_{m+1})_n, A(u_m)_n - Au_m\rangle$$
$$-\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\lambda_n{}^2\|A(u_m)_n - Au_m\|^2.$$

Since $\left(\frac{\lambda+\epsilon}{2}\right)_n \to 0, \|(u_m)_{n+1} - (u_m)_n\| \to 0$ and $\|A(u_m)_n - Au_m\| \to 0$, we obtain $\left\|(u_m)_n - (u_{m+1})_n\right\| \to 0$.Choose a subsequence $\{(u_{m+1})_{n_i}\}$ of $\{(u_{m+1})_n\}$such that

$$\limsup_{n\to\infty} \langle f(q) - q, S(u_{m+1})_n - q\rangle = \lim_{i\to\infty}\langle f(q) - q, S(u_{m+1})_{n_i} - q\rangle$$

As$\{(u_{m+1})_{n_i}\}$is bounded, we have that a subsequence $\{(u_{m+1})_{n_{ij}}\}$of $\{(u_{m+1})_{n_i}\}$converges weakly to (u_{m+3}). We mayassume without loss of generality that $(u_{m+1})_{n_i} \rightharpoonup (u_{m+3})$.

Since$\|S(u_{m+1})_n - (u_{m+1})_n\| \to 0$, we obtain $S(u_{m+1})_{n_i} \rightharpoonup (u_{m+3})$.Then we can obtain $u_{m+3} \in F(S) \cap VI(C, A)$. In fact, let us first show that $u_{m+3} \in VI(C, A)$. Let

$$T\,u_{m+1} = \begin{cases} Au_{m+1} + N_C u_{m+1}, & u_{m+1} \in C \;, \\ \emptyset, & u_{m+1} \notin C \;, \end{cases}$$

Then $T$is maximalmonotone sequence. Let $(u_{m+1}, u_{m+2}) \in G(T)$. Since $u_{m+2} - Au_{m+1} \in N_C u_{m+1}$and $(u_{m+1})_n \in C$we have

$$\langle u_{m+1} - (u_{m+1})_n, u_{m+2} - Au_{m+1}\rangle \ge 0.$$

On the other hand, from $(u_{m+1})_n = P_C((u_m)_n - \lambda_n A(u_m)_n)$, we have$\langle u_{m+1} - (u_{m+1})_n, (u_{m+1})_n - ((u_m)_n - \lambda_n A(u_m)_n)\rangle \ge 0$ and hence

$$\langle u_{m+1} - (u_{m+1})_n, \frac{(u_{m+1})_n - (u_m)_n}{\lambda_n} + A(u_m)_n\rangle \ge 0 \;\;.$$

Therefore, we have

$$\langle u_{m+1} - (u_{m+1})_{n_i}, u_{m+2}\rangle \ge \langle u_{m+1} - (u_{m+1})_{n_i}, Au_{m+1}\rangle \ge \langle u_{m+1} - (u_{m+1})_{n_i}, Au_{m+1}\rangle$$
$$-\left\langle u_{m+1} - (u_{m+1})_{n_i}, \frac{(u_{m+1})_{n_i} - (u_m)_{n_i}}{\lambda_{n_i}} + A(u_m)_{n_i}\right\rangle$$
$$= \left\langle u_{m+1} - (u_{m+1})_{n_i}, Au_{m+1} - A(u_m)_{n_i} - \frac{(u_{m+1})_{n_i} - (u_m)_{n_i}}{\lambda_{n_i}}\right\rangle$$
$$= \langle u_{m+1} - (u_{m+1})_{n_i}, Au_{m+1} - A(u_{m+1})_{n_i}\rangle$$
$$+ \langle u_{m+1} - (u_{m+1})_{n_i}, A(u_{m+1})_{n_i} - A(u_m)_{n_i}\rangle$$
$$-\left\langle u_{m+1} - (u_{m+1})_{n_i}, \frac{(u_{m+1})_{n_i} - (u_m)_{n_i}}{\lambda_{n_i}}\right\rangle$$
$$\ge \langle u_{m+1} - (u_{m+1})_{n_i}, A(u_{m+1})_{n_i} - A(u_m)_{n_i}\rangle$$
$$-\left\langle u_{m+1} - (u_{m+1})_{n_i}, \frac{(u_{m+1})_{n_i} - (u_m)_{n_i}}{\lambda_{n_i}}\right\rangle \;.$$

Hence we have $\langle u_{m+1} - u_{m+3}, u_{m+2}\rangle \ge 0$ as $i \to \infty$. since $T$is maximal monotone sequence, we have $u_{m+3} \in T^{-1}0$ andhence $u_{m+3} \in VI(C, A)$

$$\left\|(u_m)_n - S(u_m)_n\right\| \le \left\|(u_m)_n - S(u_{m+1})_n\right\| + \left\|S(u_{m+1})_n - S(u_m)_n\right\|$$
$$\le \|(u_m)_n - S(u_{m+1})_n\| + \|(u_m)_n - (u_{m+1})_n\|,$$

We have $\|(u_m)_n - S(u_m)_n\| \to 0$. In view of Lemma 1, we obtain $u_{m+3} \in F(S)$

$$\lim_{n\to\infty} sup \langle f(q) - q, S(u_{m+1})_n - q\rangle = \lim_{i\to\infty}\langle f(q) - q, S(u_{m+1})_{n_i} - q\rangle$$
$$= \langle f(q) - q, u_{m+3} - q\rangle \leq 0,$$

$$\|(u_m)_{n+1} - q\|^2 = \left\|\left(\tfrac{\lambda+\epsilon}{2}\right)_n f((u_m)_n) + \left(1-\left(\tfrac{\lambda+\epsilon}{2}\right)_n\right)S(u_{m+1})_n - q\right\|^2 = \left(\tfrac{\lambda+\epsilon}{2}\right)_n^2 \|f((u_m)_n) - q\|^2$$
$$+2\left(\tfrac{\lambda+\epsilon}{2}\right)_n(1-\left(\tfrac{\lambda+\epsilon}{2}\right)_n)\langle f((u_m)_n) - q, S(u_{m+1})_n - q\rangle$$
$$+(1-\left(\tfrac{\lambda+\epsilon}{2}\right)_n)^2\|S(u_{m+1})_n - q\|^2$$
$$\leq \left(1-2\left(\tfrac{\lambda+\epsilon}{2}\right)_n + \left(\tfrac{\lambda+\epsilon}{2}\right)_n^2\right)\|(u_m)_n - q\|^2$$
$$+\left(\tfrac{\lambda+\epsilon}{2}\right)_n^2 \|f((u_m)_n) - q\|^2$$
$$+2\left(\tfrac{\lambda+\epsilon}{2}\right)_n\left(1-\left(\tfrac{\lambda+\epsilon}{2}\right)_n\right)\langle f((u_m)_n) - f(q), S(u_{m+1})_n - q\rangle$$
$$+2\left(\tfrac{\lambda+\epsilon}{2}\right)_n(1-\left(\tfrac{\lambda+\epsilon}{2}\right)_n)\langle f(q) - q, S(u_{m+1})_n - q\rangle$$
$$\leq \left[1-2\left(\tfrac{\lambda+\epsilon}{2}\right)_n + \left(\tfrac{\lambda+\epsilon}{2}\right)_n^2 + 2(1-\epsilon)\left(\tfrac{\lambda+\epsilon}{2}\right)_n\left(1-\left(\tfrac{\lambda+\epsilon}{2}\right)_n\right)\right]\|(u_m)_n - q\|^2$$
$$+\left(\tfrac{\lambda+\epsilon}{2}\right)_n^2 \|f((u_m)_n) - q\|^2$$
$$+2\left(\tfrac{\lambda+\epsilon}{2}\right)_n(1-\left(\tfrac{\lambda+\epsilon}{2}\right)_n)\langle f(q) - q, S(u_{m+1})_n - q\rangle$$
$$= (1-\overline{\left(\tfrac{\lambda+\epsilon}{2}\right)_n})\|(u_m)_n - q\|^2 + \overline{\left(\tfrac{\lambda+\epsilon}{2}\right)_n}\bar{\beta}_n,$$

where

$$\overline{\left(\frac{\lambda+\epsilon}{2}\right)_n} = \left(\frac{\lambda+\epsilon}{2}\right)_n\left[2-\left(\frac{\lambda+\epsilon}{2}\right)_n - 2(1-\epsilon)\left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)\right],$$

$$\bar{\beta}_n = \frac{\left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_m)_n) - q\|^2 + 2(1-\left(\frac{\lambda+\epsilon}{2}\right)_n)\langle f(q) - q, S(u_{m+1})_n - q\rangle}{2-\left(\frac{\lambda+\epsilon}{2}\right)_n - 2(1-\epsilon)(1-\left(\frac{\lambda+\epsilon}{2}\right)_n)}.$$

It is easily seen that $\overline{\left(\frac{\lambda+\epsilon}{2}\right)_n} \to 0$, $\sum_{n=1}^{\infty}\overline{\left(\frac{\lambda+\epsilon}{2}\right)_n} = \infty$,and $limsup_{n\to\infty}\bar{\beta}_n \leq 0$, by Lemma 2 we obtain$(u_m)_n \to q$. This completes the proof. □

$S$is a nonexpansive sequence of mapping, $A$is an $\frac{\lambda+\epsilon}{2}$-inverse strongly monotone sequence , and $f \in \Pi_C$.Thus, by Banach contraction mapping principle, there exists a exclusive fixed point (see[11])

$$(u_{m+3})_n^f = \left(\frac{\lambda+\epsilon}{2}\right)_n f\left((u_{m+3})_n^f\right) + \left(1-\left(\frac{\lambda+\epsilon}{2}\right)_n\right)SP_C\left((u_{m+3})_n^f - \lambda_n A(u_{m+3})_n^f\right), \quad \left(\frac{\lambda+\epsilon}{2}\right)_n \in (0,1).$$

For simplicity we will write $(u_{m+3})_n for (u_{m+3})_n^f$provided no confusion occurs. Next we show the convergenceof $\{(u_{m+3})_n\}$, (see [11]) whiles they claim the existence of the $q \in F(S) \cap VI(C,A)$which solves thevariational inequality

$$\langle (I - f)q, \epsilon\rangle \leq 0, \quad f \in \Pi_C, \qquad (q-\epsilon) \in F(S) \cap VI(C,A).$$

Theorem 3.1. Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction with coefficient $(1-\epsilon)(0 < \epsilon < 1)$, A an $\frac{\lambda+\epsilon}{2}$-inverse-strongly monotone sequence of mapping of C to H and let S be a nonexpansive sequence of mapping of C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Suppose $\{(u_{m+3})_n\}$,be sequences generated by

$$(u_{m+3})_n = \left(\tfrac{\lambda+\epsilon}{2}\right)_n f((u_{m+3})_n) + (1-\left(\tfrac{\lambda+\epsilon}{2}\right)_n)SP_C((u_{m+3})_n - \lambda_n A(u_{m+3})_n), \quad \left(\tfrac{\lambda+\epsilon}{2}\right)_n \in (0,1),$$

where$\{\lambda_n\} \subset [a,b]$ and $\left\{\left(\frac{\lambda+\epsilon}{2}\right)_n\right\}$is a sequence in $[0,1)$.If $\{\left(\frac{\lambda+\epsilon}{2}\right)_n\}$and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$for some a,b with $0 < a < b < (\lambda+\epsilon)$, when $\lim_{n\to\infty}\left(\frac{\lambda+\epsilon}{2}\right)_n = 0$, $(u_{m+3})_n$ converges strongly to q, andsuch that the variational inequality

$$\langle (I - f)q, \epsilon\rangle \le 0, \quad f \in \Pi_C, \quad (q-\epsilon) \in F(S) \cap VI(C,A).$$

Proof. Put $(u_{m+1})_n = P_C((u_{m+3})_n - \lambda_n A(u_{m+3})_n)$for every $n = 0,1,2,\ldots$. Let $u_m \in F(S) \cap VI(C,A)$.We have

$$\begin{aligned}\|(u_{m+1})_n - u_m\| &= \|P_C((u_{m+3})_n - \lambda_n A(u_{m+3})_n) - P_C(u_m - \lambda_n A u_m)\|\\ &\le \|((u_{m+3})_n - \lambda_n A(u_{m+3})_n) - (u_m - \lambda_n A u_m)\|\\ &\le \|(u_{m+3})_n - u_m\|\end{aligned}$$

for every $n = 1,2,3,\ldots$. Then we have

$$\begin{aligned}\|(u_{m+3})_n - u_m\| &= \left\|\left(\tfrac{\lambda+\epsilon}{2}\right)_n f((u_{m+3})_n) + (1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n)S(u_{m+1})_n - u_m\right\|\\ &\le \left(\tfrac{\lambda+\epsilon}{2}\right)_n \|f((u_{m+3})_n) - u_m\| + (1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n)\|S(u_{m+1})_n - u_m\|\\ &\le \left(\tfrac{\lambda+\epsilon}{2}\right)_n \|f((u_{m+3})_n) - f(u_m)\|\\ &+ \left(\tfrac{\lambda+\epsilon}{2}\right)_n \|f(u_m) - u_m\| + (1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n)\|(u_{m+1})_n - u_m\|\\ &\le \left(\tfrac{\lambda+\epsilon}{2}\right)_n (1-\epsilon)\|(u_{m+3})_n - u_m\| + \left(1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n\right)\|(u_{m+3})_n - u_m\|\\ &+ \left(\tfrac{\lambda+\epsilon}{2}\right)_n \|f(u_m) - u_m\|.\end{aligned}$$

Hence,

$$\|(u_{m+3})_n - u_m\| \le \frac{1}{\epsilon}\|f(u_m) - u_m\|$$

and$\{(u_{m+3})_n\}$is bounded, $\{(u_{m+1})_n\}, \{S(u_{m+1})_n\}, \{A(u_{m+3})_n\}$ and $\{f((u_{m+3})_n)\}$ are also bounded.

$$\begin{aligned}\|(u_{m+3})_n - u_m\|^2 &= \left\|\left(\tfrac{\lambda+\epsilon}{2}\right)_n f((u_{m+3})_n) + (1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n)S(u_{m+1})_n - u_m\right\|^2\\ &\le \left(\tfrac{\lambda+\epsilon}{2}\right)_n \|f((u_{m+3})_n) - u_m\|^2 + (1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n)\|(u_{m+1})_n - u_m\|^2\\ &\le \left(\tfrac{\lambda+\epsilon}{2}\right)_n \|f((u_{m+3})_n) - u_m\|^2\\ &+ \left(1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n\right)\left[\|(u_{m+3})_n - u_m\|^2 + \lambda_n(\lambda_n - (\lambda+\epsilon))\|A(u_{m+3})_n - Au_m\|^2\right]\\ &\le \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_{m+3})_n) - u_m\|^2 + \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\|(u_{m+3})_n - u_m\|^2\\ &+ \left(1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n\right) a(b - (\lambda+\epsilon))\|A(u_{m+3})_n - Au_m\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&-\left(1 - \left(\tfrac{\lambda+\epsilon}{2}\right)_n\right) a\left(b - (\lambda+\epsilon)\right)\|A(u_{m+3})_n - Au_m\|^2\\ &\le \left(\tfrac{\lambda+\epsilon}{2}\right)_n \left(\|f((u_{m+3})_n) - u_m\|^2 + \|(u_{m+3})_n - u_m\|^2\right).\end{aligned}$$

Since $\left(\frac{\lambda+\epsilon}{2}\right)_n \to 0$ $(n\to\infty)$, and $\{f((u_{m+3})_n)\}, \{(u_{m+3})_n\}$are bounded, we obtain

$$\|A(u_{m+3})_n - Au_m\| \to 0 \quad (n\to\infty).$$

From (1) we have

$$\begin{aligned}\|(u_{m+1})_n - u_m\|^2 &= \|P_C((u_{m+3})_n - \lambda_n A(u_{m+3})_n) - P_C(u_m - \lambda_n A u_m)\|^2\\ &\le \langle (u_{m+3})_n - \lambda_n A(u_{m+3})_n - (u_m - \lambda_n A u_m), (u_{m+1})_n - u_m\rangle\end{aligned}$$

$$= \frac{1}{2}\{\|((u_{m+3})_n - \lambda_n A(u_{m+3})_n) - (u_m - \lambda_n A u_m)\|^2 + \|(u_{m+1})_n - u_m\|^2 - \|((u_{m+3})_n - \lambda_n A(u_{m+3})_n) - (u_m - \lambda_n A u_m) - ((u_{m+1})_n - u_m)\|^2\}$$

$$\leq \frac{1}{2}\{\|(u_{m+3})_n - u_m\|^2 + \|(u_{m+1})_n - u_m\|^2 - \|(u_{m+3})_n - (u_{m+1})_n\|^2 + 2\lambda_n\langle (u_{m+3})_n - (u_{m+1})_n, A(u_{m+3})_n - Au_m\rangle - {\lambda_n}^2\|A(u_{m+3})_n - Au_m\|^2\}.$$

So, we obtain

$$\|(u_{m+1})_n - u_m\|^2 \leq \|(u_{m+3})_n - u_m\|^2 - \|(u_{m+3})_n - (u_{m+1})_n\|^2 + 2\lambda_n\langle (u_{m+3})_n - (u_{m+1})_n, A(u_{m+3})_n - Au_m\rangle - {\lambda_n}^2\|A(u_{m+3})_n - Au_m\|^2.$$

So we have

$$\begin{aligned}\left\|(u_{m+3})_n - u_m\right\|^2 &\leq \left(\frac{\lambda+\epsilon}{2}\right)_n \left\|f((u_{m+3})_n) - u_m\right\|^2 + (1 - \left(\frac{\lambda+\epsilon}{2}\right)_n)\|S(u_{m+1})_n - u_m\|^2 \\ &\leq \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_{m+3})_n) - u_m\|^2 + (1 - \left(\frac{\lambda+\epsilon}{2}\right)_n)\|(u_{m+1})_n - u_m\|^2 \\ &\leq \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_{m+3})_n) - u_m\|^2 + \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\|(u_{m+3})_n - u_m\|^2 \\ &\quad - \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\|(u_{m+3})_n - (u_{m+1})_n\|^2 \\ &\quad + 2\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\lambda_n\langle (u_{m+3})_n - (u_{m+1})_n, A(u_{m+3})_n - Au_m\rangle \\ &\quad - \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right){\lambda_n}^2\|A(u_{m+3})_n - Au_m\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\left\|(u_{m+3})_n - (u_{m+1})_n\right\|^2 &\leq \left(\frac{\lambda+\epsilon}{2}\right)_n \|f((u_{m+3})_n) - u_m\|^2 - \left(\frac{\lambda+\epsilon}{2}\right)_n \|(u_{m+3})_n - u_m\|^2 \\ &\quad + 2\left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_n\right)\lambda_n\langle (u_{m+3})_n - (u_{m+1})_n, A(u_{m+3})_n - Au_m\rangle \\ &\quad - {\lambda_n}^2\|A(u_{m+3})_n - Au_m\|^2.\end{aligned}$$

Since $\left(\frac{\lambda+\epsilon}{2}\right)_n \to 0, \|A(u_{m+3})_n - Au_m\| \to 0$,we obtain$\|(u_{m+3})_n - (u_{m+1})_n\| \to 0$ $(n \to \infty)$.By the proof of Proposition 3.1we have $(u_{m+1})_{n_i} \rightharpoonup q$and $q \in F(S) \cap VI(C, A)$, so $(u_{m+3})_{n_i} \rightharpoonup q$

$$\begin{aligned}\left\|(u_{m+3})_{n_i} - q\right\|^2 &= \left\|\left(\frac{\lambda+\epsilon}{2}\right)_{n_i} f((u_{m+3})_{n_i}) + (1 - \left(\frac{\lambda+\epsilon}{2}\right)_{n_i})S(u_{m+1})_{n_i} - q\right\|^2 \\ &= \langle \left(\frac{\lambda+\epsilon}{2}\right)_{n_i}\left(f\left((u_{m+3})_{n_i}\right) - q\right) + \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_{n_i}\right)\left(S(u_{m+1})_{n_i} - q\right), (u_{m+3})_{n_i} - q\rangle \\ &= \left(\frac{\lambda+\epsilon}{2}\right)_{n_i}\langle f\left((u_{m+3})_{n_i}\right) - q, (u_{m+3})_{n_i} - q\rangle \\ &\quad + \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_{n_i}\right)\langle S(u_{m+1})_{n_i} - q, (u_{m+3})_{n_i} - q\rangle \\ &\leq \left(1 - \left(\frac{\lambda+\epsilon}{2}\right)_{n_i}\right)\left\|(u_{m+3})_{n_i} - q\right\|^2 \\ &\quad + \left(\frac{\lambda+\epsilon}{2}\right)_{n_i}\langle f((u_{m+3})_{n_i}) - q, (u_{m+3})_{n_i} - q\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\left\|(u_{m+3})_{n_i} - q\right\|^2 &\leq \langle f((u_{m+3})_{n_i}) - q, (u_{m+3})_{n_i} - q\rangle = \langle f\left((u_{m+3})_{n_i}\right) - f(q), (u_{m+3})_{n_i} - q\rangle \\ &\quad + \langle f(q) - q, (u_{m+3})_{n_i} - q\rangle \leq (1 - \epsilon)\left\|(u_{m+3})_{n_i} - q\right\|^2 + \langle f(q) - q, (u_{m+3})_n - q\rangle.\end{aligned}$$

This implies that

$$\left\|(u_{m+3})_{n_i} - q\right\|^2 \leq \frac{1}{\epsilon}\langle (u_{m+3})_{n_i} - q, f(q) - q\rangle.$$

But $(u_{m+3})_{n_i} \rightharpoonup q$,it follows that $(u_{m+3})_{n_i} \to q$.Now we show that $q$solves the variational inequality

$$\langle (I - f)q, \epsilon\rangle \leq 0, \quad f \in \Pi_C, (q - \epsilon) \in F(S) \cap VI(C, A).$$

Because

$$(u_{m+3})_n - f\,((u_{m+3})_n) = -\frac{1-\left(\frac{\lambda+\epsilon}{2}\right)_n}{\left(\frac{\lambda+\epsilon}{2}\right)_n}((u_{m+3})_n - S(u_{m+1})_n),$$

For any $(q-\epsilon) \in F(S) \cap VI(C,A)$and notice$(q-\epsilon) = P_C((q-\epsilon) - \lambda_n A(q-\epsilon))$, we infer that

$\langle (u_{m+3})_n\text{-} f\,((u_{m+3})_n), (u_{m+3})_n\text{-}\ (q-\epsilon)\rangle$

$$= -\frac{1-\left(\frac{\lambda+\epsilon}{2}\right)_n}{\left(\frac{\lambda+\epsilon}{2}\right)_n}\langle (u_{m+3})_n\text{-} SP_C\big((u_{m+3})_n\text{-}\lambda_n A(u_{m+3})_n\big), (u_{m+3})_n\text{-} q\rangle$$

$$= -\frac{1-\left(\frac{\lambda+\epsilon}{2}\right)_n}{\left(\frac{\lambda+\epsilon}{2}\right)_n}\langle (u_{m+3})_n\text{-} SP_C((u_{m+3})_n\text{-}\lambda_n A(u_{m+3})_n) - \Big((q-\epsilon)\text{-} SP_C\big((q-\epsilon)\text{-}\lambda_n A(q-\epsilon)\big)\Big), (u_{m+3})_n\text{-}\ (q-\epsilon)\rangle \le 0\,,$$

Since $I - SP_C(I - \lambda_n A)$is strong monotone sequence. Let $i \to \infty$, we have

$$\langle q - f\,(q), \epsilon\rangle \le 0. \tag{5}$$

Assume that there exists another subsequence $\{(u_{m+3})_{n_j}\}$of $\{(u_{m+3})_n\}$such that $(u_{m+3})_{n_j} \to q^*$, $soq^* \in F(S) \cap VI(C,A)$,and from$\langle (u_{m+3})_n - f((u_{m+3})_n), (u_{m+3})_n - (q-\epsilon)\rangle \le 0$,let $j \to \infty$We have

$$\langle q^* - f(q^*), q^* - (q-\epsilon)\rangle \le 0\ ,\ (q-\epsilon) \in F(S) \cap VI(C,A). \tag{6}$$

Setting $(q-\epsilon) = q^*$in (5), we have

$$\langle q - f\,(q), q - q^*\rangle \le 0, \tag{7}$$

and setting $\epsilon = 0$in (6), we obtain

$$\langle q^* - f(q^*), q^* - q\rangle \le 0. \tag{8}$$

Inequality (7) and (8) yield

$$\|q - q^*\|^2 \le \langle f\,(q) - f(q^*), q - q^*\rangle \le (1-\epsilon)\|q - q^*\|^2\ ,$$

Which implies that $q = q^*$,since$0 < \epsilon < 1$Thus, $(u_{m+3})_n \to q\ as\ n \to \infty$and $q \in F(S) \cap VI(C,A)$ is exclusive. And q is the exclusive solution of variational inequality

$$\langle q - f\,(q), \epsilon\rangle \le 0, \qquad (q-\epsilon) \in F(S) \cap VI(C,A).$$

This completes the proof. □

4.Applications

We show two theorems in a Hilbert space by using Proposition 3.1and Theorem 3.1.(see [11,10]). A mapping $T^2 : C \to C$is called strictly pseudocontractive and projection if there exists $(1-\epsilon)$with $0 \le \epsilon < 1$ such that

$$\|T^2u_m\text{-} T^2u_{m+1}\|^2 \le \|u_m\text{-} u_{m+1}\|^2 + (1-\epsilon)\|(I - T^2)u_m - (I - T^2)u_{m+1}\|^2$$

For every$u_m, u_{m+1} \in C$. If$\epsilon = 1$,then T^2 is nonexpansive of sequence. Put $A^2 = I - T^2$, where $T^2 : C \to C$is a strictly pseudocontractive and a projection mapping with $(1-\epsilon)$. Then $A^2$is $\frac{\epsilon^2}{4}$ -inverse-strongly monotone sequence. Actually, we have, for all $u_m, u_{m+1} \in C$,

$$\|(I - A^2)u_m\text{-} (I - A^2)u_{m+1}\|^2 \le \|u_m - u_{m+1}\|^2 + (1-\epsilon)\|A^2u_m - A^2u_{m+1}\|^2.$$

On the other hand, since H is a real Hilbert space, we have

$$\left\|(I - A^2)u_m\text{-} (I - A^2)u_{m+1}\right\|^2 = \left\|u_m\text{-}\ u_{m+1}\right\|^2 + \|A^2u_m - A^2u_{m+1}\|^2 - 2\langle u_m - u_{m+1}, A^2u_m - A^2u_{m+1}\rangle.$$

Hence we have

$$\langle u_m - u_{m+1}, A^2u_m - A^2u_{m+1}\rangle \ge \frac{\epsilon^2}{4}\|A^2u_m - A^2u_{m+1}\|^2.$$

Using Proposition 3.1and Theorem 3.1,we first show a strong convergence theorem (see [11]) for finding a common fixed point of a nonexpansive sequence of mapping and a strictly pseudocontractive and projection mapping.

Theorem 4.1. Let C be a closed convex set of a real Hilbert space$H$. Let f be a contractive mapping of C into itself with coefficient $0 < \epsilon < 1$,S be a nonexpansive sequence of mapping of C into itself and let T^2 be a strictly pseudocontractive and projection mapping of Cinto itself with $\left(\frac{\lambda+\epsilon}{2}\right)$, such that $F(S^2) \cap F(T^2) \neq \emptyset$ Suppose $(u_m)_1 = u_m \in C$ and $\{(u_m)_n\}$is given by

$$(u_m)_{n+1} = \left(\frac{\lambda+\epsilon}{2}\right)_n f\,((u_m)_n) + (1 - \left(\frac{\lambda+\epsilon}{2}\right)_n) S^2((1 - \lambda_n)(u_m)_n + \lambda_n T^2(u_m)_n)$$

For every $n = 1, 2, \ldots$, where$\left\{\left(\frac{\lambda+\epsilon}{2}\right)_n\right\}$is a sequence in$[0,1)$ and $\{\lambda_n\}$is a sequence in$\left[0, \frac{2-(\lambda+\epsilon)}{2}\right)$. If $\left\{\left(\frac{\lambda+\epsilon}{2}\right)_n\right\}$and $\{\lambda_n\}$are chosen so that $\lambda_n \in [a,b]$for some a, b with $0 < 2a < 2b < 2 - (\lambda + \epsilon)$,

$$\lim_{n\to\infty} \left(\frac{\lambda+\epsilon}{2}\right)_n = 0\,, \sum_{n=1}^{\infty} \left(\frac{\lambda+\epsilon}{2}\right)_n = \infty\,, \sum |\left(\frac{\lambda+\epsilon}{2}\right)_{n+1} - \left(\frac{\lambda+\epsilon}{2}\right)_n| < \infty\,, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{(u_m)_n\}$converges strongly to $q \in F(S^2) \cap F(T^2)$, such that

$$\langle f\,(q) - q, \epsilon\rangle \leq 0, \qquad (q - \epsilon) \in F(S^2) \cap F(T^2).$$

Proof. Put $A^2 = I - T^2$. Then $A^2$is $\frac{2-(\lambda+\epsilon)}{4}$ -inverse-strongly monotone sequence. We have $F(T^2) = VI(C, A^2)$and $P_C((u_m)_n - \lambda_n A^2(u_m)_n) = (1 - \lambda_n)(u_m)_n + \lambda_n T^2(u_m)_n$.So by Proposition 3.1and Theorem 3.1, (see [11]).we obtainthe desired result. □

Theorem 4.2. Let H be a real Hilbert space H. Let f be a contractive mapping of H into itself with coefficient $0 < \epsilon < 1$, S^2 be a nonexpansive sequence mapping of H into itself and let A^2be a contraction and projection of a $\left(\frac{\lambda+\epsilon}{2}\right)$-inverse strongly monotone sequence of mappings of H into itself such that $F(S^2) \cap (A^2)^{-1}0 \neq \emptyset$.Suppose$(u_m)_1 = (u_m) \in C$ and $\{(u_m)_n\}$ is given by

$$(u_m)_{n+1} = \left(\frac{\lambda+\epsilon}{2}\right)_n f\,((u_m)_n) + (1 - \left(\frac{\lambda+\epsilon}{2}\right)_n) S^2((u_m)_n - \lambda_n A^2(u_m)_n)$$

for every $n = 1, 2, \ldots$, where$\left\{\left(\frac{\lambda+\epsilon}{2}\right)_n\right\}$is a sequence in $[0,1)$ and$\{\lambda_n\}$is a sequence in $[0, \lambda + \epsilon)$. If $\left\{\left(\frac{\lambda+\epsilon}{2}\right)_n\right\}$and $\{\lambda_n\}$are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < (\lambda + \epsilon)$,

$$\lim_{n\to\infty} \left(\frac{\lambda+\epsilon}{2}\right)_n = 0, \sum_{n=1}^{\infty} \left(\frac{\lambda+\epsilon}{2}\right)_n = \infty, \sum_{n=1}^{\infty} |\left(\frac{\lambda+\epsilon}{2}\right)_{n+1} - \left(\frac{\lambda+\epsilon}{2}\right)_n| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then$\{(u_m)_n\}$ converges strongly to $q \in F(S^2) \cap (A^2)^{-1}0$, such that

$$\langle f\,(q) - q, \epsilon\rangle, \quad (q - \epsilon) \in F(S^2) \cap (A^2)^{-1}0.$$

Proof. We have $(A^2)^{-1}0 = VI(C, A^2)$. so putting $P_H = I$, by Proposition 3.1 and Theorem 3.1, weobtain the desired result. (see [11]). □

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