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Generalized Fixed Point Theorem with C*-Algebra Valued Metric Space

A. Padma

Department of Mathematics,Chaitanya Bharathi Institute of Technology, Gandipet, Hyderabad,Telangana,India

Abstract: *In this present paper based on the concept and properties of c∗-algebras valued metric spaces and gives some fixed point theorems for self-maps with contractive conditions on such spaces. as applications, existence and uniqueness results for a type of integral equation and operator equation are given.*
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I. Introduction

We begin with the concept of C∗-algebras.

Suppose that $\mathbb{A}$ is a unital algebra with the unit I. An involution on A is a conjugatelinear map a → a∗ on A such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all a, b ∈ A. The pair ($\mathbb{A}$, ∗) is called a ∗-algebra. A Banach ∗-algebra is a ∗-algebra$\mathbb{A}$together with a complete sub-multiplicative norm such that $\|a^*\|=\|a\|(a \in \mathbb{A})$. A C*-algebra is a Banach ∗-algebra such that$\|a^*a\| = \|a\|^2$

Notice that the seeming mild requirement on a C*-algebra above is in fact very strong. Moreover, the existence of the involution C*-algebra theory can be thought of as unlimiteddimensional actual analysis. Clearly that under the norm topology, L(H), the set of all bounded linear operators on a Hilbert space H, is a C*-algebra. As we have known, the Banach contraction principle is a extremelyrealistic, simple and classical tool in modern analysis. Also it is an important tool for solving existence problems in many branches of mathematics and physics. In general, the theorem has been generalized in two directions. On the one side, the usual contractive (expansive) condition is replaced by weakly contractive (expansive) condition. On the other side, the action spaces are replaced by metric spaces endowed with an ordered or partially ordered structure. In recent years, O'Regan and Petrusel [3] started the investigations concerning a fixed point theory in ordered metric spaces. Later, many authors followed this research by introducing and investigating the different types of contractive mappings. For example in [4] Caballero et al. considered contractive like mapping in ordered metric spaces and applied their results in ordinary differential equations. In 2007, Huang and Zhang [5] introduced the concept of cone metric space, replacing the set of real numbers by an ordered Banach space. Later, many authors generalized their fixed point theorems on different type of metwhich permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited. Ma et al. the authors studied the operator-valued metric spaces and gave some fixed point theorems on the spaces. In this paper, we introduce a new type of metric spaces which generalize the concepts of metric spaces and operator-valued metric spaces, and give some related fixed point theorems for self-maps with contractive or expansive conditions on such spaces.

The paper is organized as follows: Based on the concept and properties of C*-algebras, we first introduce a concept of C*-algebra-valued metric spaces. Moreover, some fixed point theorems for mappings satisfying the contractive or expansive conditions on C*-algebra-valued metric spaces are established. Finally, as applications, existence and uniqueness results for a type of integral equation and operator equation are given.

II. Preliminaries

some basic definitions, which will be used later.

Throughout this paper, A will denote an unital C*-algebra with a unit I. Set $A_h = \{x \in A : x = x^*\}$. We call an element x ∈ A a positive element, denote it by $x \geq \theta$, if $x \in A_h$ and $\sigma(x) \subset R_+ = [0,\infty)$, where σ(x) is the spectrum of x. Using positive elements, one can define a partial ordering $\leq$on$\mathbb{A}_h$as follows :$x \leq y$ if and only if $y - x \geq \theta$, where θ means the zero element in A. From now on, by A+ we denote the set $\{x \in A : x \geq \theta\}$ and $|x| =(x^*x)^{1/2}$

Remark2.1 When $\mathbb{A}$ is a unital C*-algebra, then for any x $\in \mathbb{A}_+$ we have x $\leq$I$\Leftrightarrow \|x\| \leq 1$ [1,2]
With the help of the positive element in $\mathbb{A}$, one can give the definition of a C*-algebravalued metric space.

Definition 2.1
Let X be a nonempty set. Suppose the mapping d : X × X → $\mathbb{A}$ satisfies:
(1) θ ≤ d(x, y) for all x, y ∈ X and d(x, y) = θ ⇔ x = y;
(2) d(x, y) = d(y, x) for all x, y ∈ X;
(3)d(x,y)≤ d(x, z) + d(z, y) for all x, y, z ∈ X. Then d is called a C*-algebra-valued metric on X and (X,$\mathbb{A}$,d) is called a C∗-algebravalued metric space.
It is obvious that C*-algebra-valued metric spaces generalize the concept of metric spaces, replacing the set of real numbers by $\mathbb{A}$+.

Definition2.2
Let (X,$\mathbb{A}$, d) be a C*-algebra-valued metric space. Suppose that $\{x_n\}$ ⊂ X and x ∈ X.If for any ε >0 there is N such that
1)for all n > N,$\|d|(x_n, x\|$≤ ε, then $\{x_n\}$ is said to be convergent with respect to A and $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote it by $\lim_{n\to\infty} x_n$= x.
2. for all n, m > N, d(x_n, x_m)≤ ε, then $\{x_n\}$ is called a Cauchy sequence with respect to A.
We say (X,$\mathbb{A}$, d) is a complete C*-algebra-valued metric space

NOTE: if every Cauchy sequence with respect to A is convergent..
It is obvious that if X is a Banach space, then (X,$\mathbb{A}$, d) is a complete C*-algebra-valued metric space if we set d(x, y) = $\|x - y\|$I
The following are nontrivial examples of complete C∗-algebra-valued metric space.
Example 2.3
Let X = $L^\infty(E)$and H =$L^2(E)$ where E is a Lebesgue measurable set. By L(H) we denote the set of bounded linear operators on Hilbert space H. Clearly L(H) is a C∗-algebra with the usual operator norm. Define d : X × X → L(H) by
d(f , g) = $\pi_{|f-g|}$ (∀f , g ∈ X),
where π_h : H → H is the multiplication operator defined by $\pi_h(\phi)$ = h · ϕ, for ϕ ∈ H. Then d is a C*-algebra-valued metric and (X, L(H), d) is a complete C*-algebravalued metric space.
Indeed, it suffices to verity the completeness. Let $\{f_n\}_{n=1}^{\infty}$ in X be a Cauchy sequence with respect to L(H). Then for a given ε >0 there is a natural number N(ε) such that for all n, m ≥ N(ε),

$$\|d\,(f_{n,}f_m\,)\| = \|\pi|f_{n,} - f_m|\| = \|f_{n,} - f_m\|_\infty \leq \varepsilon,$$

then $\{f_n\}_{n=1}^{\infty}$is a Cauchy sequence in the space X. Thus, there is a function f ∈ X and a natural number N_1(ε) such that $\|f_{n,} - f_m\|_\infty$ ≤ ε if n ≥ N.
It follows that

$$\|d\,(f_{n,}f\,)\| = \|\pi|f_{n,} - f|\| = \|f_{n,} - f\|_\infty \leq \varepsilon,$$

. Therefore, the sequence $\{f_n\}_{n=1}^{\infty}$converges to the function f in X with respect to L(H), that is, (X, L(H), d) is complete with respect to L(H).
Example 2.2
Let X = R and $\mathbb{A}$ = M_2(R). Define
d(x, y) = diag (|x − y|,α|x − y|) ,
where x, y ∈ℝ and α ≥0 is a constant. It is easy to verify d is a C*-algebra-valued metric and (X,M嶹(R), d) is a complete C∗-algebra-valued metric space by the completeness of R. Now we give the definition of a C*-algebra-valued contractive mapping on X.
Definition 2.4.: Suppose that (X,$\mathbb{A}$, d) is a C*-algebra-valued metric space. We call a mapping T : X → X is a C*-algebra-valued contractive mapping on X, if there exists an A belongs to $\mathbb{A}$with A <1such that
d(Tx,Ty) ≤ A*d(x, y)A, ∀x, y ∈ X.

III. Main Results

Theorem 3.1 If (X,λ, d) is a complete C*-algebra-valued metric space and T is a contractive mapping, there exists a unique fixed point in X.
Proof:. Choose x_0∈ X and set x_{n+1}= Tx_n = $T^{n+1}x_0$ n = 1,2, ... For convenience, by B we denote the element d($x_{1,}x_0$)in A.
Notice that in a C*-algebra, if a, b ∈ A_+ and a ≤ b, then for any x ∈ A both x*ax and x*bx are positive elements and x*ax ≤ x*bx [1].

Thus d(x_{n+1},x_n) = d(x_n,x_{n-1})≤ λ d(x_n,x_{n-1})≤ $\cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \lambda^n$ d($x_1 x_0$)……..(3.1.1)

For any m≥ 1, p≥ 1

d($x_{m+p,,} x_{m\,,,}$) ≤s[d($x_{m+p,,} x_{m+p-1,,}$)⊕d(d($x_{m+p-1,,} x_{m,}$)] ≤s[d($x_{m+p,,} x_{m+p-1,,}$)⊕s^2[d($x_{m+p-1,,} x_{m+p-2,}$)⊕$d($ $x_{m+p-2,}$,$x_{m,}$)]

≤s[d($x_{m+p,,} x_{m+p-1,,}$)⊕s^2[d($x_{m+p-1,,} x_{m+p-2,}$)⊕ $\ldots \ldots \ldots \ldots$ ⊕s^{p-1}[d($x_{m+2,,} x_{m+1,}$)⊕d($x_{m+1,,} x_{m,,}$)

≤ $s\lambda^{m+p-1}$[d($x_{1,,} x_{0,}$)⊕$s^2\lambda^{m+p-2} d(x_{1,}, x_{0,})$]⊕ $\ldots \ldots \ldots \ldots$ ⊕$s^{p-1}\lambda^{m+1}$[d($x_{1,,} x_{0,}$)⊕$s^{p-1}\lambda^m$ [d($x_{1,,} x_{,,}$)

≤ ($s\lambda^{m+p-1}$+$s^2\lambda^{m+p-2}$ $\ldots \ldots \ldots \ldots$ +$s^{p-1}\lambda^{m+1}$)d($x_{1,,} x_{0,}$)+$s^{p-1}\lambda^m$ d($x_{1,,} x_{0,}$)

≤ $\frac{(s\lambda^{m+p[(s\lambda^{-1})^{p-1}-1]}}{s-\lambda}$ d($x_{1,,} x_{0,}$) ⊕$s^{p-1}\lambda^m$[d($x_{1,,} x_{0,}$)

≤ $\frac{s^p\lambda^{m+1}}{s-\lambda}$ d($x_{1,,} x_{0,}$) ⊕$s^{p-1}\lambda^m$[d($x_{1,,} x_{0,}$)

Let 0≪‖ c ‖ be given

We have

$\frac{s^p\lambda^{m+1}}{s-\lambda}$ d($x_{1,,} x_{0,}$) ⊕$s^{p-1}\lambda^m$[d($x_{1,,} x_{0,}$)→ $zero$

As m tend to infinite , p= 1,2,…….(since $\lambda < 1$)by using , we can find m_0 belongs to N such that

$\frac{s^p\lambda^{m+1}}{s-\lambda}$ d($x_{1,,} x_{0,}$) ⊕$s^{p-1}\lambda^m$[d($x_{1,,} x_{0,}$) 0≪‖ c ‖ for all m ≥ m_0

We have

d($x_{m+p,,} x_{m\,,}$)≪‖ c ‖ for all m ≥ m_0

And for any p= 1,2,3,…..

so {$x_{n\,,,}$} is a Cauchy sequence in (X,d)

since (x,d) is complete fuzzy metric space there exists x* belongs to X such that

d($x_{n,,} x^*$)≪ $\frac{\|c\|}{s(\lambda+1)}$ for all n greater than equal to m_0

hence

d(Tx^*,x^*) ≤s[d(Tx^*, $Tx_{n,,}$,)⊕$d(Tx_{n,,}, x^*)$

≤s[d($x^*, x_{n,,}$,)⊕$d(Tx_{n+1,,}, x^*)$

So d(Tx^*,x^*) ≪‖ c ‖ for all n greater than equal to $x_{0,}$

Since

$\bar{o}$ ≪‖ c ‖is arbitrary , it follow that , it follows that d(Tx^*,x^*) $= \ \bar{o}$

Thus

Tx^*,= x^*. So x^*. Is a fixed point of T. if y^*. Is a another fixed point , then we have

d(x^*,x^*) ≤ **d(Tx^*,Ty^*) =$\bar{o}$ thus x^*,= y^*.**

Definition 2.5 Let X be a nonempty set. We call a mapping T is a C∗-algebra-valued expansion mapping on X, if T : X → X satisfies:

(1) T(X) = X;

(2) d(Tx,Ty) A∗d(x, y)A, ∀x, y ∈ X, where A ∈ A is an invertible element and $\|A^{-1}\| \leq 1$

Theorem 2.6:

Let (X,𝔸, d) be a complete C∗-algebra-valued metric space. Then for the expansion mapping T, there exists a unique fixed point in X.

d(x, y)A–〈〈d T–〈〈x,T–〈〈y , and thus A–

Lemma 2.7. Suppose that A is a unital C∗-algebra with a unit I.

(1) If a ∈ A₊ with $\|a\| \leq 1/2$, then I – a is invertible and $\|a(1-a)^{-1}\| \leq 1$

(2) suppose that a, b ∈ A with a, b θ and ab = ba, then ab≥ θ

Theorem 3.7: Let (X,A, d) be a complete C∗-valued metric space. Suppose the mapping T : X → X satisfies for all x, y ∈X d(Tx,Ty) , A d(Tx, y) + d(Ty, x) , where A ∈ A ₊ and A <1.. Then there exists a unique fixed point in X. Proof Without loss of generality, one can suppose that A = θ

Notice that A ∈ A +, A(d(Tx, y) + d(Ty, x)) is also a positive element.

Theorem 2.8: let (X,λ,d) be a complete C* -valued metric space .suppose T:X → X be a C ∗ −valued contractive mapping . then T has unique point .

Let Theorem 2.10: Let (X,λ, d) be a complet metric space . suppose the mapping T : X → X satisfies for all x,y, belongs to X

d(Tx,Ty)≤ A d(Tx, y) + d(Ty, x) +d(x,y) then T has a unique fixed point in .

Conclusions:

We proved a main theorem in fixed point theomrem in C*- algebra metric space and using suitable contraction condition . he ex-tenson of contractive define on C* -algbra valued metric space .we proved existence of fixed point using the concept of minimal elements in C* 0-algenra valued metric space by introduce with notation of po on X.

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