IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 5 Ser. I (sep – Oct 2019), PP 17-20
www.iosrjournals.org

Conformal Change of Douglas Special Finsler Space with Special $(\boldsymbol{\alpha}, \boldsymbol{\beta})$- Metric

Narasimhamurthy. S. K[1] and Gayathri. K[2]

Abstract: *In this paper, we find the necessary and sufficient conditions for a Finsler space with the metric $L = \alpha - \frac{\beta^2}{\alpha}$ to be a Douglas space and also to be a Berwald space, where α is a Riemannian metric and β is a differential one-form. Further, we study the conformal change of Douglas space with the above mentioned special $(\alpha, \beta) - metric$.*

Keywords: *Douglas Space, Berwald Space, Conformal change.*
AMS Subject Classification (2010): *53A20, 53B40, 53C60, 58J60.*

Date of Submission: 26-08-2019 Date of Acceptance: 10-09-2019

I. Introduction

The conformal theory of Finsler spaces has been introduced by M. S. Knebelman in 1929 and later on M. Hashiguchi developed such theory which was based on Matsumoto's approach to Finsler geometry. The conformal theory of two-dimensional Finsler space has been studied by M. Matsumoto and based on the above work, B. N. Prasad and D. K. Diwedi discussed the theory of conformal change of three-dimensional Finsler space.

Nabil L. Youssef, S. H. Abed and A. Soleiman investigated intrinsically conformal changes in Finsler geometry. Also they studied conformal change of Barthel connection and its curvature tensor, the conformal changes of Cartan and Berwald connections as well as their curvature tensors. Shun-ichi Hojo, M. Matsumoto and K. Okubo discussed the theory of conformally Berwald Finsler spaces and its applications to $(\alpha, \beta) -$ metrics.

The notion of Douglas space has been introduced by M. Matsumoto and S. Bacso as a generalization of Berwald space from the view point of geodesic equations. It is remarkable that a Finsler space is a Douglas space or is of Douglas type if and only if the Douglas tensor vanishes identically.

M. Matsumoto studied on Finsler spaces with $(\alpha, \beta) -$ metric of Douglas type. Hong-Suh Park and Eun-Seo Choi explained Finsler spaces with an approximate Matsumoto metric of Douglas type. The authors S. Bacso and I. papp studied on a generalized Douglas space. Also the team of authors Benling Li, Yibing Shen and Zhongmin Shen studied on a class of Douglas metrics.

The first part of the present paper is devoted to study the condition for the Finsler space with special $(\alpha, \beta) -$ metric to be a Douglas type (Theorem 3.1). The second part is devoted to find the condition for the Finsler space with special $(\alpha, \beta) -$ metric to be Berwald space (Theorem 4.1) and finally, we apply the conformal change of Finsler space with the special $(\alpha, \beta) -$ metric of Douglas type (Theorem 5.1). We study conformal change of Douglas space with special $(\alpha, \beta) -$ metric $\mathrm{L} = \alpha - \frac{\beta^2}{\alpha}$ and also we obtain the conditions for Finsler space with an $(\alpha, \beta) -$ metric $\mathrm{L}^2 = 2\alpha\beta$ to be conformally Berwald.

II. Preliminaries

To find the condition for the Finsler space with special $(\alpha, \beta) -$ metric to be Berwald space (Theorem 4.1) and finally, we apply the conformal change of Finsler space with the special $(\alpha, \beta) -$ metric of Douglas type (Theorem 5.1). We study conformal change of Douglas space with special $(\alpha, \beta) -$ metric $\mathrm{L} = \alpha - \frac{\beta^2}{\alpha}$ and also we obtain the conditions for Finsler space with an $(\alpha, \beta) -$ metric $\mathrm{L}^2 = 2\alpha\beta$ to be conformally Berwald.

2.1 THE CONDITION TO BE A DOUGLAS SPACE

The geodesics of a Finsler space $F^n = (M^n, L)$ are given by the differential equations:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

Where $2G^i(x,y) = \gamma_j^{\ i}{}_k(x,y)y^j y^k$ *and* $\gamma_j^{\ i}{}_k(x,y)$ are Christoffel symbols constructed from $g_{ij}(x,y)$ with respect to x^i.

A Finsler space F^n is said to be of Douglas type if

$$D^{ij} \equiv G^i(x,y)y^j - G^j(x,y)y^i \qquad (2.1)$$

are homogeneous polynomials in (y^i) of degree three.
Let $hp(r)$ denote the homogeneous polynomial in y^i of degree r.
We use the following definition in future.

Definition 2.1: *The Finsler space F^n is of Douglas type if and only if the Douglas tensor*

$$D_{i\ jk}^{h} = C_{i\ jk}^{h} - \frac{1}{n+1}(G_{ijk}y^h + G_{ij}\delta_k^h + G_{jk}\delta_i^h + G_{ki}\delta_j^h)$$

Vanishes identically, where $G_{i\ jk}^{h} = \dot{\partial}_k G_{i\ j}^{h}$ is $hv-curvature\ tensor$ *of the Berwald connection.*

The covariant differentiation with respect to the Levi-Cavita connection $\{{}_{j}{}^{i}{}_{k}\}(x)$ of R^n is denoted by (|).
We use the symbols as follows:

$$r_{ij} = \frac{1}{2}\left(b_{i|j} + b_{j|i}\right), \qquad s_{ij} = \frac{1}{2}\left(b_{i|j} + b_{j|i}\right)$$

$$s_j = b_r s^{rj}, \qquad s_j^i = a^{ir}s_{rj}$$

The functions $G^i(x,y)$ of F^n with $(\alpha,\beta)-$ metric are written in the form

$$2G^i = \left\{{}_{0}{}^{i}{}_{0}\right\} + 2B^i,$$

$$B^i = \frac{\alpha L_\beta}{L_\alpha}\ s_0^i + C^*\left[\frac{\beta L_\beta}{\alpha L}y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left(\frac{1}{\alpha}y^i - \frac{\alpha}{\beta}b^i\right)\right], \qquad (2.2)$$

Where $L_\alpha = \partial L/\partial\alpha, L_\beta = \partial L/\partial\beta, L_{\alpha\alpha} = \partial L/\partial\alpha\partial\alpha$, the subscript 0 means contraction by y^i and we put

$$C^* = \frac{\alpha\beta\,(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})},$$

Where $\gamma^2 = b^2\alpha^2 - \beta^2, b^i = a^{ij}b_j\ and\ b^2 = a^{ij}b_i b_j$.

Since $\left\{{}_{0}{}^{i}{}_{0}\right\}(x)$ are $hp(2), F^n$ with $(\alpha,\beta)-$ metric is a Douglas space if and only if $B^{ij} \equiv B^i y^j - B^j y^i$ are $hp(3)$. From (1.2.1) and (1.2.2), we have

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}\left(s_0^i y^j - s_0^j y^i\right) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*(b^i y^j - b^j y^i) \qquad (2.3)$$

We use the following lemma.

Lemma 2.1 *If $\alpha^2 \equiv 0(mod\beta), that\ is, a_{ij}(x)y^i y^j$ contains $b_i y^i$ as a factor, then the dimension n is equal to 2 and b^2 vanishes. In this case, we have 1-form $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

2.2DOUGLAS SPACE WITH SPECIAL METRIC $L = \alpha - \frac{\beta^2}{\alpha}$

In this section, we find the condition for a Finsler space F^n with a special $(\alpha,\beta)-$ metric

$$L = \alpha - \frac{\beta^2}{\alpha} \qquad (3.1)$$

to be a Douglas type. The derivatives of (3.1) are given by

$$L_\alpha = 1 + \frac{\beta^2}{\alpha^2}, \quad L_\beta = -\frac{2\beta}{\alpha}, \quad L_{\alpha\alpha} = -\frac{2\beta^2}{\alpha^3} \qquad (3.2)$$

Substituting (3.2) in (2.3), we get

$$\{\alpha^2(1-2b^2) + 3\beta^2\}\{(\alpha^2+\beta^2)B^{ij} - \alpha^2(\alpha-2\beta)\left(s_0^i y^j - s_0^j y^i\right)\}$$
$$+ \alpha^2\{r_{00}(\alpha^2+\beta^2) - 2\alpha^2 s_0(\alpha-2\beta)\}(b^i y^j - b^j y^i) = 0 \qquad (3.3)$$

Suppose that F^n is a Douglas space, that is, B^{ij} are *hp*(3). Separating rational and irrational terms of y^i in (3.3), then, we get the following two equations:

$$\{\alpha^2(1-2b^2) + 3\beta^2\}\{(\alpha^2+\beta^2)B^{ij} + 2\alpha^2\beta\left(s_0^i y^j - s_0^j y^i\right)\} + \alpha^2\{r_{00}(\alpha^2+\beta^2) + 4\beta s_0\alpha^2\}(b^i y^j - b^j y^i)$$
$$= 0 \qquad (3.4)$$

and

$$2s_0\alpha^4(b^i y^j - b^j y^i) + \alpha^2\{\alpha^2(1-2b^2) + 3\beta^2\}\left(s_0^i y^j - s_0^j y^i\right) = 0. \qquad (3.5)$$

Substituting (3.5) in (3.4), we have

$$\{\alpha^2(1-2b^2) + 3\beta^2\}(\alpha^2+\beta^2)B^{ij} + \alpha^2 r_{00}(\alpha^2+\beta^2)(b^i y^j - b^j y^i) = 0 \qquad (3.6)$$

Only the term $3\beta^4 B^{ij}$ of (3.6) does not contain α^2.
Hence, we must have *hp*(5), v_5^{ij} satisfying

$$3\beta^4 B^{ij} = \alpha^2 v_5^{ij} \qquad (3.7)$$

Now, we study the following two cases:

Case (i): $\alpha^2 \not\equiv 0(mod\ \beta)$

In this case, (3.7) is reduced to $B^{ij} = \alpha^2 v^{ij}$, where v^{ij} are *hp*(1). Thus, (3.6) gives

$$\{\alpha^2(1-2b^2)+3\beta^2\}v^{ij} + r_{00}(b^i y^j - b^j y^i) = 0 \tag{3.8}$$

Transvecting this by $b_i y_j$, where $y_j = a_{jk} y^k$, we have

$$\alpha^2\{(1-2b^2)v^{ij} b_i y_j + b^2 r_{00}\} = \beta^2(r_{00} - 3v^{ij} b_i y_j) \tag{3.9}$$

Since $\alpha^2 \not\equiv 0(mod\ \beta)$, then there exists a function *h(x)* satisfying

$$(1-2b^2)v^{ij} b_i y_j + b^2 r_{00} = h(x)\beta^2,$$
$$r_{00} - 3v^{ij} b_i y_j = h(x)\alpha^2.$$

Eliminating $v^{ij} b_i y_j$ from the above two equations, we obtain

$$(1+b^2)r_{00} = h(x)\{(1-2b^2)\alpha^2 + 3\beta^2\} \tag{3.10}$$

From (3.10), we get

$$b_{i|j} = k\{(-1+2b^2)a_{ij} - 3b_i b_j\} \tag{3.11}$$

Where $k = -h(x)/(1+b^2)$. Hence b_i is a gradient vector.

Conversely, if (3.11) holds, then $s_{ij} = 0$ and we get (3.10). Therefore, (3.3) is written as follows:

$$B^{ij} = k\{\alpha^2(b^i y^j - b^j y^i)\}$$

Which are *hp*(3), that is, F^n is a Douglas space.

Case (ii): $\alpha^2 \equiv 0(mod\ \beta)$.

In this case, there exists 1-form δ such that $\alpha^2 = \delta\beta$, $b^2 = 0$ and by lemma 2.1, the dimension is two. Therefore (3.7) is reduced to $B^{ij} = \delta w_2^{ij}$, where w_2^{ij} are *hp*(2). Thus the equation (3.5) leads to

$$2s_0\delta(b^i y^j - b^j y^i) + (\delta+3\beta)(s_0^i y^j - s_0^j y^i) = 0$$

Transvecting the above equation by $y_i b_j$, we have $s_0 = 0$. Substituting $s_0 = 0$ in the above equation, we have $s_{ij} = 0$. Therefore, (3.6) reduces to

$$(\delta+3\beta)w_2^{ij} + r_{00}(b^i y^j - b^j y^i) = 0$$

Transvecting the above equation by $b_i y_j$, wec get

$$(\delta+3\beta)w_2^{ij} b_i y_j - r_{00}\beta^2 = 0$$

Which is written as

$$\delta w_2^{ij} b_i y_j = \beta(\beta r_{00} - 3w_2^{ij} b_i y_j)$$

Therefore, there exists an *hp*(2), $\lambda = \lambda_{ij}(x)y^i y^j$ such that

$$w_2^{ij} b_i y_j = \beta\lambda,\ \ \beta r_{00} - 3w_2^{ij} b_i y_j = \delta\lambda$$

Eliminating $w_2^{ij} b_i y_j$ from the above equations, we get

$$\beta r_{00} = \lambda(3\beta+\delta) \tag{3.12}$$

Which implies there exists an *hp*(1). $v_0 = v_i(x)y^i$ such that

$$r_{00} = v_0(3\beta+\delta),\ \lambda = v_0\beta \tag{3.13}$$

From r_{00} given by (3.13) and $s_{ij} = 0$, we get

$$b_{i|j} = \frac{1}{2}\{v_i(3b_j + d_j) + v_j(3b_i + d_i)\} \tag{3.14}$$

Hence b_i is a gradient vector.

Conversely, if (3.14) holds, then $s_{ij} = 0$, which implies $r_{00} = v_0(3\beta+\delta)$. Therefore, (3.3) is written as follows:

$$B^{ij} = -v_0\delta(b^i y^j - b^j y^i)$$

Which are *hp*(3). Therefore, F^n is a Douglas space. Thus, we have

Theorem 3.1: *A Finsler space with a special (α,β)-metric $L = \alpha - \frac{\beta^2}{\alpha}$ is a Douglas space if and only if*

(1) $\alpha^2 \not\equiv 0(mod\ \beta), b^2 \neq 1$: $b_{i|j}$ *is written in the form (3.11).*

(2) $\alpha^2 \equiv 0(mod\ \beta)$: $n = 2$ *and* $b_{i|j}$ *is written in the form (3.14), where* $\alpha^2 = \beta\delta, \delta = d_i(x)y^i, v_o = v_i(x)y^i$.

III. Conclusion

M. Matsumoto and S. Bacso introduced the notion of Douglas space as a generalization of Berwald space from the view point of geodesic equations. Douglas metrics can be viewed as generalized Berwald metrics. The study on Douglas metrics will enhance our understanding on the geometrical meaning of non-Riemannian quantities. In this paper, we found the conditions for a Finsler space with special $(\alpha,\beta)-$ metric $L = \alpha - \frac{\beta^2}{\alpha}$ to be a Douglas space and also to be a Berwald space. Further, we found the conditions for a conformally transformed Douglas space with the above mentioned special $(\alpha,\beta)-$ metric to be a Douglas space.

The important findings of this paper are as follows:

1. A Finsler space with a special $(\alpha, \beta) -$ metric $L = \alpha - \frac{\beta^2}{\alpha}$ is a Douglas space if and only if

a. $\alpha^2 \not\equiv 0(mod\ \beta), b^2 \neq 1$: $b_{i|j}$ is written in the following form:

$b_{i|j} = k\{(-1 + 2b^2)a_{ij} - 3b_i b_j\}$

b. $\alpha^2 \equiv 0(mod\ \beta)$: $n = 2$ and $b_{i|j}$ is written in the following form:

$b_{i|j} = \frac{1}{2}\{v_i(3b_j + d_j) + v_j(3b_i + d_i)\}$,

where $\alpha^2 = \beta\delta, \delta = d_i(x)y^i, v_0 = v_i(x)y^i$

2. $\alpha^2 \not\equiv 0(mod\ \beta)$, then the Douglas space with a special $(\alpha, \beta) -$ metric $L = \alpha - \frac{\beta^2}{\alpha}$ $(b^2 \neq 1)$ is conformally transformed to a Douglas space if and only if the transformation is homothetic.

References

[1]. Benling Li, Yibing Shen and Zhongmin Shen, On a class of Douglas metrics, Comm. Korean Math. Soc., 14(3)(1999), 535-544. B.N Prasad. B.N Gupta and D.D. Singh, Conformal transformation in Finsler spaces with $(\alpha, \beta) -$ metric, Indian J. Pure and Appl. Math., 18(4)(1961), 290-301.

[2]. Hong-Suh Park and Eun-Seo Choi, Finsler spaces with an approximate Matsumoto metric of Douglas type, Comm. Korean Math. Soc., 14(3)(1999), 535-544.

[3]. M. Hashiguchi, S. Hojo and Matsumoto, On Landsberg spaces of dimension two with $(\alpha, \beta) -$ metric, Tensor, N.S., 57(1996), 145-153.

[4]. M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, Kaiseisha press, Otsu, Saikawa, Japan, (1986).

[5]. M. Matsumoto, The Berwald connection of a Finsler space with an $(\alpha, \beta) -$ metric, Tensor, N.S., 50(1991),18-21.

[6]. M. Matsumoto, On Finsler spaces with $(\alpha, \beta) -$ metric of Douglas type, Tensor, N.S. 60(1998).

[7]. Pradeep Kumar, S.K. Narasimhamurthy, H.G. Nagaraja, S.T. Aveesh, On a special hypersurface of a Finsler space with $(\alpha, \beta) -$ metric, Tbilisi Mathematical Journal, 2(2009), 51-60.

[8]. P.L. Antonelli, R.S. Ingarden and M. Matsumoto, The theory of sprays and Finsler spaces with applications in Physics and Biology, Kluwer academic publishers, Netherlands (1993).

[9]. S. Bacso, X. Cheng and Z. Shen, Curvature properties of $(\alpha, \beta) -$ metrics, Advanced studies in Pure Mathematics, Math. Soc. Of Japan, 48(2007), 73-110.

[10]. S. Bacso and M. Matsumoto, On Finsler spaces of Douglas type. A generalization of the notion of Berawld spae, Publ. Math. Debrecen, 51(1997), 385-406.

[11]. S. Bacso and I. Papp, On Landsberg spaces of dimension two with $(\alpha, \beta) -$ metric, Periodica Mathematica Hungarica, 48(1-2)(2004), 181-184.

[12]. S.K. Narasimhamurthy and D.M. Vasantha, Projective change between two Finsler spaces with $(\alpha, \beta) -$ metric, Kyunpook Mathematical Journal.

[13]. S.K. Narasimhamurthy and G.N. Latha Kumari, On a hypersurface of a special Finsler space with a metric $L = \alpha + \frac{\beta^2}{\alpha} + \beta$, ADJM, 9(1)(2010), 36-44.

Narasimhamurthy.S.K. " Conformal Change of Douglas Special Finsler Space with Special (α,β)- Metric." IOSR Journal of Mathematics (IOSR-JM) 15.5 (2019): 17-20.