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An Accelerated Broyden's Algorithm for Solving Systems of Nonlinear Equations

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Abstract: *Quasi Newton's methods are promising schemes for solving systems of nonlinear equations. In this paper, we continue in the spirit of quasi-Newton update and present an accelerated Broyden's-like method with improved Jacobian approximation for solving large-scale systems of nonlinear equations. The anticipation has been to further improve the performance of Broyden's update as well as reducing function values. The effectiveness of our proposed scheme is appraised through numerical comparison with some well known Newton's like methods.*
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I. Introduction

Let us consider the problem of finding the solution of nonlinear equations

$$F(x) = 0, \tag{1}$$

is continuously differentiable in a open neighborhood S of a solution $x^* \in S$ of the system (1). Assume that, there exists a solution x^* where $F(x^*) = 0$ and $F'(x^*) \neq 0$ the prominent method for finding the solution to (1) is the Newton's method which generates a sequence of iterates $\{x_k\}$ from a given initial point x_0 using:

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k), \quad \text{k = 0, 1, 2 . . .} \tag{2}$$

Where $F'(x_k)$ is the Jacobian of F.

The pleasant features of this method are; rapid convergence and easy to implement. Nevertheless, Newton's method has some numerous shortcomings, which attract the attention of many researchers over time. This includes; the computation and storage of Jacobian matrix as well as solving n linear equations in each iteration [1, 2, 4]. It is well known that, Jacobian computation entails first-order derivatives of the systems, but some functions derivatives are quite costly and sometime not available or could not be done precisely [3, 10]. In this case Newton's method cannot be directly used. Therefore, to do away with this crucial issue some effort need to surface.

It has been suggested that the Jacobian matrix be evaluated either once and for all or once every few iterations, instead of at every iteration as is strictly required [6, 5, 7]. The simplest method to tackle this very crucial issue is Fixed Newton's method. This method saves a lot the computational burdens of the Jacobian matrix $F'(x_k)$, by approximating the Jacobian with the Jacobian at x_0 (Initial guess) i.e

$$F'(x_0)s_k = -F(x_k) \tag{3}$$

for all k.

Algorithm 1 (Fixed Newton's Method)

Given x_0

Solve s_k for k = 0, 1, 2,...

$$F'(x_0)s_k = -F(x_k)$$

Update $x_{k+1} = x_k + s_k$

Quasi-Newton method is another variant of Newton-type methods, it replaces the Jacobian or its inverse with an approximation, which can be updated at each iteration and is given as:

$$x_{k+1} = x_k - B_k^{-1} F(x_k), \tag{4}$$

Where the matrix B_k is the approximation of the Jacobian at x_k. The main idea behind quasi-Newton's method is to eliminate the evaluation cost of the Jacobian matrix. The famous Quasi Newton's update is Broyden's update, this method is effective and most at times numerically stable [8, 9]. In this paper, we introduce an accelerated parameter based on Andrei [12] into the Broyden's methed. The anticipation has been to further improve the performance of Broyden's update, by reducing the number of iterations and matrix requirements respectively. We organized the paper as follows: In the next section, we present the details of the proposed method. Some numerical results are reported in Section 3. Finally, conclusions are made in Section 4.

II. Derivation process

It is well known that, it is not always feasible to compute the full elements of the Jacobian matrix of the given nonlinear function or it may be very expensive, we often have to approximate the Jacobian matrix by some other approach, the famous method of doing so is quasi-Newton's method [11] and Broyden's method in particular. The basic idea underline this method has been to reduce the evaluation cost of the Jacobian matrix. Moreover, in some situations analytic derivatives could not be done precisely, or not available to obtain, the promising method designed to embark upon this situation is Broyden's scheme. The method is an iterative procedure that generates a sequence of points $\{x_k\}$ from a given initial guess x_0 via the following updating scheme:

$$x_{k+1} = x_k - B_k^{-1} F(x_k) \tag{5}$$

where B_k is an approximation to the Jacobian which can be updated at each iteration using a rank-one-matrix for k = 0, 1, 2 . . . . The appealing feature of this method is that, it requires only one function evaluation per iteration. The update formulae for the Broyden matrix B_k is given in [7] as:

$$B_{k+1} = B_k + \frac{(y_k - B_k s_k) s_k^T}{s_k^T s_k} \tag{6}$$

The method propose in this work has the advantage of reducing the computational cost of Newton's method and Broyden's method. This is made possible by employing the accelerated parameter proposed by Andrei [12] given as:

$$\xi_k = -\frac{a_k}{b_k}, \qquad a_k = F(x_k)^T d_k,\ b_k = -(F(x_k) - F(z_k))d_k \text{ where } d_k = -B_k^{-1} F(x_k)$$

and $z = x_k + d_k$, if $b_k > 0$, then $x_{k+1} = x_k + \xi_k d_k$.

Now, we can describe the algorithm for our propose method as follows:

Algorithm 2 (Accelerated Broyden's method (ABM))

Step I: Given x_o, Ɛ and B_o compute $F_o = F(x_o)$.

Step II: Check stopping criteria. If yes stop, else go to Step III

Step III: Compute $Z_k = x_k - B_k^{-1} F(x_k)$ and $F(x_k) - F(Z_k)$

Step IV: Compute $a_k = -F(x_k)^T B_k^{-1} F(x_k),\ b_k = -y_k^T B_k^{-1} F(x_k)$

Step V: Acceleration scheme, if $b_k \neq 0$, set $\xi_k = -a_k / b_k$ and then update

$x_{k+1} = x_k - \xi_k B_k^{-1} F(x_k)$ *or* Otherwise Set $x_{k+1} = x_k - B_k^{-1} F(x_k)$

Step VI: Compute $S_k = x_{k+1} - x_k$ and $y_k = F(x_{k+1}) - F(x_k)$.

Step VII: Compute $B_{k+1} = B_k + \frac{(y_k - B_k S_k) S_k^T}{S_k^T S_k}$

Step VIII: Set $x_k = x_{k+1}$ and go back to step II.

III. Numerical Results

This section is devoted to the implementation of our proposed method (ABM method) in comparison with the Broyden Method (BM) and Newton's method (NM) respectively. All the experiments are implemented on a PC using MATLAB 7.9.0 (R2009b), with double precision arithmetic. For each test
Problem, we performed an experiments with different initial guess in order to evaluate the efficiency of the methods. As regards the stopping criteria used in our experiments, in all the algorithms, convergence is assumed if:

$$\|F(x_k)\| \leq 10^{-3} \qquad (7)$$

We forced the algorithm to stop whenever the number of iterations exceeds 2000, and the symbol "-" is used to represent the failure. The identity matrix has been chosen as an initial approximate Jacobian. Three (3) benchmark problems are considered: In the following, details on the benchmarks test problems are presented:

Problem 1 **System of nonlinear equations**:

$$F_1 = x_1^2 - 1$$
$$F_2 = x_2^2 - 1$$
$$x_o = (4.5, 4.5)$$

Problem 2 **System of nonlinear equations:**

$$F_1 = 2x_1 + x_2$$
$$F_2 = x_1^2 + 2x_2$$
$$x_0 = (1,0)$$

Problem 3 **System of nonlinear equations:**

$$F_1 = x_1^2 + x_1 - 2$$
$$F_2 = x_2^2 + x_2 - 2$$
$$x_0 = (0.5, 0.5)$$

Table 1
Results of Problems (1-3)
(Number of iteration)

Problem	NM		BM		ABM	
	NI	x^*	NI	x^*	NI	x^*
1	3	(1, 1)	11	(1, 1)	2	(1, 1)
2	5	(0, 0)	6	(0, 0)	5	(0, 0)
3	3	(1, 1)	5	(0.9, 0.9)	3	(1, 1)

The numerical results of Tables 1 show the performance of all the algorithms considered in these implementation, the numerical results indicates that our algorithm has made significant performance among these algorithms. In addition, it is worth mentioning that, the ABM method does not require more storage locations than classic Broyden's methods. Moreover, ABM method will also be a promising method for large-scale systems.

IV. Conclusion

In this paper we present an accelerated Broyden's method for solving Systems of non- linear equations (ABM). The Jacobian approximation considered was Broyden's updates. The anticipation has been to further improve the performance of Broyden's method via incorporating accelerated parameter. It is also worth mentioning that the method is capable of significantly reducing the number of iteration, as compared to NM and BM methods, while maintaining good accuracy of the numerical solution to some extent. Another fact that makes the ABM method appealing is that throughout the numerical experiments it never fails to converge.

Hence we can claim that our method (ABM) is a good alternative to Newton-type methods for solving large-scale systems of nonlinear

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