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A Comparative Investigation on Numerical Solution of Initial Value Problem by Using Modified Euler's Method and Runge-Kutta Method

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ABSTRACT: *In this article, Modified Euler's Method and Runge- Kutta Methods have been used to find the numerical solutions of ordinary differential equations with initial value problems. By using MATLAB we determined the solutions of some numerical problems and at the same time calculated the exact analytic solution. Then, the numerical approximate solutions were compared with the exact solutions for validating the accuracy. We found that, the solution become more precise when the step size is very small. Here, the difference between the numerical approximate solutions and analytic solutions is the relative error. We found that, between the two proposed methods the relative error is nominal for Runge-Kutta fourth order method.*

Keywords: *Initial value problem, Modified Euler's method, Runge- Kutta method, Error analysis.*

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I. Introduction

Differential Equations are of great help for solving complex mathematical problems in almost every section of Engineering, Science and Mathematics. In mathematics a number of real problems arise in the form of differential equations. These differential equations are either in the form of ordinary differential equation or partial differential equation. Usually, most of the problems which are modeled by these differential equations are so complicated that it is hard to determine the exact solution and one of two approaches is taken to approximate the solution. The first technique we will use is reducing the differential equations into a form which can be solved exactly and the results can be used to approximate the solution of the original problems. Another technique that we will use in this article is the approximation method which gives a more perfect result and less relative error. To solve those mathematical problems where it is very difficult or nearly impossible to determine the exact solution, numerical methods are used. Only a limited number of differential equations can be solved analytically. The solutions of a large number of differential equations cannot be determined using the familiar analytical methods. So in these cases we need to apply numerical methods to solve a differential equation under certain initial restriction or restrictions. To find the solution of initial value problem of ordinary differential equations there are a number of numerical methods. In this research paper we will present two numerical methods such as Modified Euler's method and Runge-Kutta method to find the solution of initial value problems for ordinary differential equations.

From the literature review analysis, we can understand that a number of works have been performed to find the numerical solutions of initial value problems by using Modified Euler's method and Runge-Kutta method. Also, to determine high correctness, various authors attempted to solve initial value problem by using Modified Euler's method and Runge-Kutta method. In[1] the author described how to find the accurate solutions of initial value problems for ordinary differential equation by using Runge-Kutta fourth order method. In[2-3] the authors suggested some numerical methods to solve initial value problems for ordinary differential equations. Also [4-16] studied a variety of numerical methods for finding the solutions of initial value problems for ordinary differential equations. In this paper, Euler modified method and Runge Kutta method are to solve initial value problems of ordinary differential equations without any alteration, discretization or limiting assumptions. The Runge Kutta method is the most commonly used numerical method as it gives consistent initial values and is also very suitable when calculation of higher derivatives are very complicated. The numerical outcomes are very positive. To finalize, for validating the anticipated formula couple of examples which includes different kinds of ordinary differential equations are given The solutions of each examples illustrates that the convergence and error analysis which are performed demonstrate the effectiveness of the methods. In the case of Euler modified method it is less convenient to find the solution of the differential equation numerically since it requires h to be very small for finding logical accuracy. In Runge-Kutta method, it is not required to find the derivatives of the superior order and the method also has the advantage of needing the functional values at some preferred points on the sub-interval thus providing better accuracy. We found that in case of Euler modified method very small step size converges to analytical solution and thus larger number of approximations is required. On the other hand, the results found in Runge Kutta method converges closer to analytical solution and it needs lees iteration to give accurate solutions.

II. Problem Formulation

In this segment we consider three numerical methods for obtaining the numerical solutions of the initial value problem (IVP) of the first-order ordinary differential equation is of the form

$$y' = \mathrm{f}(\mathrm{x}, \mathrm{y}(\mathrm{x})), \mathrm{x}\epsilon(\mathrm{x}_0, \mathrm{x}_\mathrm{n}),\ \mathrm{y}(\mathrm{x}_0) \tag{1}$$

where $y' = dy\ /\ dx$ and f(x, y(x)) is the given function and y(x) is the solution of the equation (1). In this article, we find the solution of this equation on a finite interval (x_0, x_n), starting with the initial point x_0. A continuous approximation to the solution y(x) will not be obtained; instead, approximations to y will be generated at various values, called mesh points, in the interval (x_0, x_n). Numerical methods employ the Equation (1) to obtain approximations to the values of the solution corresponding to various selected values of the solution corresponding to different selected values of , n = 0,1,2, … … . The parameter h is called the step size. The numerical solution of (1) is given by a set of points $\{(x_0, x_n)$: n = 0,1,2,3, … … n$\}$ and each point (x_0, x_n) is an approximation to the corresponding point $\{x_n, y(x_n)\}$ on the solution curve.

2.1 Modified Euler Method

In this In this method the curve in the interval (x_0, x_1) where $x_1 = x_0 + h$ is approximated by the line through (x_0, y_0) with the slope f $(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))$ which is the slope as the middle point whose abscissa is average of x_0 and x_1 . A generalized form of Euler's Modified formula is

$$y_{n+1} = y_n + hf\{x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\}$$

2.2 Runge-Kutta Method

This method was devised by two German mathematicians, Runge about 1894 and extended by Kutta a few years later. The Runge-Kutta method is most familiar because it is pretty accurate, steady and simple to program. This method is notable by their order in the logic that they concur with Taylor's series solution up to terms of h^{r} where r is the order of the method. It do not require previous computational of higher derivatives of y(x) as in Taylor's series method. The fourth order Runge-Kutta method (RK4) is broadly used for solving initial value problems (IVP) for ordinary differential equation (ODE). The general formula for Runge-Kutta method is

$$y_{n+1} = y_n + (k_1 + 2k_2 + 2k_3 + k_4)/6 \qquad \text{n} = 0,1,2, \dots \dots$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

III. Error Analysis

There exist two kinds of errors in numerical solution of ordinary differential equations. Round-off errors and Truncation errors happen when ordinary differential equations are solved numerically. Rounding errors initiate from the verity that computers can only characterize numbers using a fixed and restricted number of important figures. Thus, such numbers cannot be represented accurately in computer memory. The inconsistency introduced by this restriction is call Round-off error. Truncation errors in numerical study occur when approximations are used to determine a number of quantities. The exactness of the solution will rely on how miniature we make the step size, h. A numerical method is said to be convergent if

$$\lim_{h \to 0} max_{1 \le n \le N} |y(x_n) - y_n|$$

where y(xn) denotes the approximate solution and yn denotes the exact solution. In this work we consider two initial value problems to examine accuracy of the proposed methods. The

Approximated solution is determined by using MATLAB software for three proposed numerical methods at different step size. The maximum error is defined by

$$E_R = \lim_{h\to 0} max_{1\le n\le N} \,|y(x_n) - y_n|$$

IV. Numerical Examples

In this section we consider two numerical examples to verify which numerical methods converge faster to analytical solution. Numerical solutions and errors are computed.

Example 1: We consider the initial value problem y^'(x) = x2 + xy, y(0) = 1 on the interval $0 \le x \le 1$. Then the exact solution of the given problem is as y(x) $=\sqrt{\frac{\pi}{2}e^{\frac{x^2}{2}}}$ erf($\frac{x}{\sqrt{2}}$+$e^{\frac{x^2}{2}} - x$) The approximate solutions and maximum errors are obtained and shown in tables 1(a)-(d).

Table 1 (a) Numerical approximations and maximum errors for step; size h=0.1; (b) Numerical approximations and maximum errors for step size h=0.05; (c) Numerical approximations and maximum errors for step size h=0.025; (d) Numerical approximations and maximum errors for step size h=0.0125

1(a): Numerical approximations and maximum errors for step; size h=0.1;

Euler's Modified Method		Runge-Kutta Method		Exact Solution
	ER		ER	
1.005250000000000	9.65218E-05	1.0053464802083334	4.16045E−08	1.0053465218128410
1.022661643750000	0.000227819	1.0228893798037348	8.26716E−08	1.0228894624752929
1.054783850254688	0.000408114	1.0551918407370900	1.23029E−07	1.0551919637660336
1.104662546534985	0.000656406	1.1053187896458685	1.63325E−07	1.1053189529706604
1.175976557420941	0.000998415	1.1769747667144460	2.05805E−07	1.1769749725189769
1.273209735845547	0.001469256	1.2746787363539485	2.55624E−07	1.2746789919776722
1.401871127660406	0.002117191	1.4039879953710888	3.23030E−07	1.4039883184007750
1.568778873945045	0.003008896	1.5717873427344033	4.26941E−07	1.5717877696756601
1.782428926401787	0.004236927	1.7866652528501639	6.00769E−07	1.7866658536190383
2.053477058070325	0.005930347	2.0594065035273252	9.01815E−07	2.059407405342576

1(b) : Numerical approximations and maximum errors for step size h=0.05;

Euler's Modified Method		Runge-Kutta Method		Exact Solution
	ER		ER	
1.0052500 00000000	9.65218E-05	1.00534648 0208333	4.1605E-08	1.00534652 18128410
1.022661643750000	0.000227819	1.0228893798037348	8.2672E-08	1.0228894624752929
1.054783850254688	0.000408114	1.0551918407370900	1.2303E-07	1.0551919637660336
1.104662546534985	0.000656406	1.1053187896458685	1.6332E-07	1.1053189529706604
1.175976557420941	0.000998415	1.1769747667144460	2.058E-07	1.1769749725189769
1.273209735845547	0.001469256	1.2746787363539485	2.5562E-07	1.2746789919776722
1.401871127660406	0.002117191	1.4039879953710888	3.2303E-07	1.4039883184007750
1.568778873945045	0.003008896	1.5717873427344033	4.2694E-07	1.5717877696756601
1.782428926401787	0.004236927	1.7866652528501639	6.0077E-07	1.7866658536190383
2.053477058070325	0.005930347	2.0594065035273252	9.0182E-07	2.059407405342576

1(c): Numerical approximations and maximum errors for step size h=0.025;

Euler's Modified Method		Runge-Kutta Method		Exact Solution
	ER		ER	
1.0053404 3378088	6.08803E-06	1.0053465216505684	1.6228E-10	1.005346521812841
1.022875020804634	1.44417E-05	1.0228894621535374	3.2176E-10	1.0228894624752929
1.055165966239071	2.59975E-05	1.0551919632892464	4.7679E-10	1.0551919637660336
1.105276938021313	4.20149E-05	1.105318952342068	6.286E-10	1.1053189529706604
1.176910764603475	6.42079E-05	1.1769749717346203	7.8435E-10	1.1769749725189769
1.274584063574121	9.49284E-05	1.2746789910151588	9.6252E-10	1.2746789919776722
1.403850894637291	0.000137424	1.4039883171991199	1.2017E-09	1.403988318400775
1.571591569133329	0.000196201	1.5717877681003285	1.5753E-09	1.5717877696756601
1.786388311742869	0.000277542	1.786665851402671	2.2164E-09	1.7866658536190383
2.059017158076531	0.000390247	2.0594074019860655	3.3565E-09	2.059407405342576

1 (d): Numerical approximations and maximum errors for step size h=0.0125;

Euler's Modified Method		Runge-Kutta Method		Exact Solution
	ER		ER	
1.0053404 3378088	6.08803E-06	1.0053465218027011	1.01399E−11	1.005346521812841
1.022875020804634	1.44417E-05	1.0228894624551952	2.00999E−11	1.0228894624752929
1.055165966239071	2.59975E-05	1.0551919637362754	2.97600E−11	1.0551919637660336
1.105276938021313	4.20149E-05	1.1053189529314784	3.91900E−11	1.1053189529706604
1.176910764603475	6.42079E-05	1.1769749724701777	4.88001E−11	1.1769749725189769
1.274584063574121	9.49284E-05	1.274678991917932	5.97400E−11	1.2746789919776722
1.403850894637291	0.000137424	1.403988318326378	7.44000E−11	1.403988318400775
1.571591569133329	0.000196201	1.571787769578305	9.73599E−11	1.5717877696756601
1.786388311742869	0.000277542	1.786665853482107	1.36930E−10	1.7866658536190383
2.059017158076531	0.000390247	2.0594074051349014	2.07670E−10	2.059407405342576

Example 2: We consider the initial value problem $y' = x^2 - xy$, $y(0) = 1$ on the interval $0 \le x \le 1$. The exact solution of the given problem is $y(x) = \frac{2e^{\frac{x^2}{2}}}{\sqrt{2\pi}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)+2}$

The approximate solutions and maximum errors are obtained and shown in the table

2(a): Numerical approximations and maximum errors for step; size h=0.1;

Euler's Modified Method		Runge-Kutta Method		Exact Solution
	ER		ER	
0.9145000 0000000	0.000990872	0.9135089320204528	1.95878E−07	0.913509127898782
0.850700872600742	0.001482354	0.8492181710604544	3.47642E−07	0.849218518702443
0.803539722296333	0.001716324	0.8018229448618213	4.53096E−07	0.8018233979576023
0.769598236661347	0.001814651	0.7677830621260258	5.24033E−07	0.7677835861595071
0.746530852829335	0.001841152	0.7446891282464022	5.72232E−07	0.7446897004786337
0.732718569364669	0.001830167	0.730887796150559	6.06628E−07	0.7308884027785085
0.727051749410847	0.00180045	0.725250665872579	6.33400E−07	0.7252512992720983
0.728789350891342	0.001762265	0.7270264295821635	6.56635E−07	0.7270270862176577
0.737464721311611	0.001721133	0.7357429095800243	6.78965E−07	0.7357435885449581
0.752820255668837	0.001679904	0.751139649932897	7.02025E−07	0.7511403519579868

2(b): Numerical approximations and maximum errors for step size h=0.05;

Euler's Modified Method		Runge-Kutta Method		Exact Solution
	ER		ER	
0.9137386 3169801	0.000229504	0.9135091213176563	6.58113E−09	0.913509127898782
0.849562976531398	0.000344458	0.8492185048488676	1.38536E−08	0.849218518702443
0.802223388565394	0.000399991	0.8018233782217195	1.97359E−08	0.8018233979576023
0.768207590571715	0.000424004	0.7677835620326319	2.41269E−08	0.767783586159507
0.745120887406124	0.000431187	0.7446896730961782	2.73825E−08	0.744689700478633
0.731317887808574	0.000429485	0.7308883728944706	2.98840E−08	0.730888402778508
0.725674565817607	0.000423267	0.7252512673367656	3.19353E−08	0.725251299272098
0.727442024142068	0.000414938	0.7270270524626191	3.37550E−08	0.727027086217657

2(c) Numerical approximations and maximum errors for step size h=0.025;

Euler's Modified Method		Runge-Kutta Method		Exact Solution
	ER	ER		
0.9137386 3169801	0.000229504	0.9135091276397532	2.59029E−10	0.913509127898782
0.849562976531398	0.000344458	0.8492185180509749	6.51469E−10	0.849218518702443
0.802223388565394	0.000399991	0.8018233969598182	9.97784E−10	0.8018233979576023
0.768207590571715	0.000424004	0.7677835848899629	1.26955E−09	0.7677835861595071
0.745120887406124	0.000431187	0.7446896989999765	1.47866E−09	0.7446897004786337
0.731317887808574	0.000429485	0.7308884011344937	1.64401E−09	0.7308884027785085
0.725674565817607	0.000423267	0.7252512974898684	1.78223E−09	0.7252512992720983
0.727442024142068	0.000414938	0.7270270843117828	1.90588E−09	0.7270270862176577

0.736149394556421	0.000405806	0.7357435865210894	2.02387E−09	0.7357435885449581
0.751536906604233	0.000396555	0.7511403498157103	2.14228E−09	0.7511403519579868

2(d): Numerical approximations and maximum errors for step size h=0.0125;

Euler's Modified Method		Runge-Kutta Method		Exact Solution
	ER		ER	
0.9135226 4673271	1.35188E-05	0.9135091278869912	1.17910E−11	0.913509127898782
0.849238872455001	2.03538E-05	0.8492185186679586	3.44851E−11	0.849218518702443
0.801847096047129	2.36981E-05	0.8018233979021255	5.54771E−11	0.8018233979576023
0.767808764293526	2.51781E-05	0.7677835860871474	7.23600E−11	0.7677835861595071
0.744715355453644	2.5655E-05	0.7446897003930565	8.55770E−11	0.7446897004786337
0.730913999935655	2.55972E-05	0.7308884026823441	9.61640E−11	0.7308884027785085
0.725276562956256	2.52637E-05	0.7252512991670091	1.05089E−10	0.7252512992720983
0.727051884359835	2.47981E-05	0.7270270861045542	1.13103E−10	0.7270270862176577
0.735767867647858	2.42791E-05	0.7357435884242071	1.20751E−10	0.7357435885449581
0.751164100109758	2.37482E-05	0.751140351829581	1.28405E−10	0.7511403519579868

V. Discussion Of Results

The results which are obtained shown in Tables 1(a)-(d) and Tables 2(a)-(d). The numerical approximate solution is calculated with step sizes 0.1, 0.05, 0.025 and 0.0125 and utmost errors also are evaluated at particular step size. From the tables for each technique we state that a numerical solution converges to the accurate solution if the step size leads to decreased errors such that in the limit when the step size to zero the errors also tends to zero. We saw that the Euler method, Euler modified method iterations using the step size 0.1 and 0.05 didn't converge to exact solution but for step size 0.025 and 0.0125 converge gradually to exact solution. Also we observed that the Runge-Kutta approximations for same step size converge firstly to exact solution. This shows that the small step size gives the improved estimation. The Runge-Kutta method of order four requires four evaluations per step, so it should give more accurate results in compare with Euler method, Euler modified method with one-fourth the step size if it is to be superior. Lastly we observed that the fourth order Runge- Kutta method is converging quicker than the Euler method, Euler modified method and it is the best efficient method for solving initial value problems for ordinary differential equations.

VI. Conclusion

In this paper, Euler improved, Euler modified method and Runge-Kutta method are used for solving ordinary differential equation (ODE) in initial value problems (IVP). Finding more accurate results needs the step size smaller for all methods. From the figures we can see the accuracy of the methods for decreasing the step size h. The numerical solutions obtained by the three proposed methods are in good agreement with exact solutions. Comparing the results of the three methods under investigation, we observed that the rate of convergence of Euler improved, Euler modified method is O(h) and the rate of convergence of fourth-order Runge Kutta method is $O(h^4)$. The Euler improved, Euler modified method was found to be less accurate due to the inaccurate numerical results that were obtained from the approximate solution in comparison to the exact solution. From the study the Runge-Kutta method was found to be generally more accurate and also the approximate solution converged faster to the exact solution when compared to the Euler improved, Euler modified method. It may be concluded that the Runge Kutta method is powerful and more efficient in finding numerical solutions of initial value problems (IVP).

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