IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 4 Ser. II (Jul – Aug 2019), PP 25-34
www.iosrjournals.org

Statistical Order Convergence and Statistically Relatively Uniform Convergence in Riesz Space of Double Sequence

[1] M. M. Plata, [2]A. M. Brono, [3]A. G. K. Ali
[1] Department of Mathematics and Statistics University of Maiduguri, Borno State Nigeria
[2]Department of Mathematics and Statistics University of Maiduguri, Borno State Nigeria
[3]Department of Mathematics Kashim Ibrahim College of Education Maiduguri, Borno State Nigeria

Abstract: *In this paper, the study seeks to introduce the concept of statistical order convergence and statistically relatively uniform convergence in Riesz spaces of double sequences. Which is an extension of that of recently introduced for single sequences. We shall also give the analogous definitions of statistical order convergence, statistically relatively uniform convergence and norm statistical convergence of double sequences. Finally, we shall explore and establish some inclusion relations among these concepts.*

Keywords and phrases: *Statistical convergence, Riesz space, double sequence, relatively uniform ,order convergence.*
Mathematics subject classification: Primary 40F50, 40J05, 40G05

Date of Submission: 18-07-2019 Date of acceptance: 03-08-2019

I. Introduction

The first idea of statistical convergence goes back to Zygmund monograph (1952), where Zygmund called it almost convergence. The concept of statistical convergence was formally introduced by Steinhaus (1951) and Fast (1951) independently and later reintroduced by Schoenberg (1959). Statistical convergence as a generalization of the usual notion of convergence has attracted much attention since it was introduced over sixty years ago. This concept has been applied in various areas such as number theory and probability theory. Furthermore, generalizations have appeared in Hausdorff topological spaces and functions spaces, locally convex space and Banach spaces.

The concepts of order convergence and relatively uniform convergence are two fundamental concepts and basic tools in the study of Riesz spaces. The two big books Luxemburg and Zaanen (1971) and Zaanen (1983) present a good detailed investigation of these two concepts. Sencimenand Pehlivan (2012) introduced the concepts of statistical order convergence which is a natural generalization of order convergence. Meanwhile some basic definitions and results are established. Ercan (2009) introduced the notion of statistically relatively u-uniformly convergent sequences in Riesz spaces, which is a statistical analogue of u-uniform convergence sequence.

Xue and Tao (2018) recently introduced the concept of statistical order convergence and statistically relatively uniform convergence in Riesz spaces and established inclusion relations between these two concepts. In this paper, we shall further introduced the concept of statistical order convergence and statistically relatively uniform convergence in Riesz spaces of double sequences and explore some inclusion relations in these concepts of double sequences.

Notations: throughout this paper, $\mathbb{N}$ will denote the set of all positive integers and $\mathbb{R}$ will denote the set of real numbers. Let E be a Riesz space and the set of all positive elements of E is denoted by E_+. Let$\left(x_{m,n}\right)_{m,n\in\mathbb{N}}$ be a double sequence in a Riesz space E. If $\left(x_{m,n}\right)_{m,n\in\mathbb{N}}$ is increasing (decreasing), we shall write $x_{m,n}\uparrow(x_{m,n}\downarrow)$. If moreover, $\sup_{m,n\in\mathbb{N}}x_{m,n}=x$ $(\inf_{m,n\in\mathbb{N}}x_{m,n}=x)$ exist, then we denote this by $x_{m,n}\uparrow x\left(x_{m,n}\downarrow x\right)$. Given a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in a normed Riesz space E and an infinite subset $A\subseteq\mathbb{N}$, we say that $(x_{m,n})_{m,n\in\mathbb{N}}$ converges in norm to $x\in E$ along A if, for every $\epsilon>0$, there exists a natural number N such that for each $m,n>N, m,n\in A$ we have $\left\|x_{m,n}-x\right\|<\epsilon$. We denote this by $\lim_{m,n\in A}x_{m,n}=x$. If a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in a Reisz space E satisfies property P for all m,n except a natural density zero, then we say that $(x_{m,n})_{m,n\in\mathbb{N}}$ satisfies property P for almost all m,n following the concept of Fridy (1985) we abbreviate this by $a.a.m,n$. A Reisz space E is called a $Dedekind\ \sigma\text{-}complete$ if every nonempty finiteor countable subset which is order bounded from above has a supremum and every nonempty finite or countable subset which is order bounded from below has an infimum.

II. Statistically Relatively Uniform Convergence of Double Sequence

Let E be a Reisz space and let $\mu \in E_+$. A double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in E is said to converge μ-uniformly to $x \in E$ whenever, for every $\epsilon > 0$, there exists a natural number N such that $|x_{m,n} - x| \leq \epsilon\mu$ holds for all $m, n \geq N$. We denote this by$x_{m,n} \xrightarrow{\mathrm{ru}_2} x(\mu)$. It is said that the double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in E converges relatively uniformly to x whenever $(x_{m,n})_{m,n\in\mathbb{N}}$ converges to x for some $\mu \in E_+$. We shall write $x_{m,n} \to x(\mathrm{ru}_2)$.

It should be noted that, in an Archimedean space E, a double sequence $x_{m,n} \xrightarrow{\mathrm{ru}_2} x(\mu)$ if and only if there exists a double sequence $(\epsilon_{m,n})_{m,n\in\mathbb{N}}$ of positive real with $\epsilon_{m,n} \downarrow 0$ such that
$|x_{m,n} - x| \leq \epsilon_{m,n}\mu$ for all $m, n \in \mathbb{N}$.
We now, give the definition of μ-uniformly convergent double sequences in Riesz spaces as follows:
Definition 2.1: Let E be a Riesz space and $\mu \in E_+$.A double sequence$(x_{m,n})_{m,n\in\mathbb{N}}$ in E is said to be statistically μ-uniformly convergent to $x \in E$ if

$$\lim_{m,n\to\infty} \frac{1}{m,n} \left|\left\{j,k: j \leq m, k \leq n, \left(|x_{j,k} - x| - \epsilon\mu\right)^+ > 0\right\}\right| = 0$$

For each $\epsilon > 0$. Equivalently, $|x_{m,n} - x| \leq \epsilon\mu$, a.a.m,n. for each $\epsilon > 0$. We write $x_{m,n} \xrightarrow{\mathrm{st}_2} x(\mu)$. We say that a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in E converges statistically relatively uniformly to x provided that $x_{m,n} \xrightarrow{\mathrm{st}_2} x(\mu)$ for some $\mu \in E_+$. We write $x_{m,n} \xrightarrow{\mathrm{sru}_2} x(\mu)$.
The following lemma about sets of natural density is the key to establishing many results regarding statistically convergent of double sequences.
Lemma 2.1. Let $\{A_{i,j}\}_{i,j\in I}$ be a countable collection of subsets of $\mathbb{N}$ such that $\delta(A_{i,j}) = 1$ for each $i, j \in I$. Then there is a subset $A \subseteq \mathbb{N}$ such that $\delta_2(A) = 1$ and $|A \backslash A_{i,j}| < \infty$ for all $i, j \in I$.
Theorem 2.1: Let E be an Archimedean Riesz space and $\mu \in E_+$. The following are equivalent for a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in E and $x \in E$:
(1) $x_{m,n} \xrightarrow{\mathrm{st}_2} x(\mu)$
(2) There exist a double sequence $(\epsilon_{m,n})_{m,n\in\mathbb{N}}$ of positive reals with $\epsilon_{m,n} \downarrow 0$ such that
$|x_{m,n} - x| \leq \epsilon_{mn}\mu$, a.a.$m, n$.
Proof:
(1) $\Longrightarrow$ (2). Suppose that $x_{m,n} \xrightarrow{\mathrm{st}_2} x(\mu)$. For each $j, k \in \mathbb{N}$, we set

$A_{j,k} = \left\{(m,n) \in \mathbb{N} \times \mathbb{N}: |x_{m,n} - x| \leq \frac{1}{j,k}\mu\right\}$.
Then $\delta_2(A_{j,k}) = 1$ and $A_{j,k} \supseteq A_{j+1,k+1}$ for each $j, k \in \mathbb{N}$.
Let E_μ be the principal ideal generated by the singleton $\{\mu\}$.Wedefine

$$\|x\|_\mu = inf\{\lambda > 0: |x| \leq \lambda\mu\}$$

for each $x \in E_\mu$. Then $\|.\|_\mu$ is a lattice norm on E_μ and $|x| \leq \|x\|_\mu\mu$ for all $x \in E_\mu$.
By lemma 2.1, we get a subset $A \subseteq \mathbb{N}$ such that $\delta_2(A) = 1$ and $|A \backslash A_{j,k}| < \infty$ for all $j, k \in \mathbb{N}$.
Claim: $\lim_{m,n\in A\cap A_{11}} \|x_{m,n} - x\|_\mu = 0$.
Indeed, for every $\epsilon > 0$, we choose $j_0, k_0 \in \mathbb{N}$with $\frac{1}{j_0k_0} < \epsilon$. We set
$N = max_{n\in A\backslash A_{j_0,k_0}} m, n$. Then for each $m, n > N, m, n \in A$, we have

$$\|x_{m,n} - x\|_\mu \leq \frac{1}{j_0k_0} < \epsilon$$

Let $A \cap A_{11} = \{j, k: j, k \in \mathbb{N}\}$. Then $\delta(A \cap A_{11}) = 1$.
We let $\epsilon_{m,n} = \sup_{j>m,k>n} \left\|x_{j_p,k_q - x}\right\|_\mu$ for each $m, n \in \mathbb{N}$. Then $\epsilon_{m,n} \downarrow 0$ and

$$\left|x_{j_p,k_q} - x\right| \leq \left\|x_{j_p,k_q} - x\right\|_\mu \mu \leq \epsilon_{m,n}\mu, \quad m, n = 1,2, \ldots$$

Let $\eta_{m,n} = \epsilon_{j,k}$ for $j_{j-1} \leq m \leq k_{k-1}, j_j \leq n \leq k_k, (j, k = 1,2, \ldots)$, where $k_0 = 0$. Then $\eta_{m,n} \downarrow 0$ and

$$\left|x_{j_p,k_q} - x\right| \leq \epsilon_{j,k}\mu = \eta_{j_{p,q}k_{r,s}}\mu, \quad p, q, r, s = 1,2, \ldots$$

This implies

$$A \cap A_{11} \subseteq \{m, n \in \mathbb{N}: |x_{m,n} - x| \leq \eta_{m,n}\mu\}.$$

Hence

$$\delta_2(\{(m,n) \in \mathbb{N} \times \mathbb{N}: |x_{m,n} - x| \leq \epsilon_{m,n}\mu\}) = 1.$$

(2) $\Longrightarrow$ (1). Let $(\epsilon_{m,n})_{m,n\in\mathbb{N}}$ be the double sequences of positive reals as stated (2). Let $K = \{(m,n) \in \mathbb{N} \times$ *N:xm,n−x≤∊m,nμ* and *∊>0.* Choose *k0,j0* with *∊j0,k0<∊.* Then, for *m,n∈K* and *m>j0* and *n>k0* we have

$$|x_{m,n} - x| \le \epsilon_{m,n}\mu \le \epsilon_{j_0,k_0} \le \epsilon\mu$$

Since

$\delta_2(\{(m,n) \in K \times K : m > j_{0,}n > k_0\}) = 1$,

we get

$$\delta_2(\{(m,n) \in \mathbb{N} \times \mathbb{N} : |x_{m,n} - x| \le \epsilon\mu\}) = 1.$$

This completes the proof.

Note that a subsequence of statistically relatively uniform convergent need not be statistically relatively uniform convergent and a statistically relatively uniform convergent of double sequence need not be relatively uniform convergent. These facts can be seen in the following easy example.

Example 2.1: Let $(x_{m,n})_{m,n\in\mathbb{N}}$ be a double sequence in $\mathbb{R}$ defined by

$$x_{m,n} = \begin{cases} mn, & m,n = j^2, k^2 \quad (j,k \in \mathbb{N}) \\ \dfrac{1}{mn}, & otherwise \end{cases}$$

Then, $x_{m,n} = \frac{1}{m,n}$, a.a.m,n and hence, by theorem 2.1, we have $x_{m,n} \xrightarrow{\text{st}_2} 0$, but, it is obvious that the subsequence $(x_{m,n})_{m,n\in K}$ of $(x_{m,n})_{m,n\in\mathbb{N}}$, where $K = \{j^2, k^2 : j,k \in \mathbb{N}\}$, is not statistically relatively uniform convergent. Since the sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ is not order bounded.$(x_{m,n})_{m,n\in\mathbb{N}}$is not relatively uniform convergent.

Remark 2.1: statistically relatively uniform convergence is stable that is if $x_{m,n} \xrightarrow{\text{sru}\,2} 0$, then there exists a double sequence $(\lambda_{m,n})_{m,n\in\mathbb{N}}$ of reals with $0 \le \lambda_{m,n} \uparrow \infty$ such that $\lambda_{m,n}x_{m,n} \xrightarrow{\text{sru}\,2} 0$. Indeed, there exist $\mu \in E_+$ and a double sequence $(\epsilon_{m,n})_{m,n\in\mathbb{N}}$ of reals with $\epsilon_{m,n} \downarrow 0$such$|x_{m,n}| \le \epsilon_{m,n}\mu$, a.a.$m,n$.

That is

$$\frac{1}{\sqrt{\epsilon_{mn}}}|x_{m,n}| \le \sqrt{\epsilon_{mn}\,\mu}, \text{ a.a.} m,n.$$

Hence,

$$\frac{1}{\sqrt{\epsilon_{mn}}}x_n \xrightarrow{\text{sru}\,2} 0 .$$

We give another useful characterization of statistically μ-uniform convergence.

Theorem 2.2. Let E be an Archimedean Riesz space and $\mu \in E_{+.}$ The following are equivalent for a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in $E$and $x \in E$.

(1) $x_{m,n} \xrightarrow{\text{st}_2} x(\mu)$

(2) There exists a subset $K = \{j_p, k_q : p,q \in \mathbb{N}\}$ of $\mathbb{N}$ with $\delta_2(K) = 1$ such that $x_{j_p k_{q,}} \xrightarrow{\text{ru}\,2} x(\mu)$.

(3) There exists a double sequence $(y_{m,n})_{m,n\in\mathbb{N}}$ in E such that $x_{m,n} = y_{m,n}$ a.a.m,n. and $y_{m,n} \xrightarrow{\text{ru}\,2} x(\mu)$.

Proof: the implication (1) $\Longrightarrow$ (2) follows from theorem 2.1.

(2) $\Longrightarrow$ (3) Assume that there exist a subset $K = \{j_p, k_q : p,q \in \mathbb{N}\}$ of $\mathbb{N}$ with $\delta_2(K) = 1$ such that$x_{j_p k_{q,}} \xrightarrow{\text{ru}\,2} x(\mu)$ then there exist a double sequence $(\epsilon_{p,q})_{p,q\in\mathbb{N}}$ of positive reals with $\epsilon_{p,q} \downarrow 0$ such that $|x_{j_p,k_q} - x| \le \epsilon_{p,q}\mu$ for all $p,q \in \mathbb{N}$.

We set

$$y_{m,n} = \begin{cases} x_{m,n}, & m,n \in K \\ x, & m,n \notin K \end{cases}$$

And $\eta_{m,n} = \epsilon_{p,q}forj_{p-1} < m < k_{q-1}, j_p < n < k_q$ $(p,q = 1,2,\ldots)$, where $j_0 = 0,\ k_0 = 0$.

Then, $x_{m,n} = y_{m,n}$ a.a.m,n., $\eta_{m,n} \downarrow 0$, and $|y_{m,n} - x| \le \eta_{m,n}\mu$ for all $m,n \in \mathbb{N}$.

Hence,

$$y_{m,n} \xrightarrow{\text{ru}\,2} x(\mu).$$

(3) $\Longrightarrow$ (1)is trivial.

Corollary 2.1: Let E be an Archimedean Riesz space, then every monotone statistically relatively uniform convergent double sequence in E is a relatively uniformly convergent.

Proof: suppose that $(x_{m,n})_{m,n\in\mathbb{N}}$ is an increasing double sequence in E and $x_{m,n} \xrightarrow{\text{st}_2} x(\mu)$ for some $x \in E$and $\mu \in E_+$. Followings the concept of Ercan (2009) we have that $\sup_{m,n\in\mathbb{N}}x_{m,n}$ exists and equals x. By Theorem

2.2 There exists a subset $K=\{j_p,k_q:p,q\in\mathbb{N}\}$ of $\mathbb{N}$ with $\delta_2(K)=1$ such that $x_{j_p,k_q}\overset{\mathrm{ru}_2}{\longrightarrow}x(\mu)$. Thus, we get a double sequence $(\epsilon_{p,q})_{p,q\in\mathbb{N}}$ of positive reals with $\epsilon_{p,q}\downarrow 0$ such that $\left|x_{j_p,k_q}-x\right|<\epsilon_{p,q}\mu$for $p,q\in\mathbb{N}$. We set

$$\eta_{m,n}=\begin{cases}\epsilon_{11}, & 1\le m\le j_1, 1\le n\le k_1\\ \epsilon_{p-1,q-1}, & j_{p-1}<m\le j_p, k_{q-1}<n\le k_q\end{cases}$$

Then $\eta_{m,n}\downarrow 0$. We set $u=(x-x_{11})\vee\mu$. Then, for $1\le m\le j_1, 1\le n\le k_1$, we have

$$x-x_{m,n}\le x-x_{11}\le \eta_{11}u.$$

For $k_{p-1,q-1}<m,n\le k_{p,q}$ $\quad(p,q=1,2,\dots)$, we have

$$x-x_{m,n}\le x-x_{j_{p-1},k_{q-1}}\le \eta_{m,n}\mu\le\eta_{m,n}u.$$

Hence, $x_{m,n}\overset{\mathrm{ru}_2}{\longrightarrow}x(u)$. This completes the proof.

Let E be a Riesz space and let $\mu\in E_+$. Following Luxemburg and Zaanen (1971), Recall that a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in E is called a μ-uniform Cauchy double sequence whenever, for every $\epsilon>0$,there exist a natural numbersM,Nsuch that $\left|x_{m,n}-x_{p,q}\right|\le\epsilon\mu$ holds for all $m,p\ge M$ and $n,q\ge N$. We say that a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ is uniform Cauchy if it is μ-uniform Cauchy for some $\mu\in E_+$.

Definition 2.1: Let E be a Riesz space and let $\mu\in E_+$. We say that a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in E is statistically μ-uniform Cauchy double sequence whenever, for every $\epsilon>0$, there exist a natural numbersM,N such that $\left|x_{m,n}-x_{M,N}\right|\le\epsilon\mu$, a.a.$m,n$. We say that a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ is statistically uniform Cauchy if it is statistically μ-uniform Cauchy for some $\mu\in E_+$.

Following the concepts of Sencimen and Pehlivan (2012), a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in Riesz space E is said to be statistically order bounded if there exist $y,z\in E$ such that $y\le x_{m,n}\le z$, a.a.m,n. it is easy to see that every statistically μ-uniform Cauchy double sequence is statistically order bounded for all $\mu\in E_+$. It should be noted that a statistically uniform Cauchy double sequence need not be uniform Cauchy as can be seen in the following example.

Example 2.2: Let $K=\{j^2,k^2:j,k\in\mathbb{N}\}$. For each $m,n\in\mathbb{N}$, define $f_{m,n}\in C[0,1]$ by

$$f_{m,n}=\begin{cases}m,n, & m,n\in K\\ \frac{\mu}{mn}, & m,n\notin K\end{cases}$$

Where $\mu(t)=t, t\in[0,1]$. Then, $f_{m,n}=\frac{\mu}{mn}$, a.a.m,n. and hence $f_{m,n}\overset{\mathrm{st}_2}{\longrightarrow}0(\mu)$. Thus, we see that the double sequence $(f_{m,n})_{m,n\in\mathbb{N}}$ is statistically μ-uniform Cauchy, but not uniform Cauchy since $\sup_{m,n\in\mathbb{N}}=\left\|f_{m,n}\right\|=\infty$.

Theorem 2.3: Let E be a Riesz space and $\mu\in E_+$, the following are equivalent for a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in E.

(1) $(x_{m,n})_{m,n\in\mathbb{N}}$ is statistically μ-uniform Cauchy double sequence in E.

(2) There exists a subset $K=\{j_p,k_q:p,q\in\mathbb{N}\}$ of $\mathbb{N}$ with $\delta_2(K)=1$ such that $(x_{j_p,k_q})_{p,q\in\mathbb{N}}$ is μ-uniform Cauchy sequence.

Proof: (1) $\Longrightarrow$ (2). For each $j,k\in\mathbb{N}$, by (1), we get a natural number N_k such that

$$\delta_2\left(\left\{m,n\in\mathbb{N}:\left|x_{m,n}-x_{N_{j,k}}\right|\le\frac{1}{jk}\mu\right\}\right)=1.$$

Let $A_{j,k}=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\left|x_{m,n}-x_{N_{j,k}}\right|\le(\frac{1}{jk})\mu\right\}(j,k=1,2,\dots)$. It follows from Lemma (2.1), that there exists a subset $K\subseteq\mathbb{N}\times\mathbb{N}$ with $\delta_2(K)=1$ such that $\left|K\backslash A_{j,k}\right|<\infty$ for all $j,k\in\mathbb{N}$. Let $\epsilon>0$, choose$j_{0,}k_0$ such that $\frac{1}{j_0k_0}<\frac{\epsilon}{2}$ since the set $K\backslash A_{j_0,k_0}$ is finite, there exists a natural number N such that for all $m,n>N, m,n\in K$, we have

$$\left|x_{m,n}-x_{N_{j_0,k_0}}\right|\le\frac{1}{j_0k_0}\mu\le\frac{\epsilon}{2}\mu,$$
$$\left|x_{p,q}-x_{N_{j_0,k_0}}\right|\le\frac{1}{j_0k_0}\mu\le\frac{\epsilon}{2}\mu.$$

Hence, we get

$$\left|x_{m,n}-x_{p,q}\right|\le\epsilon\mu\ .$$

This means that the double sequence$(x_{m,n})_{m,n\in\mathbb{N}}$ is a μ-uniform Cauchy double sequence along K.

(2) $\Longrightarrow$ (1). Let K be stated as in (2). Let $\epsilon>0$. There exists a natural number j_0,k_0 such that

$$\left|x_{j_p,k_q}-x_{j_r,k_s}\right|\le\epsilon\mu,\quad\forall p,q,r,s\ge j_{0,}k_0.$$

This implies that $\{j_r,k_s:r\ge j_0,s\ge k_0\}\subseteq\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\left|x_{m,n}-x_{j_{j_0,}k_{k_0}}\right|\le\epsilon\mu\right\}$.

Hence we get that $\left|x_{m,n}-x_{j_{j_0,}k_{k_0}}\right|\le\epsilon\mu$, a.a.$m,n$. this completes the proof.

Corollary 2.2: Let E be a Riesz space. Then every monotone statistically μ-uniform Cauchy double sequence is μ-uniform Cauchy for all $\mu\in E_+$.

Proof: suppose that $(x_{m,n})_{m,n\in\mathbb{N}}$ is an increasing statistically μ-uniform Cauchy double sequence in E. It follows from Theorem 2.3 that there exists a strictly increasing double sequence $\left(j_p k_q\right)_{p,q\in\mathbb{N}}$ of positive integers such that $(x_{j_r,k_s})_{r,s\in\mathbb{N}}$ is a μ-uniform Cauchy double sequence. Let $\epsilon > 0$. Then there exists a natural number r_0, s_0 such that

$$\left|x_{j_p k_q} - x_{j_r k_s}\right| \leq \epsilon\mu \; for\; all\; p,r \geq r_{0,}q,s \geq s_0.$$

For all , $h > m \geq j_{r_0}, n \geq k_{s_0}$, if $j_r \leq m, k_s \leq n < g, h < j_{r+1,}k_{s+1}$ for some $r \geq r_0, s \geq s_{0.}$ Then we have

$$0 \leq x_{g,h} - x_{m,n} \leq x_{j_{r+1},k_{s+1}} - x_{j_r,k_s} \leq \epsilon\mu.$$

If$j_r, k_s \leq m, n < j_{r+1}, k_{s+1}$ for some $r > r_0, s > s_0$ and $j_p, k_q \leq g, h < j_{p+1,}k_{q+1}$ for some $p > r, q > s$ then we have

Hence, $(x_{m,n})_{m,n\in\mathbb{N}}$ is μ-uniform cauchy.

Let E be a Riesz space and $0 < \mu \in E$. Following the concept of Ercan (2009), we have that a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in E is said to be statistically μ-uniform pre-cauchy if

$$\lim_{m,n\to\infty} \frac{1}{m^2,n^2}\left|\left\{(p,i)(q,j): p,i > m \text{ and } q,j > n\left(\left|x_{p,q} - x_{i,j}\right| - \epsilon\mu\right)^+ > 0\right\}\right| = 0.$$

For every $\epsilon > 0$.We say that a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ is statistically uniform pre-Cauchy if it is statistically μ-uniform pre-cauchy for some $0 < \mu \in E$.

Corollary 2.3: Every statistically μ-uniform Cauchy double sequence in Riesz space E is statistically μ-uniform pre-Cauchy for all $0 < \mu \in E$.

Proof: Let $(x_{m,n})_{m,n\in\mathbb{N}}$ be a statistically μ-uniform Cauchy double sequence in E. By Theorem 2.3, there exists a subset $K \subseteq \mathbb{N}\times\mathbb{N}$ with $\delta_2(K) = 1$ such that $(x_{m,n})_{m,n\in K}$ is a μ-uniform Cauchy along K. Let $\epsilon > 0$, there exists a subset $A \subseteq K$ such that $K\backslash A$ is finite and

$$A \times A \subseteq \left\{(p,j),(q,k): \left|x_{p,q} - x_{j,k}\right| \leq \epsilon\mu\right\}.$$

Note that, for each $m, n \in \mathbb{N}$, we have

$$\left|A_{m,n}\right|^2 = |\{(p,j),(q,k) \in A \times A: p,j \leq m, q,k \leq n\}|.$$

Hence

$$\frac{1}{m^2,n^2}\left|A_{m,n}\right|^2 \leq \frac{1}{m^2,n^2}\left|\left\{(p,j),(q,k): p,j \leq m, q,k \leq n, \left|x_{p,q} - x_{j,k}\right| \leq \epsilon\mu\right\}\right|$$

Since $\delta_2(A) = 1$, we get, by letting $m, n \to \infty$,

$$\lim_{m,n\to\infty} \frac{1}{m^2,n^2}\left|\left\{(p,j),(q,k): p,j \leq m, q,k \leq n, \left|x_{p,q} - x_{j,k}\right| \leq \epsilon\mu\right\}\right| = 1$$

This completes the proof.

The following example shows that a statistically uniform pre-Cauchy double sequence need not be statistically uniform Cauchy.

Example 2.3: Let

$$x_{j,k} = \sum_{p,q=1,1}^{m,n} \frac{1}{pq} m! \leq j < (m+1)!, n! \leq k < (n+1)!, \qquad j,k,m,n = 1,2,...$$

Then $(x_{j,k})_{j,k\in\mathbb{N}}$ is increasing and tends to ∞. Hence, the double sequence $(x_{j,k})_{j,k\in\mathbb{N}}$ has no convergent double subsequences and so is not statistically convergent. It follows from the concept of Fridy (1985) that $(x_{j,k})_{j,k\in\mathbb{N}}$ is not statistically Cauchy and hence is not statistically uniform Cauchy. Following the concept of Connor, Fridy and Kline (1994) we have that $(x_{j,k})_{j,k\in\mathbb{N}}$ is statistically pre-cauchy. This means that $(x_{j,k})_{j,k\in\mathbb{N}}$ is statistically I-uniform pre-Cauchy.

Theorem 2.4: Let E be an Archimedean Riesz space and $\mu \in E_+$. Then

(1) Every statistically μ-uniformly convergent double sequence in E is statistically μ-uniform Cauchy.

(2) If, moreover, E is Dedekind σ-complete, then every monotone statistically μ-uniform Cauchy double sequence in E is statistically μ-unifromly convergent.

Proof: (1) Suppose that$x_{m,n} \xrightarrow{\text{st}_2} x(\mu)$. Let $\epsilon > 0$, then $\delta_2\left(\left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \left|x_{m,n} - x\right| \leq (\epsilon/2)\mu\right\}\right) = 1$. Let $K = \left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \left|x_{m,n} - x\right| \leq (\epsilon/2)\mu\right\}$ and $N = \min_{m,n\in K} m,n$. Then, for every $m, n \in K$, we have

$$\left|x_{m,n} - x_{M,N}\right| \leq \left|x_{m,n} - x\right| + \left|x_{M,N} - x\right| \leq \frac{\epsilon}{2}\mu + \frac{\epsilon}{2}\mu = \epsilon\mu.$$

This implies that $K \subseteq \left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \left|x_{m,n} - x_{M,N}\right| \leq \epsilon\mu\right\}$ and hence $\delta_2\left(\left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \left|x_{m,n} - x_{M,N}\right| \leq\right.\right.$ $\epsilon\mu$=1.

(2) Let $(x_{m,n})_{m,n\in\mathbb{N}}$ be an increasing statistically μ-uniform Cauchy sequence in E.Then$(x_{m,n})_{m,n\in\mathbb{N}}$ is statistically order bounded. Since $(x_{m,n})_{m,n\in\mathbb{N}}$ is increasing, we see that $(x_{m,n})_{m,n\in\mathbb{N}}$ is order bounded. Since E is Dedekind σ-complete, then $\sup_{m,n} x_{m,n}$ exists and it is denoted by x.

Let $\epsilon > 0$. Then there is a natural number M, N such that $|x_{m,n} - x_{M,N}| \leq \epsilon\mu$, a.a.m,n. Let $K = \{(m,n) \in \mathbb{N} \times \mathbb{N}: x_{m,n} - x_{M,N} \leq \epsilon\mu$ and $A = \{(m,n) \in K \times K: m \geq M, n \geq N\}$. Then $\delta_2 K = \delta_2 A = 1$.
Let us fix any $m, n \in A$. Then, for any $p \geq m, q \geq n, p, q \in A$, we have

$$x_{p,q} - x_{m,n} \leq x_{p,q} - x_{M,N} \leq \epsilon\mu.$$

That is,

$$x_{p,q} \leq x_{m,n} + \epsilon\mu,\ \forall\, p,q \in \{(j,k) \in A \times A: j \geq m, k \geq n\}.$$

Since $(x_{m,n})_{m,n\in\mathbb{N}}$ is an increasing, we get $x = \sup_{p,q\in\{j,k\in A: j\geq m,k\geq n\}} x_{p,q}$.
Hence, we get

$$x \leq x_{m,n} + \epsilon\mu, \forall\, m,n \in A.$$

This means

$$0 \leq x - x_{m,n} \leq \epsilon\mu, \quad \text{a.a.}m,n.$$

Thus,

$$x_{m,n} \xrightarrow{\text{st}_2} x(\mu).$$

It follows from Theorem 2.4 and Corollary 2.3 that every statistically μ-unifromly convergent double sequence is statistically μ-uniform pre-cauchy for all $\mu \in E_+$. The next result suggests that, under certain circumstances, the converse is also true.

Theorem 2.5: Let E be an Archimedean Riesz space and $\mu \in E_+$. Assume that $(x_{m,n})_{m,n\in\mathbb{N}}$ is statistically μ-uniform pre-cauchy double sequence in E. If $(x_{m,n})_{m,n\in\mathbb{N}}$ has a double subsequence$(x_{m,n})_{m,n\in K}$ which converges μ-uniformly to $x \in E$ and $\liminf_{m,n\to\infty}(|K_{m,n}|/m,n) > 0$.Then $x_{m,n} \xrightarrow{\text{st}_2} x(\mu)$.

Proof: Let $\epsilon > 0$. Since $(x_{m,n})_{m,n\in K}$ converges μ-uniformly to x. There exists a subset $A \subseteq K$ such that K/A is finite and $|x_{p,q} - x| \leq \frac{\epsilon}{2}\mu \ \ \forall\, p,q \in A$.
We set $B = \left\{(j,k) \in \mathbb{N} \times \mathbb{N}: \left(|x_{j,k} - x| - \epsilon\mu\right)^+ \neq 0\right\}$. Then, for $(p,j),(q,k) \in A \times B$, we have

$$\left(|x_{p,q} - x_{j,k}| - \frac{\epsilon}{2}\mu\right)^+ \neq 0.$$

That is,

$$A \times B \subseteq \left\{(p,j),(q,k): \left(|x_{p,q} - x_{j,k}| - \frac{\epsilon}{2}\mu\right)^+ \neq 0\right\}.$$

This implies

$$\frac{|A_{m,n}|}{mn}\frac{|Bm,n|}{mn} \leq \frac{\left|\left\{(p,j),(q,k): \left(|x_{p,q} - x_{j,k}| - (\epsilon/2)\mu\right)^+ \neq 0, \quad p,j \leq m, q,k \leq n\right\}\right|}{m^2n^2}$$

Letting $m^2, n^2 \to \infty$, we get $\delta_2(B) = 0$. This completes the proof.

III. Relationship to Statistical Order convergence and Norm Statistical Convergence of Double Sequence.

It follows from the concept of Luxemburg and Zaanen (1971), a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in a Riesz space E is order convergent to $x \in E$ whenever there exists a double sequence $p_{m,n} \downarrow 0$ in E such that $|x_{m,n} - x| \leq p_{m,n}$ hold for all m, n.

Definition 3.1: Let $(x_{m,n})_{m,n\in\mathbb{N}}$ be a double sequence in Riesz space E. If there exists a set $K = \{k_{p,q}\}_{p,q\in\mathbb{N}}$ with $\delta_2(K) = 1$ such that $(x_{j_p,k_q})_{p,q\in\mathbb{N}}$ is increasing and $\sup_{p,q\in\mathbb{N}} x_{j_p,k_q} = x$ for some $x \in E$, then one writes $x_{m,n} \uparrow^{\text{st}_2} x$. Similarly, if $(x_{j_p,k_q})_{p,q\in\mathbb{N}}$ is decreasing and $\inf_{p,q\in\mathbb{N}} x_{j_p k_q} = x$ for some $x \in E$, then we write $x_{m,n} \downarrow^{\text{st}_2} x$. If $x_{m,n} \uparrow^{\text{st}_2} x$`cor $x_{m,n} \downarrow^{\text{st}_2} x$, then one said that $(x_{m,n})_{m,n\in\mathbb{N}}$ is statistically monotone convergent to x.

Definition 3.2: Let Ebe a Riesz. A double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in E is statistically order convergent to $x \in E$ provided that there exists a double sequence $(p_{m,n})_{m,n\in\mathbb{N}}$ such that $p_{m,n} \downarrow^{\text{st}_2} 0$ and a subset K of $\mathbb{N}$with$\delta_2(K) = 1$. Such that $|x_{m,n} - x| \leq p_{m,n}$ for every $m, n \in K$. One writes, $x_{m,n} \xrightarrow{\text{st}_2-\text{ord}} x$.

Remark 3.1: It is easy to see that $x_{m,n} \xrightarrow{\text{st}_2-\text{ord}} x$ if and only if there exists a double sequence $(p_{m,n})_{m,n\in\mathbb{N}}$ in E with $p_{m,n} \downarrow^{\text{st}_2} 0$ such that $|x_{m,n} - x| \leq p_{m,n}$, a.a.m,n.

Theorem 3.1: Let $(x_{m,n})_{m,n\in\mathbb{N}}$be a double sequence in $E, x \in E$ and μE_+. Then $x_{m,n} \xrightarrow{\text{ru}_2} x(\mu)$ if and only if $x_{m,n} \xrightarrow{\text{st}_2-\text{ord}} x$ and $(x_{m,n})_{m,n\in\mathbb{N}}$is a$\mu$-uniform Cauchy double sequence.

Proof: the necessary part is trivial. We only prove sufficient part.
Suppose that $x_{m,n} \xrightarrow{\text{st}_2-\text{ord}} x$ and $(x_{m,n})_{m,n\in\mathbb{N}}$ is μ-uniform Cauchy. By remark 3.1, there exists a double sequence $(p_{m,n})_{m,n\in\mathbb{N}}$ in E with $p_{m,n} \downarrow 0$ such that $|x_{m,n} - x| \leq p_{m,n}$, a.a.m,n. let $K = \{(m,n) \in \mathbb{N} \times$

$\mathbb{N}: |x_{m,n} - x| \leq p_{m,n}\}$. Then $\delta_2(K) = 1$. Let $\epsilon > 0$, then there exists a natural number M, Nsuch that $|x_{p,q} - x_{m,n}| \leq \epsilon\mu$ for all $p, m \geq M, q, n \geq N$. Let us fix any $m, n \geq M, N$. Then, for any $p, q \in \{j, k \in K: j \leq m, k \leq n\}$, we have

$$|x_{m,n} - x| \leq |x_{m,n} - x_{p,q}| + |x_{p,q} - x| \leq \epsilon\mu + p_{m,n}.$$

Note that $\inf_{p,q\in K,p,q\geq m,n} p_{m,n=0.}$ Hence, we get $|x_{m,n} - x| \leq \epsilon\mu$ for all $m, n \geq N$. This completes the proof.

Theorem 3.2: Let $(x_{m,n})_{m,n\in\mathbb{N}}$ be a double sequence in $E, x \in E$and $\mu\epsilon E_+$. Then $x_{m,n} \xrightarrow{\text{st}_2} x(\mu)$ if and only if $x_{m,n} \xrightarrow{\text{st}_2-\text{ord}} x(\mu)$ and $(x_{m,n})_{m,n\in\mathbb{N}}$ is a statistically μ-uniform Cauchy sequence.

Proof: It suffices to prove the sufficient part. Assume that$x_{m,n} \xrightarrow{\text{st}_2-\text{ord}} x$and $(x_{m,n})_{m,n\in\mathbb{N}}$ is statistically μ-uniform Cauchy double sequence. Following the concept of Sencimen and Pehlivan (2012) and Theorem(2.2), we obtain a subset $K = (j_r, k_s)_{r,s\in\mathbb{N}}$ of $\mathbb{N}$ with $\delta_2(K) = 1$ such that the double sequence $(x_{j_r,k_s})_{r,s\in\mathbb{N}}$ converges in order to x and is μ-uniform Cauchy. Let $\epsilon > 0$. There exists a natural number r_0, s_0 such that

$$|x_{j_u,k_v} - x_{j_r,k_s}| \leq \epsilon\mu, \quad \forall u, r \geq r_0, v, s \geq s_0.$$

Let us fix any $r, s \geq r_0, s_0$. Let $u, v \to \infty$, we get $|x - x_{j_r,k_s}| \leq \epsilon\mu$.

This means that $x_{j_r,k_s} \xrightarrow{\text{ru}_2} x(\mu)$. It follows from Theorem 2.2 that $x_{m,n} \xrightarrow{\text{st}_2} x(\mu)$.

Theorem 3.3: Let E be an Archimedean Riesz space. The following statements are equivalent:

(1) Statistical order convergence of double sequence is stable

(2) Statistical order convergence and statistically relatively uniform convergence of double sequence are equivalent.

Proof: (1)⟹ (2) suppose that $x_{m,n} \xrightarrow{\text{st}_2-\text{ord}} 0$. Then, by (1) there exists a double sequence$(\lambda_{m,n})_{m,n\in\mathbb{N}}$ of reals with $0 \leq \lambda_{m,n} \uparrow \infty$ such that $\lambda_{m,n}x_{m,n} \xrightarrow{\text{st}_2-\text{ord}} 0$. By Remark 3.1 there exists a double sequence $(p_{m,n})_{m,n\in\mathbb{N}}$ with $p_{m,n} \downarrow 0$ such that

$$\lambda_{m,n}|x_{m,n}| \leq p_1 \text{a.a}, m, n.$$

$$|x_{m,n}| \leq \frac{1}{\lambda_{m,n}} p_1 \text{a.a. } m, n.$$

Hence, $x_{m,n} \xrightarrow{\text{st}_2} 0(p_1)$ and then $x_{m,n} \xrightarrow{\text{sru}_2} 0$.

(2) ⟹(1) assume that, $x_{m,n} \xrightarrow{\text{st}_2-\text{ord}} 0$. By (2), we see that $x_{m,n} \xrightarrow{\text{sru}_2} 0$. Then, there exists$\mu \in E_+$ and a double sequence $\epsilon_{m,n} \downarrow 0$ such that

$|x_{m,n}| \leq \epsilon_{m,n}\mu$, a.a. m, n.

Thus, we get $(1/\sqrt{\epsilon_{mn}})x_{m,n} \xrightarrow{\text{st}_2-\text{ord}} 0$. And we are done.

Note that statistical order convergence and statistically relatively uniform convergence of double sequences are not equivalent in general as can be seen in the following example.

Example 3.1 Let us take a counter example, for each $m, n \in \mathbb{N}$, we define $f_{m,n} \in C[0,1]$ by $f_{m,n}(t) = 1 \left[0, \frac{1}{mn}\right]$ and $f_{m,n}(t) = 0$ on $\left[\frac{2}{mn}, 1\right]$ and $f_{m,n}(t)$ is linear on $\left[\frac{1}{mn}, \frac{2}{mn}\right]$. Clearly, $f_{m,n} \downarrow 0$, moreover, there is no double sequence $0 \leq \lambda_{m,n} \uparrow \infty$ such that $\lambda_{m,n}f_{m,n} \xrightarrow{\text{st}_2-\text{ord}} 0$. Otherwise, it follows from the concept of Sencimen and Pehlivan (2012) there exists a subset $K = \{j_p, k_q: p, q \in \mathbb{N}\}$ of $\mathbb{N}$ with $\delta_2(K) = 1$ such thatthe double sequence $(\lambda_{j_p,k_q}f_{j_p,k_q})_{p,q\in\mathbb{N}}$ is order convergent 0. Hence there exists a double sequence $(p_{p,q})_{p.q\in\mathbb{N}}$ in E with $p_{p,q} \downarrow 0$ so that $\lambda_{j_p,k_q}f_{j_p,k_q} \leq p_{p,q}$ for all $p, q \in \mathbb{N}$. This implies

$$1 = f_{j_p,k_q}(0) \leq \frac{1}{\lambda_{j_p,k_q}} p_{p,q}(0) \leq \frac{1}{\lambda_{j_p,k_q}} p_1(0) \to 0 \qquad (p, q \to \infty)$$

This contradiction suggest that statistical order convergence in $C[0,1]$ is not stable. Then Theorem 3.3 gives the conclusion.

Following the concept of Connor, Ganichev and Kadets (2000), a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in a Banach space E is said to be norm statistically convergent to $x \in E$ provided that $\delta_2(\{(m,n) \in \mathbb{N} \times \mathbb{N}: \|x_{m,n} - x\| \geq$ *ϵ=0* for all *ϵ>0*.it is clear that a double sequence *(xm,n)m,n∈ℕ* in a Banach lattice$E$ is norm statistically convergent whenever it is statistically relatively uniformly convergent. The double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ is said to be weakly statistically convergent to x provided that the double sequence $(\langle x^*, x_{m,n} - x\rangle)_{m,n\in\mathbb{N}}$ converges statistically to 0 for each $x^* \in X^*$. Following Connor, Ganichev and Kadets (2000). It is noted that if a double sequence $(x_{m,n})_{m,n\in\mathbb{N}}$ in a Banach space E is norm statistically convergent to $x \in E$, then there exists a subset $K = (k_{p,q})_{p,q\in\mathbb{N}} \subseteq \mathbb{N}$ with$\delta_2(K) = 1$ such that $(x_{m,n})_{m,n\in\mathbb{N}}$ converges in norm to x. Combining this observation, Theorem 2.3 and Theorem 2.2with the concept of Zaanen (1983) we obtain the following result.

Theorem 3.4: Let $(x_{m,n})_{m,n\in\mathbb{N}}$ be a double sequence in a Banach lattice $E, x \in E$ and $\mu \in E_{+}$. Then $x_{m,n} \xrightarrow{\mathrm{st}_2} x(\mu)$ if and only if $(x_{m,n})_{m,n\in\mathbb{N}}$ is a norm statistically convergent to x and $(x_{m,n})_{m,n\in\mathbb{N}}$ is statistically μ-unifrom Cauchy sequence.

Following the concept of Schwarz (1984), in $C(K)$-spaces, the relatively uniform convergence of double sequence coincides with the norm convergence of double sequence. It follows from Theorem 2.2 that, in $C(K)$-spaces, the statistically relatively uniform convergence coincides with the norm statistical convergence of double sequence. In Example 3.1, utilizing Theorem 3.3, we show that statistical order convergence of double sequence and statistically relatively uniform convergence of double sequence are not equivalent in general. Now we construct a statistical order convergent of double sequence that is not statistically relatively uniformly convergent as follows.

Example 3.2: Let $K = \{j^2, k^2 : j, k \in \mathbb{N}\}$. For each $m, n \in \mathbb{N}$, we define $f_{m,n} \in C[0,1]$ by

$$f_{m,n} = \begin{cases} 1, & m,n \in K \\ g_{m,n}, & m,n \notin K, \end{cases}$$

Where $g_{m,n}(t) = t^{mn}$, $t \in [0,1]$, $m, n = 1,2, \dots$

Obviously, $g_{m,n} \downarrow 0$ and hence $f_{m,n} \xrightarrow{\mathrm{st}_2-\mathrm{ord}} 0$. Since $\|g_{m,n}\| = 1$ for all $m, n \in \mathbb{N}$, we see that the double sequence $(f_{m,n})_{m,n\in\mathbb{N}}$ is not norm statistically convergent and hence not statistically relatively uniformly convergent.

Following the concept of Sencimen and Pehlivan (2012), we suggest that the statistical order convergence need not be norm statistically convergent. The following example suggests that the norm statistical convergence need not be statistically order convergent and hence not be statistically relatively uniformly convergent.

Example 3.3: We take the double sequence $(f_{m,n})_{m,n\in\mathbb{N}}$. Let E be the normed Riesz space of all real continuous functions on [0,1] with $\|f\| = \int_0^1 |f|\,dt$. We set

$$f_{1,1}\left(\tfrac{1}{2}\right) = 1,$$
$$\|f_{1,1}\| = \tfrac{1}{2} ,$$
$$0 \le f_{1,1} \le 1 ,$$
$$f_{2,2}\left(\tfrac{1}{3}\right) = f_{2,2}\left(\tfrac{2}{3}\right) = 1 ,$$
$$\|f_{2,2}\| = \tfrac{1}{2^2} ,$$
$$0 \le f_{2,2} \le 1 ,$$

And generally

$$f_{m,n}\left(\tfrac{1}{m+1,n+1}\right) = f_{m,n}\left(\tfrac{2}{m+1,n+1}\right) = \cdots = f_{m,n}\left(\tfrac{m,n}{m+1,n+1}\right) = 1 ,$$
$$\|f_{m,n}\| = \tfrac{1}{2^{mn}} ,$$
$$0 \le f_{m,n} \le 1$$

For $m, n = 1,2, \dots$

Following the concept of Zaanen (1983), a double sequence $(f_{m,n})_{m,n\in\mathbb{N}}$ converges to 0 in norm and no subsequence of it converges to 0 in order.

We set $K = \{j^2, k^2 : j, k \in \mathbb{N}\}$ and let

$$g_{m,n} = \begin{cases} 1, & m,n \in K \\ f_{m,n}, & m,n \notin K. \end{cases}$$

Clearly, $(g_{m,n})_{m,n\in\mathbb{N}}$ is norm statistically convergent to 0. But $(g_{m,n})_{m,n\in\mathbb{N}}$ is not statistically order convergent. Indeed, if $(g_{m,n})_{m,n\in\mathbb{N}}$ is statistically order convergent to 0, using the concept Sencimen and Pehlivan (2012), there exists a subset A of $\mathbb{N}$ with $\delta_2(A) = 1$ such that $(g_{m,n})_{m,n\in A}$ is order convergent to 0. Hence, the subsequence $(f_{m,n})_{m,n\in A\cap(\mathbb{N}\setminus K)} = (g_{m,n})_{m,n\in A\cap(\mathbb{N}\setminus K)}$ is order convergent to 0. This is a contradiction.

Example 3.4: We take the double sequence $(x^{r,s})_{r,s\in\mathbb{N}}$ as follows. Let

$$E = \left\{(x_{m,n})_{m,n\in\mathbb{N}} \in \left(\textstyle\sum_{m,n} \oplus l_\infty^2\right)_\infty : x_{m,n} \in c_0 \text{ except for finitely many } m,n\right\}.$$

Then E is ideal in $\left(\sum_{m,n} \oplus l_\infty^2\right)_\infty$. Define $x^{(r,s)} = \left(x_{m,n}^{(r,s)}\right)_{m,n\in\mathbb{N}} \in E$ for $r, s = 1,2, \dots$ by

$$x_{m,n}^{(r,s)}(K) = \tfrac{1}{rs}\,(j,k = 1,2,..), \text{ for } m,n \le r,s,$$
$$x_{m,n}^{(r,s)} = 0, \text{ for } m,n > r,s.$$

Then $\|x^{(r,s)}\| = \frac{1}{rs} \to 0\ (r,s \to \infty)$, but no subsequence of $\left(x^{(r,s)}\right)_{r,s\in\mathbb{N}}$ is relatively uniformly convergent.

Then, as in Example 3.3, we set $K = \{j^2, k^2 : j, k \in \mathbb{N}\}$ and let

$$y^{(r,s)} = \begin{cases} 0, & r,s \in K \\ x^{(r,s)}, & r,s \notin K. \end{cases}$$

Clearly, $\left(y^{(r,s)}\right)_{r,s\in\mathbb{N}}$ is norm statistically convergent to $0.$ An argument similar to Example 3.3 shows that $\left(y^{(r,s)}\right)_{r,s\in\mathbb{N}}$ is not relatively uniformly convergent.

The following example illustrates that even a combination of statistical order convergence and norm statistical convergence of double sequence does not imply the statistically relatively uniform convergence.

Example 3.5: Pick $g \in L_1[0,1]$ such that g is continuous on $(0,1], g(t) > 0$ for all $t \in [0,1]$, and $\lim_{t\to 0^+} g(t) = \infty$. Let E be the Riesz subspace of $L_1[0,1]$ consisting of all f such that, for some $\delta_2(f) > 0$ and $y(f) \in \mathbb{R}$, we have $f(t) = y(f)g(t)$for all $0 \le t < \delta_2(f)$ Define a Riesz space $\|\cdot\|$ 0n E by setting

$$\|f\| = \|f\|_{L_1} + |y(f)|.$$

We now show that a double sequence$\left(f_{m,n}\right)_{m,n\in\mathbb{N}}$ in E such that $f_{m,n} \downarrow 0$ and $\lim_{m,n\to\infty}\left\|f_{m,n}\right\| = 0$,but $f_{m,n} \nrightarrow 0(\mathrm{ru}_2)$, as follows.

Let $f_{0,0} = g.$ Define $f_{m,n}$ by induction so that

(a) $f_{m,n}$is contineous on (0,1]

(b) $0 \le f_{m,n} \le f_{m-1,n-1}$,

(c) $f_{m,n}(1/jk) = (1/jk)f_0(1/jk), j,k = 1,2,\dots,m,n,$

(d) $f_{m,n}(t) = \left(1/(m+1,n+1)\right)g(t),\ t \in (0,1/(m+1,n+1)],$

(e) $\left\|f_{m,n}\right\|_{L_1} \le \left(1/(m+1,n+1)\right)\|g\|_{L_1}$

This implies that no subsequence of $\left(f_{m,n}\right)_{m,n\in\mathbb{N}}$ converges relatively uniformly to 0.

Similarly, we set $K = \{j^2, k^2 : j,k \in \mathbb{N}\}$ and let

$$g_{m,n} = \begin{cases} 0, & m,n \in K \\ f_{m,n}, & m,n \notin K, \end{cases}$$

Clearly, $\left(g_{m,n}\right)_{m,n\in\mathbb{N}}$ is norm statistically convergent to 0and $g_{m,n} \xrightarrow{\mathrm{st}_2-\mathrm{ord}} 0$. But Theorem 2.2 yields that $\left(g_{m,n}\right)_{m,n\in\mathbb{N}}$ is not statistically relatively uniformly convergent to 0. This is because no subsequence of $\left(f_{m,n}\right)_{m,n\in\mathbb{N}}$ converges relatively uniformly to 0.

Theorem 3.5: Let $\left(x_{m,n}\right)_{m,n\in\mathbb{N}}$ be a monotone double sequence in Banach lattice $E.$ Then

(1) $\left(x_{m,n}\right)_{m,n\in\mathbb{N}}$is norm statistically convergent if and only if it is norm convergent.;

(2) $\left(x_{m,n}\right)_{m,n\in\mathbb{N}}$ is weakly statistically convergent if and only if it is weakly convergent;

(3) $\left(x_{m,n}\right)_{m,n\in\mathbb{N}}$is norm statistically convergent if and only if it is weakly statistically convergent.

In order to prove Theorem 3.5, we need two simple Lemmas.

Lemma 3.1: If a double sequence $\left(x_{m,n}\right)_{m,n\in\mathbb{N}}$in a Banach lattice E is increasing and converges norm statistically to $x \in E$,then $x = \sup_{m,n\in\mathbb{N}} x_{m,n}$

Actually, Lemma 3.1 also holds for weakly statistical convergence of double sequence as shown in the following.

Lemma 3.2: If a double sequence $\left(x_{m,n}\right)_{m,n\in\mathbb{N}}$ in a Banach lattice E is increasing and converges weakly statistically to $x \in E$,then $x = \sup_{m,n\in\mathbb{N}} x_{m,n}$.

Proof: Let $x^* \in (E^*)_+$. Then there exists a strictly increasing double sequence $\left(j_p, k_q\right)_{p,q\in\mathbb{N}}$ of natural numbers so that $\lim_{p,q\to\infty}\langle x^*, x_{j_p,k_q}\rangle = \langle x^*, x\rangle$. For each $m,n \in \mathbb{N}$, we choose j_{p_0}, k_{q_0} with$j_{p_0}k_{q_0} > m,n$. Then,we have

$$\langle x^*, x_{m,n}\rangle \le \langle x^*, x_{j_{p_0}k_{q_0}}\rangle, \quad \forall p,q > p_0, q_0$$

This means that

$$\langle x^*, x_{m,n}\rangle \le \langle x^*, x\rangle,$$

Hence, x is an upper bound of $\left(x_{m,n}\right)_{m,n\in\mathbb{N}}$.

Assume that u is an arbitrary upper bound of $\left(x_{m,n}\right)_{m,n\in\mathbb{N}}$. Then, for every $x^* \in (E^*)_+$, we have

$$\langle x^*, x_{m,n}\rangle \le \langle x^*, u\rangle, \quad \forall m,n \in \mathbb{N}.$$

Since the double sequence $\left(\langle x^*, x_{m,n}\rangle\right)_{m,n\in\mathbb{N}}$ converges statistically to $(\langle x^*, x\rangle)_{m,n\in\mathbb{N}}$ we have

$$\langle x^*, x\rangle \le \langle x^*, u\rangle.$$

This implies that $x \le u$ and we are done.

Proof of Theorem 3.4: (1) suppose that $\left(x_{m,n}\right)_{m,n\in\mathbb{N}}$. Is increasing and norm statistically convergent to $x \in E$. By Lemma 3.2, we get $x = \sup_{m,n\in\mathbb{N}} x_{m,n}$. Let $\left(k_{p,q}\right)_{p,q\in\mathbb{N}}$ be a strictly increasing double sequence of natural numbers so that $\lim_{p,q\to\infty} x_{j_p,k_q} = x$ in norm. Let $\epsilon > 0$. Choose a natural number p_0, q_0 such that $\left\| x_{j_{p_0}k_{q_0}} - x_{\|} < \epsilon$. Then, for any $m, n > j_{p_{0,}} k_{q_0}$, we have

$$0 \leq x - x_{m,n} \leq x - x_{j_{p_0}k_{q_0}}.$$

This yields

$$\|x - x_{m,n}\| \leq \left\| x - x_{j_{p_0}k_{q_0}} \right\| < \epsilon.$$

Hence, $\lim_{m,n\to\infty} x_{m,n} = x$ in norm.

(2). Let $(x_{m,n})_{m,n\in\mathbb{N}}$ be increasing and weakly statistically convergent to $x \in E$. By Lemma 3.1, we get $x = \sup_{m,n\in\mathbb{N}} x_{m,n}$. Let $x^* \in (E^*)_+$. since $(x_{m,n})_{m,n\in\mathbb{N}}$ converges weakly statistically to x, $(\langle x^*, x_{m,n}\rangle)_{m,n\in\mathbb{N}}$ converges statistically to $\langle x^*, x\rangle$. There exists a subset $K \subseteq \mathbb{N}$ with $\delta_2(K) = 1$ such that $\lim_{m,n\in K}\langle x^*, x_{m,n}\rangle = x*,x$. Let $\epsilon>0$. Choose $M,N\in K$ such that $x*,xMN-x*,x<\epsilon$. Then, for all $m,n\geq M,N$, we have

$$0 \leq x - x_{m,n} \leq x - x_{M,N}.$$

This implies that

$$0 \leq \langle x^*, x - x_{m,n}\rangle \leq \langle x^*, x - x_{M,N}\rangle < \epsilon.$$

Hence, $(x_{m,n})_{m,n\in\mathbb{N}}$ converges weakly to x.

(3). Follows from (1) and (2).

References

[1]. Connor, J. Fridy, J., & Kline, J. (1994). Statistically pre-Cauchy sequences, *Analysis International Mathematical Journal of Analysis and its Applications*,*14*(4), 311-317.

[2]. Connor, J. Ganichev, M. &Kadets, V. (2000). A characterization of Banach spaces with separable duals via weak statistical convergence, *Journal of Mathematical Analysis and Applications*, *244*(1), 251-261.

[3]. Ercan, Z. (2009). A characterization of u-uniformly completeness of Riesz spaces in terms of statistical u-uniformly pre-completeness, *Demonstratio Mathematica*, *42*(2), 381-385.

[4]. Fridy. J. A., (1985). On Statistical Convergence, *International Journal of Analysis and Applications*, *5*(4), 301-313.

[5]. Fast, H. (1951). Sur la convergence statistique,*Colloquium Mathematicum*, *2*, 241-244

[6]. Luxemburg, W. A. J., &Zaanen, A. C. (1971). *Riesz Spaces I*, North- Holland, Amsterdam-London.

[7]. Şençimen, C., &Pehlivan, S. (2012). Statistical order convergence in Riesz spaces.*MathematicaSlovaca*, *62*(2), 257-270.

[8]. Steinhaus, H. (1951). sur la convergence ordinaire et la convergecneasymptotique, colloquium Mathematicum.2, 73-74.

[9]. Schoenberg, I. J. (1959). The integrability of certain functions and related summability methods, *The American Mathematical Monthly*. *66*, 361-375.

[10]. Schwarz, H. –U. (1984), Banach lattices and operators, Teubner-TextezurMathematik.

[11]. Xue, X., & Tao, J. (2018). Statistical Order Convergence and Statistically Relatively Uniform Convergence in RieszSpaces, *Journal of Function Spaces*.

[12]. Zaanen, A. C. (1971). *Riesz Spaces II*, North- Holland, Amsterdam-London, 30.

[13]. Zygmund, A. (1952). Trigonometrical series, Chelsea publishing Co.,NY, USA, 2nd edition.

M. M. Plata. " Statistical Order Convergence and Statistically Relatively Uniform Convergence in Riesz Space of Double Sequence." IOSR Journal of Mathematics (IOSR-JM) 15.4 (2019): 25-34.