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Modeling and Analysis of Traffic Flows using Four Dimensional Non-linear Dynamical System of Ordinary Differential Equations

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Abstract: *Earlier many researchers devoted to describe the nature of traffic flow using PDE and ODE mostly using the concept of conservation law in fluid mechanics. Here we have* described *the nature of vehicles flow using nonlinear dynamical system. We have formulated a new model that show us the causes of delay time as a consequence of many vehicles involved in the way. That is, the qualitative behavior of the flow of vehicles with inflow, outflow, and blocking effects was presented. We observed freely flow vehicles move with the allowable speed whereas blocked vehicles move with the restricted speed to reduce congestions on the road. The rate of flow is high for free vehicles and less for blocked vehicles. This model shows the problem of congestion and delay on the road can be solved by considering all possible options to use the available route ahead of vehicles motions. The maximum possible usage of time is to get free vehicles and the maximum target moving with vehicles under influence of blocking is to save life with reductions of congestions. We have checked the posedness of the model. The local and Global stability analysis of equilibrium point carried out using retardation number. Further, we have also presented the sensitivity analysisof parameters. The simulation of the model was carried out with the help of MATLAB. Some impartant results and observations have been drawn and are presented lucidly in the text of this paper.*

Key terms: *Nonlinear dynamical systems, Equilibrium point, Traffic flow, Stability analysis, Retardation number, Well-Posedness,Sensitivity analysis.*

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I. Introduction

Traffic flow of vehicles on the roads is one of the sensitive issues both for developing and developed countries. The great problem in traffic flows is the occurrence of jams as a result of busy or crowded roads. Some of the reasons foroccurrenceoftraffic congestions may be put as: car blockings, not having traffic lights, indisciplineofdrivers, over capacity of roads, occurrences of illegal parkings, and improper application of sideways. Though the traffic control organizations have been usingvarious controlling mechanisms the traffic congestions could not have been achied even to a satisfactory level.The intricate traffic congestion problems are solved to some extent by using intelligent transportation system or ITS [1].

An intelligent transportation system consists of many other subsystems and those are arranged in all busy traffic roads. The sensors of each subsystem gather traffic data such as average speed and number of vehicles that pass through a certain point per unit time and send that data to ITS centers. These collected data will undergo a simulation processintherserver. The simulation results obtained will helpto adjust the cycle time of traffic lights in every road intersection, suggest alternative paths to avoid congestions, and predict when and where congestions would occur. This process of collecting data and simmulation remains as a continuous effort to build an integrated intelligent transportation system.

Since the beginning of the twentieth century, the researchers have begun to develop mathematical models that could describe the behavior of traffics in urban areas.As a result, there have been two classes of models developed to describe traffic flows viz., microscopic and macroscopic models.

In microscopic model, the traffic flow is described by the behavior of each single vehicle on the road. The movement of each vehicle is represented by ordinary differential equation where the dynamical equation depends on velocity and position of the next vehicle. However, the microscopic model results in complicated when road has a large number of vehicles[2, 3, 4].

In macroscopic model, the traffic flow is described by mimic the movement of vehicles with the motion of fluid [5, 6]. The macroscopic model was first introduced by Lighthill and Whitham [7] in 1955 and independently by Richards in 1956. The mathematical model introduced by Lighthill-Whitham-Richards is formulated as a nonlinear partial differential equation derived by using the conservation of vehicles in a single road and called as LWR model.

Even though different models have been used the researchers have not so far used the concept of four dimensional nonlinear dynamical systems to illustrate the flow of vehicles and the blocking effects.Blocking is one of the serious problems on a crowded road. Hence, it is important to identify the locations on a road where this blocking occurs and increases.

The vehicles running on a road have been classified into four categories: Free, Slow, Blocked, and Discharged vehicles.

Motivation to carryout the present research work

Comparion of "SEIR models of epidemiology" with "FSBD models of Traffic flow"

Generally, the four dimensional nonlinear dynamical system of ODE is applied for modeling the dynamics of human population in epidemiology. In most of the epidemiological models the total human population will be classified in to four compartments i.e., Susceptible – Exposed – Infected – Recovered (SEIR).

(i) The Susceptible compartment contains normal people. These people have yet not either exposed to or infected by the disease. However, these are potential people and are likely to be effected by the disease.

(ii) The Exposed compartment contains people who have already exposed to the disease. These people having been exposed to the disease exhibit the symptoms of the disease but they are not capable of propagating the disease to other humanbeings.

(iii) The Infected compartment contains people who have already been infected by the disease. While being suffered from the disease, these people are also capable of propagating the disease to other humanbeings. That is, the exposed humans show up only symptoms of the disease and cannot transfer the disease to others but, the infected humans show up the symptoms and can transfer the disease to others.

(iv) The Recovered compartment contains people who have already got effected by the disease and have got recovered from that by some means viz., natural cure, treatment etc.

In just similar to the epidemiological models where humans are classified into Susceptible – Exposed – Infected – Recovered (SEIR) compartments here in this study the vehicles on a road (or drivers driving vehicles) are classified into four compartments i.e., Free – Slow – Bolcked – Discharged (FSBD).

(i) The Free compartment contains normal vehicles. That is, these vehicles have yet not either slowed or blocked because of any other vehicle. However, these are potential vehicles and are likely to be either slowed or blocked by any other vehicle.

(ii) The Slow compartment contains vehicles who have already exposed to the blocking. That is, these vehicles having been exposed partially to the blocking exhibit slowness in the speed but they are not capable of blocking other vehicles.

(iii) The Blocked compartment contains vehicles who have already been blocked by other vehicles. While being blocked by other vehicles, these vehicles are also capable of propagating the blocking to other vehicles. That is, the slow vehicles are experiencing partially blockings but they can not block other vehicles. However, the blocked vehicles are experiencing complete blockings andthey can block other vehicles.

(iv) The Discharged compartment contains vehicles who have already got blocked by other vehicles and have got discharged from that by some means viz., the blocking vehicle moves and gives a way to the blocked vehicle.

Four Dimensional Nonlinear Dynamical System

***Free vehicles*:**These vehicles are free of all types of blockings.They run with a speed depending on driver's interest within the allowable speed range.

***Slow vehicles*:** These vehicles move with reduced speed because of blocking by other vehicles on the road under consideration.

Blocked vehicles:These vehicles are totally stopped or tend to stop because either these blocked by othervehiclesrunning on the road or the pathisclosed.

Discharged vehicles: These are vehicles that start moving from almost stopping place. These vehicles may be free when there is no blocking and slow when there is blocking.

Thus, considering the blocking effects in traffic flow is worthfulenough as it inputs some useful informationfor designing of transportation system. In the present study,the model of blocking effects in traffic flow is constructedusing nonlinear dynamical system of first order ODE.In this study, the blocking effectsareobserved during the peak hours when many vehicles join the road. The simulation study is performed considering various combinations of inflows, outflows and blocking effects of vehicles.

II. Model Formulations

The model formulated here classifies the total number of vehicles running on a road $N(t)$ into four compartments viz., Free vehicles $F(t)$, Slow vehicles $S(t)$, Blocked vehicles $B(t)$, and Discharged vehicles $R(t)$. So that

$$N(t) = F(t) + S(t) + P(t) + R(t)(1)$$

Here in (1), (i) $N(t)$ denotes total population size of vehicles under consideration, (ii) $F(t)$denotes the population size of Free Vehicles that are flowing freely without any influence of blocking effects but there is a possibility to face blocking in future, (iii) $S(t)$denotes the population size of slow vehicles which are partially blocked and are moving under the influence of blockings, (iv) $B\ (t)$denotes the population size of Blocked Vehicles which are totally blocked and are almost stopped and (v) $D\ (t)$ denotes the population size of Discharged vehicles which are just released from blockings and have a chance of experiencing blockings again.

The model equation for describing the rate of change of free vehicles on a road is constructed considering the following assumptions: (i) The rate of free vehicles increase by a constant rate τ as new vehicles are expected to enter on to the road with this rate. Thus, the parameter $\tau$is just similar to the constant birth rate in epidemiology, (ii) Vehicles are assumed to be blocked by blocked vehicles at a rate of α. This parameter α is just similar to the transfer rate of infection in epidemiology, (iii) Vehicles are assumed to be released from blockage at a rate of r_2. This parameter r_2 is just similar to the recovery rate from infection in epidemiology and (iv) Let μ_d be the rate of vehicles leaving from the present road tofollow another road. That is, with this rate the vehicles of each compartment decreases as the vehicles change the road. Thus, μ_d is just similar to natural death rate in epidemiology. Thus, the rate of change of free vehicles on a road is given by

$$dF/dt = \tau - \alpha FB + r_2 D - \mu_d F.$$

The rate of Slow vehicles is assumed to increase due to blocking of free vehicles at the rate ofα andblocking of discharged vehicles at the rateof δ.But, this rate is assumed to decrease by discharging ofslow vehicles at rate γ and blocking of slow vehicles at rateη. Thus, the rate of change of slow vehicles on a road is given by

$$dS/dt = \alpha FB + \ \delta D - \gamma S - \eta S - S\mu_d.$$

The rate of blocked vehicles is assumed to increase at a rate of η due to slow vehicles but decreases at the rate ofr_1 due to discharging of blocked vehicles. Hence,the rate of change of blocked vehicles on a road is given by

$$dB/dt = \eta S - r_1 B - B\mu_d.$$

The rate of change of discharged vehicles is assumed to increase by discharging ofblockedvehicles at the rate $r_1$and discharging of slow vehicles at the rate γ. However, this rate is assumed to decrease at the rate ofδdue toblocking of discharged vehicles and also at the rate r_2 due to releasing of vehicles from blockage . Thus, the rate of change of discharged vehicles is given by

$$dD/dt = Br_1 + \gamma S - r_2 D - \delta D - D\mu_d\ .$$

Table 1: Description of ModelVariables

Variable	Description pertaining to traffic flow	Description pertaining to Epidemiology
$F(t)$	Population size of FREE vehicles	size of Suceptible population
$S(t)$	Population size of SLOW vehicles	size of Exposed population
$B(t)$	Population size of BLOCKED vehicles	size of Infected population
$D(t)$	Population size of DISCHARGED vehicles	size of Recovered population

Table 2: Description of Model Parameters

Parameter	Description pertaining to traffic flow (FSBD)	Description pertaining to Epidemiology (SEIR)
τ	Rate of new vehicles joining the road (OR) growth rate of free vehicles	Constant birth rate of susceptiable population
α	Rate of free vehicles becoming slow vehicles. That is, with this rate the free vehicles are experiencing partialblockings and are running with slower speeds.	Transfer rate of infection to Susceptiable population
η	Rate of slow vehicles becoming blocked vehicles. That is, with this rate the slow vehicles who have been experiencing partial blockings are now experiencecomplet blockings and are now stop running.	Transfer rate from Exposed to Infected
r_1	Rate of blocked vehicles becoming discharged vehicles.That is, with this rate the blocked vehicles who have been experiencing complet blockings are now released from all types of blockings and are running freely.	Transfer rate from Infected to Recovered
γ	Rate of slow vehicles becoming discharged vehicles.That is, with this rate the slow vehicles who have been experiencing partial blockings are now released from blockings and are running freely.	Transfer rate from Exposed to Recovered
δ	Rate of discharged vehicles becoming slow vehicles. That is, with this rate the discharged vehicles who have been released from all	Transfer rate from Recovered to Exposed

	blockings are again experience partial blockings and as a result are running with slower speeds.	
r_2	Rate of discharged vehicles becoming free vehicles. That is, with this rate the discharged vehicles who have been released from all blockings do not experience any blockings and as a result are running freely with speeds as the driver likes.	Recovery rate from infection
μ_d	Rate of vehicles following another routes. That is, with this rate vehicles go off the road under study for some reason viz., destination is reached, route is shorter, to escape from the blockings etc.	Natural death rate of all categories of populations

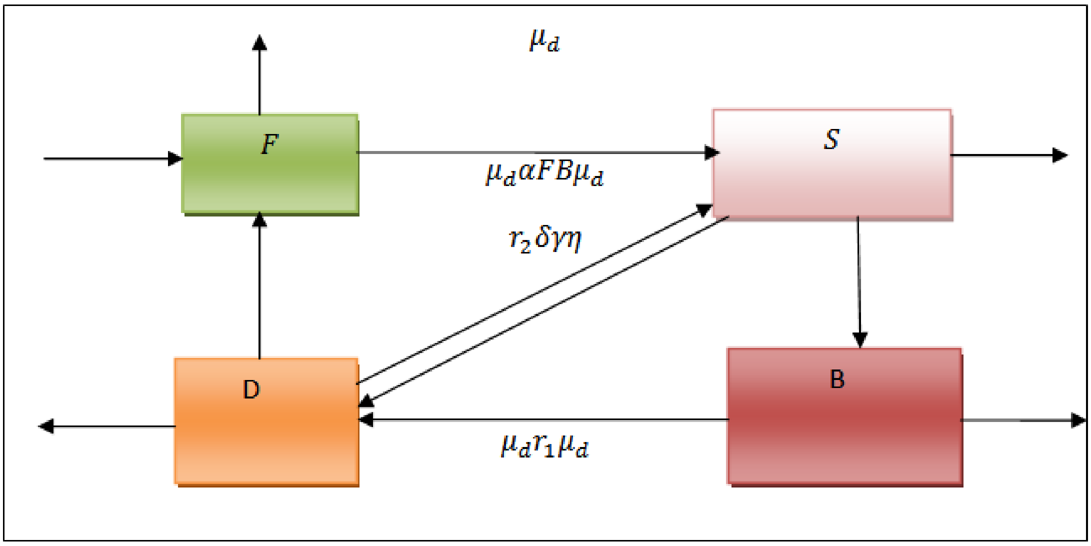

Figure 1: Progression of vehicles among Free (F), Slowed (S), Blocked (B), and Discharged(D) compartments

Based on the assumptions listed above and the diagram given in Figure 1, the mathematical model describing the dynamics of population sizes of various vehicles pertaining to a traffic flow on a road can be expressed as a system of nonlinear differential equations as

$$dF/dt = \lambda - \alpha FB + r_2 D - \mu_d F \quad (2)$$
$$dS/dt = \alpha FB + \delta D - \gamma S - \eta S - \mu_d S \quad (3)$$
$$dB/dt = \eta S - r_1 B - \mu_d B \quad (4)$$
$$dD/dt = Br_1 + \gamma S - r_2 D - \delta D - \mu_d D \quad (5)$$

With initial conditions, $F(0), S(0), B(0)$, and $D(0)$.

III. Analysis of the Model

In this section mathematical analysis of the model (2) – (5) is carried out. The analysis comprises of the following features: (i) Existence, positivity and boundedness of solutions (ii) Equilibrium points (iii) Blocking Free equilibrium points (iv)Endemic equilibrium points (v) Basic retardation number(vi)Stability analysis of the blocking free equilibrium points(vii)Local stability of blocking free equilibrium point (viii) Global stability of blocking free equilibrium point. These mathematical aspects are presented and explained in the following sub-sections respectively.

3.1 Existence, Positivity and Boundedness of solution

In order that the model equations (2) – (5)haveaphysicallyvalidmeaning for the modeled problem of traffic flow and well posed, thenecessarystepisto show that the state variables are non-negative. Theorem 1 followed by proof confirms the non-negativity of the state variables.

3.1.1 Positivity of the solutions

Theorem 1: If the initial conditions $F(0), S(0), B(0),$ and $D(0)$ are non-negative then the solution region$R = \{F(t), \quad S(t), \quad B(t), \quad D(t)\}$ of the system of equations (2) – (5) is non-negative.

Proof: To show that the solution of (6) is non-negative here each model equation of the dynamical systemisconsidered separately and shown thatit has a non-negative solution as follows:

Positivity of $F(t)$*:* Consider the model equation (2) given by$dF/dt = \lambda - \alpha FB + r_2 D - \mu_d F$ which without loss of generality, after discarding the positive terms $(\lambda + r_2 D)$, can be expressed as an inequality as $dF/dt \geq -(\alpha B + \mu_d)F$. This differential inequality, being first order and linear,can be solved easily to find its solution as $F(t) \geq F(0)\ Exp\left\{-\int_0^t[\alpha B(t)+\mu_d]dt\right\}$. Here the integral constant $F(0)$ represents initial population size and by definition it is a positive quantity. Also, as per the definition of exponential functions, the exponential factor $Exp\left\{-\int_0^t[\alpha B(t)+\mu_d]dt\right\}$ is always non-negative for any value of the exponent. Hence, it can be concluded that $F(t)$is a non-negative quantiy i.e., $F(t) \geq 0$.

Positivity of $S(t)$*:* Consider the model equation (3) given by $dS/dt = \alpha FB + \delta D - \gamma S - \eta S - \mu_d S$ which without loss of generality, after discarding the positive terms $(\alpha FB + \delta D)$, can be expressed as an inequality as $dS/dt \geq -(\gamma+\eta+\mu_d)S$. This differential inequality, being first order and linear,can be solved easily to find its solution as $S(t) \geq S(0)\ Exp\{-(\gamma+\eta+\mu_d)t\}$. Here the integral constant $S(0)$ represents initial population size and by definition it is a positive quantity. Also, as per the definition of exponential functions, the exponential factor $Exp\{-(\gamma+\eta+\mu_d)t\}$ is always non-negative for any value of the exponent. Hence, it can be concluded that $S(t)$is a non-negativequantiy i.e., $S(t) \geq 0$ for all $t \in [0,\infty)$.

Positivity of $B(t)$*:* Consider the model equation (3) given by $dB/dt = \eta S - r_1 B - \mu_d B$ which without loss of generality, after discarding the positive terms (ηS), can be expressed as an inequality as $dB/dt \geq -(r_1+\mu_d)B$. This differential inequality, being first order and linear,can be solved easily to find its solution as $B(t) \geq B(0)\ Exp\{-(r_1+u_d)t\}$. Here the integral constant $B(0)$ represents initial population size and by definition it is a positive quantity. Also, as per the definition of exponential functions, the exponential factor $Exp\{-(r_1+u_d)t\}$ is always non-negative for any value of the exponent. Hence, it can be concluded that $B(t)$is a non-negative quantiy i.e., $B(t) \geq 0$ for all $t \in [0,\infty)$.

Positivity of $D(t)$*:* Consider the model equation (3) given by $dD/dt = Br_1 + \gamma S - r_2 D - \delta D - \mu_d D$ which without loss of generality, after discarding the positive terms $(Br_1 + \gamma S)$, can be expressed as an inequality as $dD/dt \geq -(r_2+\delta+\mu_d)D$. This differential inequality, being first order and linear,can be solved easily to find its solution as $D(t) \geq D(0)\ Exp\{-(r_2+\delta+\mu_d)t\}$. Here the integral constant $D(0)$ represents initial population size and by definition it is a positive quantity. Also, as per the definition of exponential functions, the exponential factor $Exp\{-(r_2+\delta+\mu_d)t\}$ is always non-negative for any value of the exponent. Hence, it can be concluded that $D(t)$is a non-negative quantiy i.e., $D(t) \geq 0$ for all $t \in [0,\infty)$.

Since, obviously exponential expressions are positive and initial conditions are non-negative it can be concluded that the solutions region is a set containingnon-negativequantities. Thus,$R = \{[F(t),\ \ S(t),\ \ B(t),\ \ D(t)]\ \exists\ F(t) \geq 0,\ S(t) \geq 0,\ B(t) \geq 0,\ D(t) \geq 0,\ \forall t \in [0,\infty)\}$

3.1.2.Boundedness of the solutions region

In order to make the formulated model is valid and well posed it is also necessary to show that the solutions region is bounded. This fact has been stated as Theorem 2 and verified in its proof following [8, 9, 10, 11].

Theorem 2 The non-negative solutions region$R = \{F(t),\ \ S(t),\ \ B(t),\ \ D(t)\}$ of the system of equations (3) – (5) is boundedi.e., $N(t) \leq (\tau/\mu_d)$.

Proof: To show that the solutions region of the system of equations (2) – (5) is bounded, it is sufficient if it is shown that the total population is bounded. Now adding all terms on the right and left sides of equations in dynamical system (2) – (5)gives the resultant equation as

$$(dF/dt) + (dS/dt) + (dB/dt) + (dD/dt) = \tau - \alpha FB + r_2 D - \mu_d F + \alpha FB + \delta D - \gamma S - \eta S - \mu_d S + \eta S - r_1 B - \mu_d \mathrm{B} + Br_1 + \gamma S - r_2 D - \delta \mathrm{D} - \mu_d \mathrm{D}$$

Now, on the left hand side of the above equation, some pairs of terms with same expressions but with opposite signs can be found. Hence, the equation can be expressed in terms of such pairs as

$$(dF/dt) + (dS/dt) + (dB/dt) + (dD/dt) = \tau + (r_2 D - r_2 D) - \mu_d F + (\alpha FB - \alpha FB) + (\delta D - \delta \mathrm{D}) - \mu_d S + (\eta S - \eta S) - \mu_d \mathrm{B} + (r_1 B - r_1 B) + (\gamma S - \gamma S) - \mu_d \mathrm{D}$$

After discarding the zero-valued-pairs enclosed in the braces and using the fact $N(t) = F(t) + S(t) + P(t) + R(t)$ as given in (1), the foregoing equation reduces to a simplified form as follows:

$$\begin{aligned} dN/dt \\ &= \tau - \mu_d F - \mu_d S - \mu_d \mathrm{B} - \mu_d \mathrm{D} \\ &= \tau - \mu_d(F+S+B+D) \\ &= \tau - \mu_d N \end{aligned}$$

Now, for the first order linear equation with constant coefficients $dN/dt = \tau - \mu_d N$ it is straight forward to find the complete solution as $N(t) = (\tau/\mu_d) - [(\tau/\mu_d) - N(0)]e^{-\mu_d t}$. Here $N(0)$is the initial size of all categories of vehicles on the road.

It follows that $N(t)$ is bounded as $t \to \infty$ i.e., $N(t) \leq (\tau/\mu_d)$ provided that the condition $N(0) \leq (\tau/\mu_d)$is satisfied.Thus, it can be concluded that the solutions region$R = \{F(t),\ \ S(t),\ \ B(t),\ \ D(t)\}$is boundedi.e., $N(t) \leq (\tau/\mu_d)$.

3.1.3 Existence and Uniqueness of the solutions

Here it is to show that asolution for the system (2) – (5)exists and is uniquefollowing the procedure given in Derric and Grossman 1976 the solution exists and is unique. The existence and uniqueness of the solution can bestated as shown in Theorem 3.

Theorem 3: Consider a system of n first order differential equationsof the type $x_i' = f_i(x_1, x_2, x_3 \dots, x_n, t)$ together with the initial conditions $x_i(t_0) = x_{i0}$ where $i = 1, \dots, n$. Let R denotesa region in $(n+1)$-dimensional space among which one dimension is for t and n dimensions are for the vector x. If all the partial derivatives $\partial f_i / \partial x_j$ for all $i, j = 1,2, \dots n$ are continuous in $R = \{(x,t), |t - t_0| \le a, |x - x_0| \le b\}$ then there is exists a constant $\delta > 0$such that there an unique continuous vector solution$x^* = [x_1(t), x_2(t), \dots, x_n(t)]$in the interval $|t - t_0| \le \delta$for the system of n equations.

Now accordingly let state the theorem and prove

Theorem 4: There exist an unique solution to thesystem of equations (2) – (5).

Proof: The statement here is proved following the procedure given in Theorem 3. Now, thesystem of equations (3) – (5) together with the initial conditions can be expressed as

$$dF/dt = \tau - \alpha SB + r_2 D - \mu_d F \equiv f_1 \ , \qquad F(t_0) = F_0$$
$$dS/dt = \alpha SB + \delta D - \gamma S - \eta S - \mu_d S \equiv f_2 \ , \qquad S(t_0) = S_0$$
$$dB/dt = \eta S - r_1 B - \mu_d B \equiv f_3 \ , \qquad B(t_0) = B_0$$
$$dD/dt = B r_1 + \gamma S - r_2 D - \delta D - \mu_d D \equiv f_4 \ , \qquad D(t_0) = D_0$$

Let $R = \{(F, S, B, D, t): |F - F_0| \le a, |S - S_0| \le b, |B - B_0| \le c, |D - D_0| \le d, |t - t_0| \le e\}$

And, $|(\partial f_i / \partial j)|$, $i = 1, 2, 3, 4$ and $j = F, S, B, D$ are continuous and bounded then then equation (11) has unique solution.

$$\partial f_1/\partial F = -\mu_d; \quad \partial f_1/\partial S = -\alpha B; \quad \partial f_1/\partial B = -\alpha S; \quad \partial f_1/\partial D = r_2$$
$$\partial f_2/\partial F = 0; \quad \partial f_2/\partial S = \alpha B - \gamma - \eta - \mu_d; \quad \partial f_2/\partial B = \alpha S; \quad \partial f_2/\partial D = \delta$$
$$\partial f_3/\partial F = 0; \quad \partial f_3/\partial S = \eta; \quad \partial f_3/\partial B = -r_1 - \mu_d; \quad \partial f_3/\partial D = -r_2 - \delta - \mu_d$$
$$\partial f_4/\partial F = 0; \quad \partial f_4/\partial S = \gamma; \quad \partial f_4/\partial B = r_1; \quad \partial f_4/\partial D = -(r_2 + \mu_d + \delta)$$

In theorem 2 we have shown that the solutions are bounded. Thus, $|\partial f_i / \partial j|$ for $i = 1, 2, 3, 4$and $j = F, S, B, D$ are continuous and bounded. That is,

Table 3 Showing continuity of the partial derivatives

$\lvert\partial f_1/\partial F\rvert = \lvert-\mu_d\rvert = \mu_d < \infty$ $\lvert\partial f_1/\partial S\rvert = \lvert-\alpha B\rvert = \alpha\lvert B\rvert < \infty$ $\lvert\partial f_1/\partial B\rvert = \lvert-\alpha S\rvert = \alpha\lvert S\rvert < \infty$ $\lvert\partial f_1/\partial D\rvert = \lvert r_2\rvert < \infty$	$\lvert\partial f_2/\partial F\rvert = 0 < \infty$ $\lvert\partial f_2/\partial S\rvert = \lvert\alpha B - \gamma - \eta - \mu_d\rvert < \infty$ $\lvert\partial f_2/\partial B\rvert = \lvert\alpha S\rvert < \infty$ $\lvert\partial f_2/\partial D\rvert = \lvert\delta\rvert = \delta < \infty$
$\lvert\partial f_3/\partial F\rvert = 0 < \infty$ $\lvert\partial f_3/\partial S\rvert = \eta < \infty$ $\lvert\partial f_3/\partial B\rvert = r_1 + \mu_d < \infty$ $\lvert\partial f_3/\partial D\rvert = r_2 + \delta + \mu_d < \infty$	$\lvert\partial f_4/\partial F\rvert = 0 < \infty$ $\lvert\partial f_4/\partial S\rvert = \gamma < \infty$ $\lvert\partial f_4/\partial B\rvert = r_1 < \infty$ $\lvert\partial f_4/\partial D\rvert = r_2 + \delta + \mu_d < \infty$

Hence, by Derric and Grossman 1976 the solutionexistsandis unique.

3.2 Equilibrium points

In order to have a better understanding about the dynamics of a model, the equilibrium points of the solution regionare to be identified and their stability analysis is to be conducted. In this section such identification and analysis is conducted.

An equilibrium solution is a steady state solution of the model equations (2) – (5) in the sense that if the system begins at such a state, it will remain there for all times as long as any disturbance occurs. In other words, the population sizes remain unchanged and thus the rate of change for each population vanishes. Equilibrium points of the model are found, categorized, stability analysis is conducted and the results have been presented in the following:

3.2.1 Blocking free equilibrium BFE

At blocking free equilibrium vehicles flow freely without any interference of any kind of blockings. That is, at this equilibrium vehicles will run freely with speeds as per the wish of their drivers. Furthermore, at this equilibrium no vehicle is forced either to run with slower speeds or to stop completely. That is, $S = B = 0$. Thus, under this assumption thesystem of equations (2) – (5) reduced to the following form

$$dF/dt = \lambda + r_2 D - \mu_d F = 0$$
$$\delta D = 0$$
$$dD/dt = -r_2 D - \delta D - \mu_d D = 0$$

Further, the solution of these reduced form of equations can be obtained with simple algebraic operations as $D = 0$ and $F = \tau/\mu_d$. Here, $D = 0$ imples that no car is released from blockings. In fact there is no blocking here. Thus, blocking free equilibrium BFE of the model is obtained as

$$E_0 = (\tau/\mu_d, \quad 0, \quad 0, \quad 0)$$

3.2.2 The endemic equilibrium point

Let $X^* = (F^*, \quad S^*, \quad B^*, \quad D^*)$ be an endemic equilibrium point. In order to obtain endemic equilibrium E^* point of the model the left hand sidesof the equations (2) – (5) are set equal to zero.Thus, the model equations reduce to the form as

$$\begin{aligned} &\tau - \alpha F^* B^* + r_2 D^* - \mu_d F^* = 0 \\ &\alpha F^* B^* + \delta D^* - \gamma S^* - \eta S^* - \mu_d S^* = 0 \\ &\eta S^* - r_1 B^* - \mu_d B^* = 0 \\ &B^* r_1 + \gamma S^* - r_2 D^* - \delta D^* - \mu_d D^* = 0 \end{aligned} \quad (6)$$

The endemic equilibrium is the solution ofthe set of equations (6). On employing simple algebraic manipulations the solution can be obtained as

$$F^* = (\tau/\mu_d), \qquad B^* = 0, \qquad D^* = 0, \qquad S^* = 0$$

Hence, the endemic equilibrium point of the model is given by $X^* = (\lambda/\mu_d, \quad 0, \quad 0, \quad 0)$. This shows the only equilibrium point is the blocking free equilibrium point.

3.3 Derivation of basic Retardation number(R_0)

Basic retardation number is the average number of slow or blocked vehicles generated by each blocked vehicle.Calculating retardation number is important to analyze the local stability of nonlinear system of equations (2) – (5).

The retardation number is the largest eigenvalue of the matrix$K = \mathcal{F}V^{-1}$. Where,

$$\mathcal{F} = \left(\partial f/\partial x_j\right)|_{E_0}, \; V = \left(\partial v/\partial x_j\right)|_{E_0} \qquad (7)$$

Here, $f$is the newly blocking terms and v is non-singular matrix of the remaining transfer terms. Now, the basic retardation number R_0 of the model (2) – (5) is computed using the next generation matrix in similar procedure as reproduction number in epidemiological concept used to be computed. Thus, the next generation matrices$f, \; v, \; \mathcal{F}, \; V, \; V^{-1}, \; \mathcal{F}\, V^{-1}$are constructed respectively as follows:

$$f = \begin{bmatrix} \alpha FB \\ 0 \end{bmatrix}$$

$$v = \begin{bmatrix} -\delta D + (\mu_d + \gamma + \eta)S \\ -\eta S + Br_1 + \mu_d B \end{bmatrix}$$

$$\mathcal{F} = \begin{bmatrix} 0 & \alpha\tau/\mu_d \\ 0 & 0 \end{bmatrix} (8)$$

,

$$V = \begin{bmatrix} \gamma + \eta + \mu_d & 0 \\ -\eta & r_1 + u_d \end{bmatrix} \qquad (9)$$

$$V^{-1} = \begin{bmatrix} 1/(\gamma + \eta + \mu_d) & 0 \\ \eta/(r_1 + u_d)(\gamma + \eta + u_d) & 1/(r_1 + \mu_d) \end{bmatrix}$$

$$K = \mathcal{F}\, V^{-1} = \begin{bmatrix} \alpha\lambda\eta/[\mu_d(r_1 + \mu_d)(\gamma + \eta + \mu_d)] & \alpha\tau/(r_1 + \mu_d)\mu_d \\ 0 & 0 \end{bmatrix}$$

Clearly, eigenvalues of the next generation matrix$K$ are $\lambda_1 = 0$ and$\lambda_2 = \alpha\eta\tau/[\mu_d(r_1 + \mu_d)(\gamma + \eta + \mu_d)]$; among which $\lambda_2$is larger. Hence, the Retardation number of the model is given by

$$R_0 = \rho(\mathcal{F}V^{-1}) = \{\alpha\eta\tau/[\mu_d(r_1 + \mu_d)(\gamma + \eta + \mu_d)]\}$$

3.3 Stability analysis of the blocking free equilibrium

In the absence of blockings, the traffic flow model will have a unique blocking free steady stateE_0. To find the local stability ofE_0, the Jacobian matrix of the model equations valued at blocking free equilibrium point$E_0$is used. It is already shown that the BFE of model (2) – (5) is given by $E_0 = \{\tau/\mu, \quad 0, \quad 0, \quad 0\}$. Now, the stability analysis of BFE is conducted and the results are presented in the form of theorems and proofs in the following.

3.4.1 Local stability of blocking free equilibrium

Let J be the Jacobian matrix formed from system of equations (2) – (5). Thus, following the procedures given in the literature [9, 10, 11, 12] local stability of blocking free equilibrium point is found. Now, the Jacobian matrix that is constructed from the model equations (2) – (5) is

$$J(F, \quad S, \quad B, \quad D) = \begin{bmatrix} -\alpha B - \mu_d & 0 & -\alpha F & r_2 \\ \alpha B & -(\gamma + \eta + \mu_d) & \alpha F & \delta \\ 0 & \eta & -(r_1 + \mu_d) & 0 \\ 0 & \gamma & r_1 & -(r_2 + \delta + \mu_d) \end{bmatrix}$$

Similarly, the Jacobian matrix J reduces to the form, at the blocking free equilibrium point, as

$$J(\tau/\mu_d, \quad 0, \quad 0, \quad 0) = \begin{bmatrix} -\mu_d & 0 & -\alpha\tau/\mu_d & r_2 \\ 0 & -(\gamma + \eta + \mu_d) & \alpha\tau/\mu_d & \delta \\ 0 & \eta & -(r_1 + \mu_d) & 0 \\ 0 & \gamma & r_1 & -(r_2 + \delta + \mu_d) \end{bmatrix}$$

Now,

(i) Trace of $J(\tau/\mu_d, \quad 0, \quad 0, \quad 0) = -\mu_d - (\gamma + \eta + \mu_d) - (r_1 + \mu_d) - (r_2 + \delta + \mu_d) = -[4\mu_d + \gamma + r_1 +$ *η+r2+δ<0.* That is, Considering positive parametric values we can conclude that the trace of jacobian matrix at disease free equilibrium point is negative.

(ii) det$(J(\tau/\mu_d, \quad 0, \quad 0, \quad 0))$

$$= [\gamma\mu_d^3 + r_2\mu_d^3 + \delta\mu_d^3 + \eta\mu_d^3 + r_1\mu_d^3 + \mu_d^4 + \delta r_1\mu_d^2 + \eta r_1\mu_d^2 + \gamma r_2\mu_d^2 + r_2\eta\mu_d^2 + \gamma r_1\mu_d^2 + r_1 r_2\mu_d^2 + \delta\eta\mu_d^2 - \delta\eta\alpha\tau - \eta\alpha\tau\mu_d + \gamma r_2 r_1\mu_d - r_2\eta\alpha\tau + r_2\eta r_1\mu_d]$$

$$= [\mu_d^4 + (\gamma + r_2 + \delta + \eta + r_1)\mu_d^3 + (\delta r_1 + \eta r_1 + \gamma r_2 + r_2\eta + \gamma r_1 + r_1 r_2 + \delta\eta)\mu_d^2 + (\eta\alpha\tau + \gamma r_2 r_1 +$$ *r2ηr1μd−ηατδ+r2.*

Let $a_1 = [\mu_d^4 + (\gamma + r_2 + \delta + \eta + r_1)\mu_d^3 + (\delta r_1 + \eta r_1 + \gamma r_2 + r_2\eta + \gamma r_1 + r_1 r_2 + \delta\eta)\mu_d^2 + (\eta\alpha\tau + \gamma r_2 r_1 +$ *r2ηr1μd*and *a2=ηατδ+r2.*Now, we can observe that derminant of Jacobian matrix *Jτμd,0,0,0* at blocking free equilibrium point is positive provided that $a_1 > a_2$. Hence by Hourth Ruth theorem we can conclude that all eigenvalues of jacobian matrix are negative at blocking free equilibrium point. Further, using[13] it can be concluded that blocking free equilibrium point is locally asymptotically stable for $R_0 < 1$ and unstable for $R_0 > 1$.

3.4.2 Global stability of blocking free equilibrium

Let blocking expression be rewritten as the followings from which we state the next theorem in [17] for global stability.

$$x'^{(t)} = -(V - \mathcal{F})x(t) - \begin{pmatrix} \eta(F_0 - F)B \\ 0 \end{pmatrix}$$
$$dF/dt = \tau - \alpha FB + r_2 D - \mu_d F(22)$$
$$dD/dt = Br_1 + \gamma S - r_2 D - \delta \mathrm{D} - \mu_d \mathrm{D}$$

Theorem 3.4.2 If $(V - \mathcal{F})$ is non-singular M-matrix and $(F_0 - F)B \geq 0$, then the blocking free equilibrium is globally asymptotically stable for $\rho(\mathcal{F}V^{-1}) < 1$.

Proof: From (8) and (9) we can observe that $\mathcal{F}$is nonnegative and $V$is non-singular M-matrix. Thus using [13] we can conclude that $(V - \mathcal{F})$ is non-singular M-matrixfor $\rho(\mathcal{F}V^{-1}) < 1$. Also from theorem 2 we can observe that $B(t)$ is bounded and non-negative. Now, to show that the blocking-free equilibrium is globally asymptotically stable for $R_0 < 1$, it is sufficient to show that $F \leq F_0$. From the total population $N(t)$ we have, $N(t) = F(t) + S(t) + B(t) + D(t)$ which satisfies $N'(t) = \tau - \mu_d N(t)$, so that $N(t) = F_0 - (F_0 - N(0))e^{-ut}$, with $F_0 = \tau/\mu_d$ .If$N(0) \leq F_0$, then $F(t) \leq N(t) \leq F_0$ for all time. If, on the other hand, $N(0) > F_0$, then N(t) decays exponentially to F_0, and either $F(t) \to F_0$, or there is some time T after which $F(t) < F_0$. Thus, the blocking free equilibrium is globally asymptotically stable for $\rho(\mathcal{F}V^{-1}) < 1$.

IV. Numerical Simulations

The numerical simulation study of nonlinear dynamical system has been carried-out using MATLAB. The following numberical values are assigned to parameters and variables and are used to describe the blocking effects on flow of vehicles. This simulation study describes the blocking effect on the motions of vehicles of all four categoris on a road. The simulated vehicles flowshave been observed with different time intervals.

Table 4: List of values assigned to the parameters of the model

Parameter	Value	Source
τ	40	Assumed
α	0.01	Assumed
η	0.0001	Assumed
r_1	0.5	Assumed
γ	0.7	Assumed
δ	0.001	Assumed
r_2	0.5	Assumed
μ_d	0.1	Assumed

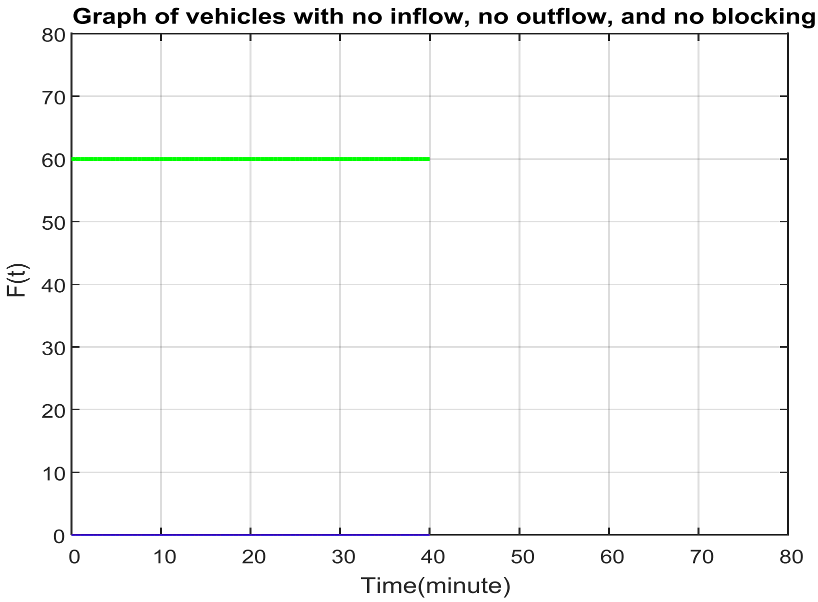

Figure 1 Population size of Free vehicles in absence of Inflow, Outflow and Blockings

The simulated graph of the model equations (2) – (5) shown in Figure 1 is obtained by using the parametric values given in Table 4 together with $F = 60,\ S = 0,\ B = 0,\ D = 0,\ \tau = 0,\ \ \mu_d = 0$. This figure shows that the population size of the Free vehicles remain constant at $F = 60$ in absence of Inflow, Outflow and Bolckings of vehicles. Further, it can also be observed that as there are no blockings on the road all vehicles can move freely with the speeds asdrivers desire. Here, 60 vehicles flow freely from the beginning to the end of the road.

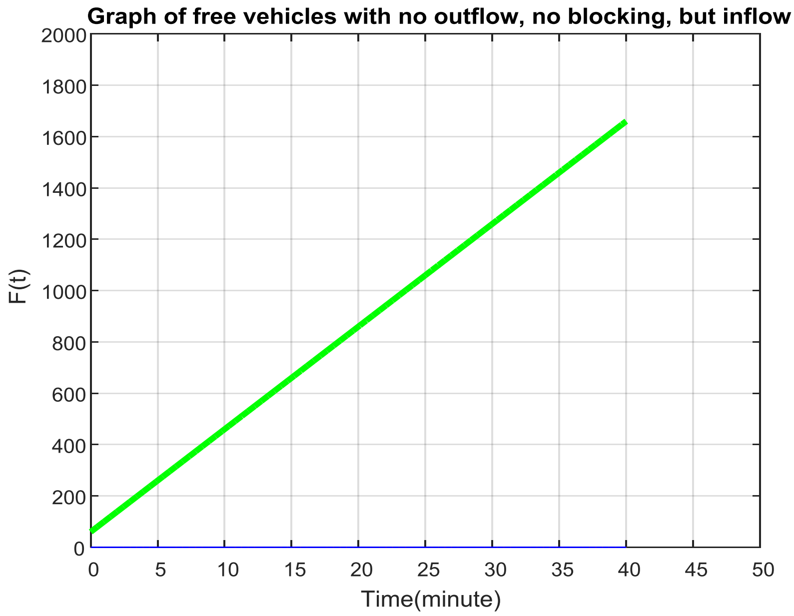

Figure 2 Population dynamics of Free vehicles in absence of Outflow and Blockings

The simulated graph of the model equations (2) – (5) shown in Figure 2 is obtained by using the parametric values given in Table 4 together with $F = 60,\ S = 0,\ B = 0,\ D = 0,\ \tau = 40,\ \mu_d = 0$. This figure shows that the population size of the Free vehicles grow linearly from the initial size$\mathrm{F} = 60$ in absence of Outflow and Bolckings of vehicles. However, here inflow of the vehicles is allowed.That is, as time increases the number of freely flowing vehicles increases linearly from the beginning to the end of the road because there are no blockings.

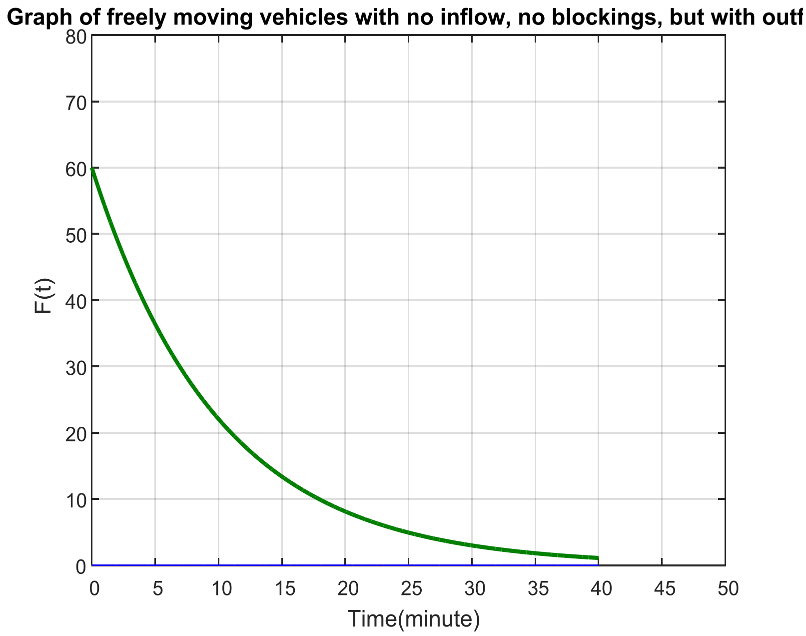

Figure 3 Dynamics of Free vehicles in absence of Inflow and Blockings but with Outflow

The simulated graph of the model equations (2) – (5) shown in Figure 3 is obtained by using the parametric values given in Table 4 together with $F = 60,\ S = 0,\ B = 0,\ D = 0, \tau = 0,\ \mu_d = 0.1$. This figure shows that the population size of the Free vehicles decreases from the initial size $\mathrm{F} = 60$ to $\mathrm{F} = 0$ in absence of Inflow and Bolckings of vehicles but with outflow. That is, in presence of outflow together with no inflow and no blockings, the number of freely moving vehicles decreases exponentially till the road becomes empty of vehicles.

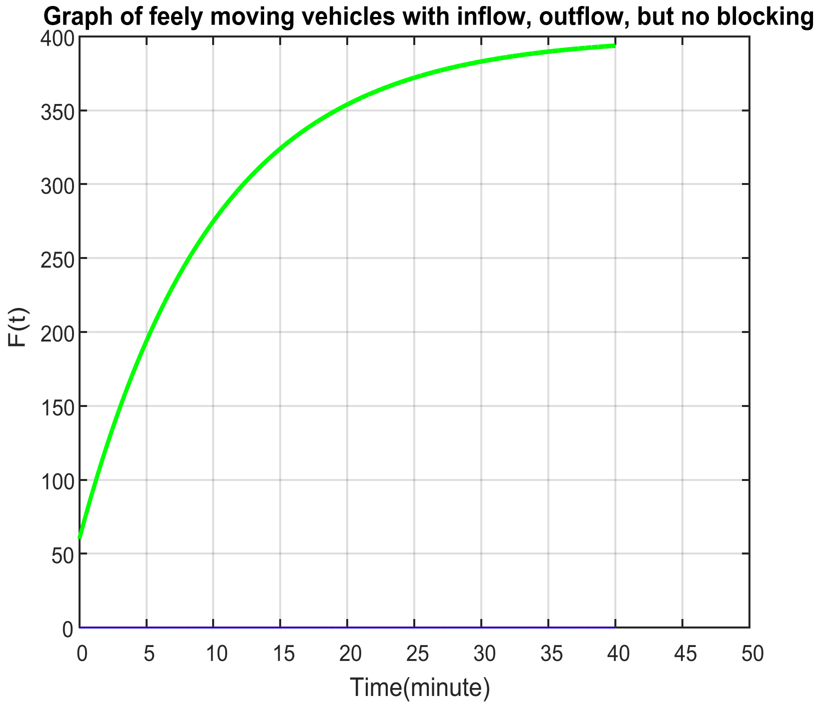

Figure 4 Dynamics of Free vehicles in absence Blockings but with Outflow and Inflow

The simulated graph of the model equations (2) – (5) shown in Figure 4 is obtained by using the parametric values given in Table 1 together with $F = 60,\ S = 0,\ B = 0,\ D = 0,\ \tau = 40,\ \mu_d = 0.1$. This figure shows that the population size of the Free vehicles increases from the initial size $\mathrm{F} = 60$ to the upper bound $\mathrm{F} = 400$ in absence of Bolckings of vehicles but with Inflow and outflow. That is, in presence of inflow and outflow together with no blockings, the number of freely moving vehicles increase exponentially till the number of vehicles on the road reaches its upper boundary value $\mathrm{F} = 400$.

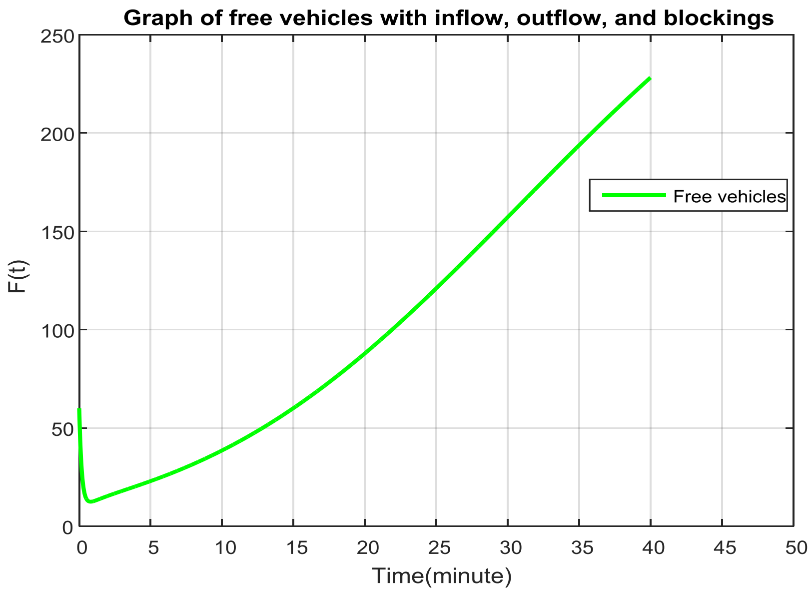

Figure 5 Dynamics of Free vehicles with Inflow, Outflow and Blockings

The simulated graph of the model equations (2) – (5) shown in Figure 5 is obtained by using the parametric values given in Table 4 together with $\mathrm{F} = 60, \mathrm{S} = 120,\ \mathrm{B} = 140,\ \mathrm{D} = 50,\ \tau = 40, \mu_d = 0.1$. This figure shows that in presence of Inflow, outflow and Bolckings of vehicles, the population size of Free vehicles initially decreases from the initial size $\mathrm{F} = 60$ to a minimum value and then from there it increases.

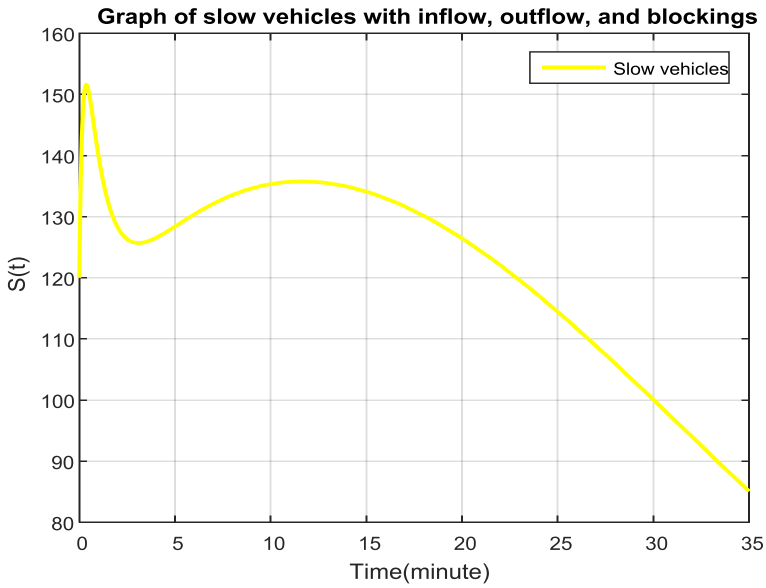

Figure 6 Dynamics of Slow vehicles with Inflow, Outflow and Blockings

The simulated graph of the model equations (2) – (5) shown in Figure 6 is obtained by using the parametric values given in Table 4 together with $F = 60,\ S = 120,\ B = 140,\ D = 50, \tau = 40,\ \mu_d = 0.1$. This figure shows that in presence of Inflow, outflow and Bolckings of vehicles, the population size of Slow vehicles initially increases from the initial size $S = 120$ to a maximum value and then from there it decreases. Furthermore, it can be observed that as there are inflows, outflows, and blockings with given initial conditions slow vehicles increases because of the blocking of restricting the speed of the vehicles.But the number of slow vehicles decreases as the road becomes free because of outflow.

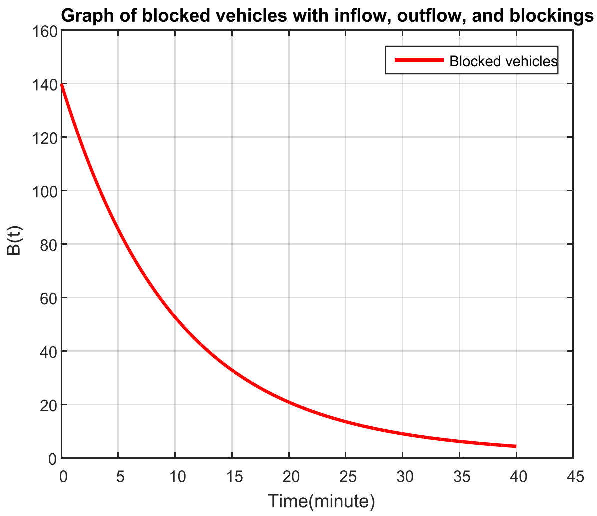

Figure 7 Dynamics of Blocked vehicles in presence of Inflow, Outflow and Blockings

The simulated graph of the model equations (2) – (5) shown in Figure 7 is obtained by using the parametric values given in Table 4 together with $F = 60,\ S = 120,\ B = 140,\ D = 50,\ \tau = 40,\ \mu_d = 0.1$. This figure shows that in presence of Inflow, outflow and Bolckings of vehicles, the population size of blocked vehicles decreases from the initial size $B = 140$ to the maximum value $B = 0$. Further, it can be interpreted that as there are inflows, outflows, and blockings with the given initial conditions the number of blocked vehicles gets decreased exponentially because as time increases most vehicles move freely.

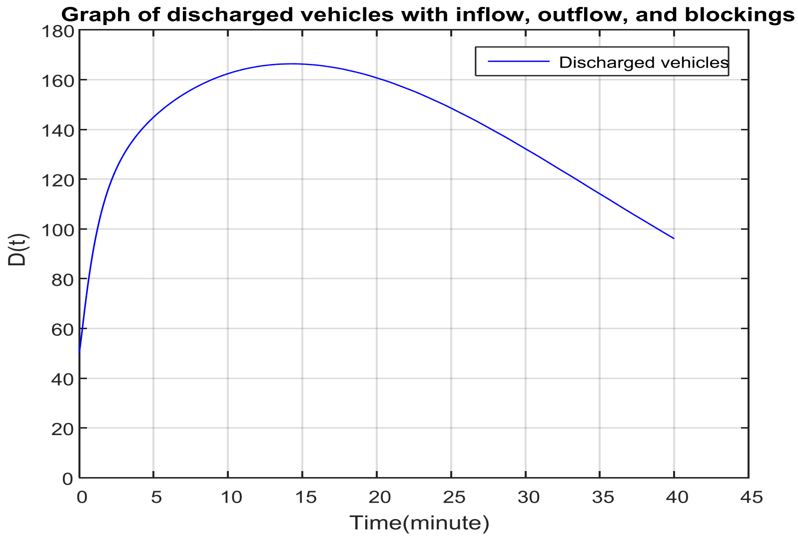

Figure 8 Dynamics of Discharged vehicles in presence of Inflow, Outflow and Blockings

The simulated graph of the model equations (2) – (5) shown in Figure 8 is obtained by using the parametric values given in Table 4 together with $F = 60$, $S = 120$, $B = 140$, $D = 50$, $\tau = 40$, $\mu_d = 0.1$. This figure shows that in presence of Inflow, outflow and Bolckings of vehicles, the population size of Discharged vehicles increases from the initial size $D = 50$ to a maximum value and from there monotonically decreasee.Further more, it can be interpreted that as there are inflows, outflows, and blockings with the given initial conditions discharged vehicles increases as vehicles are released from blockings but in presence of outflow and increment of freely moving vehicles the number of discharged vehicles decrease.

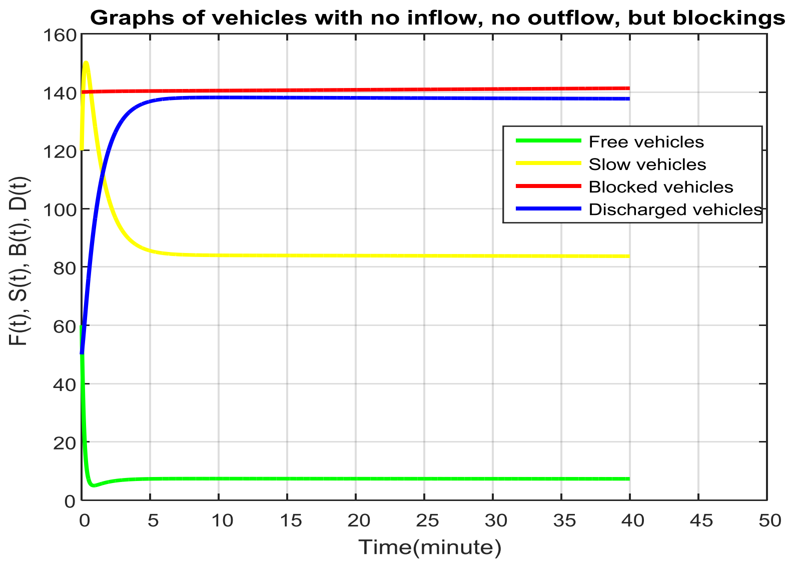

Figure 9 Dynamics of vehicles in presence of Blockings but in absense of Inflow and Outflow

The simulated graph of the model equations (2) – (5) shown in Figure 9 is obtained by using the parametric values given in Table 4 together with $F = 60$, $S = 120$, $B = 140$, $D = 50$, $\tau = 0$, $\mu_d = 0$. It can be interpreted that (i) freely moving vehicles decrease because of crowdedness of the road (ii) discharged vehicles increases to the number of blocked vehicles (iii) Slow vehicles initially increases because of blockings, then decreases and finally remain stable (iv) Blocked vehicles remain the same because the road is so crowded so almost the same of vehicles blocked as time increases.
Hence, this road is not advised for the passengers to travel.

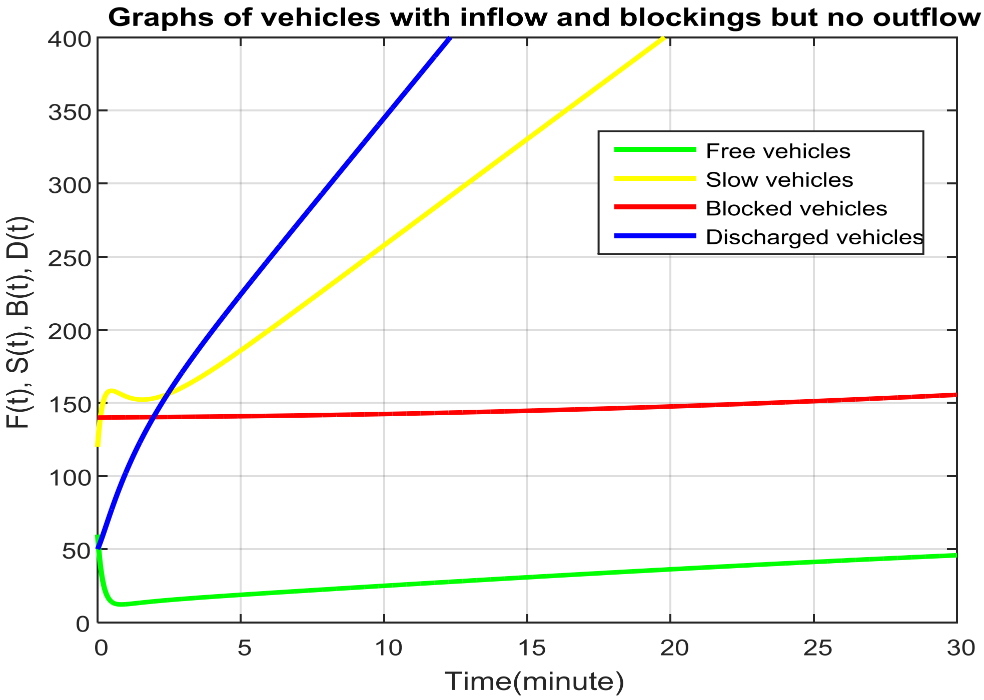

Figure 10 Dynamics of vehicles in the presence of inflow and Blockings but no outflow

The simulated graph of the model equations (2) – (5) shown in Figure 10 is obtained by using the parametric values given in Table 4 together with F = 60, S = 120, B = 140, D = 50, $\tau = 40$, $\mu_d = 0$. It can be interpreted that (i) Free vehicles decrease because of crowdedness of the road that slow free vehicles and increases because of inflow of free vehicles and discharged vehicles (ii) Discharged vehicles increases as the the result of more vehicles discharged from blockings (iii) Slow vehicles initially increases because of blockings, then decreases and finally increases as inflow Free vehicles and discharged vehicles Slowed (iv) Blocked vehicles increases gradually as more vehicles discharged and Blocked vehicles remain almost the same because the road is more under slow and discharging condition.

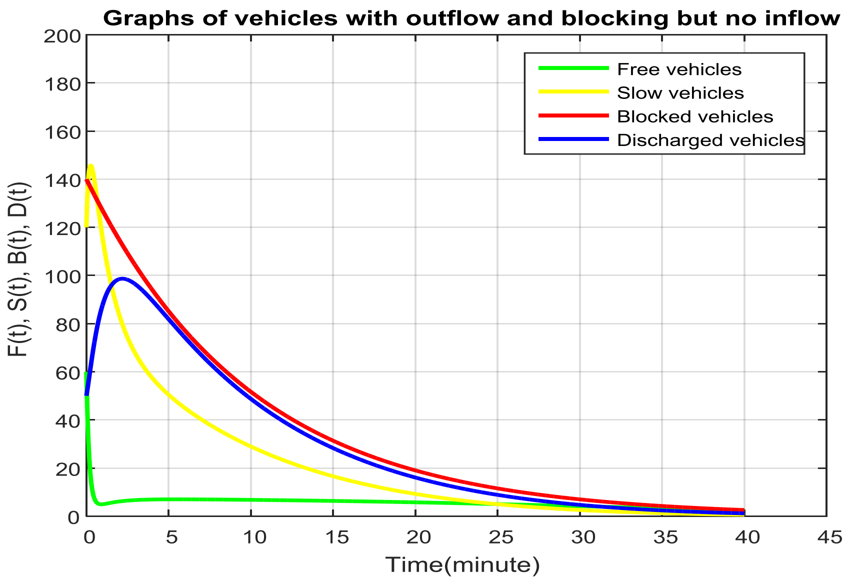

Figure 11 Dynamics of vehicles in the presence of outflow and Blockings but no inflow

The simulated graph of the model equations (2) – (5) shown in Figure 11 is obtained by using the parametric values given in Table 1 together with F = 60, S = 120, B = 140, D = 50, $\tau = 0$, $\mu_d = 0.1$. It can be interpreted that (i) Free vehicles decrease because of crowdedness of the road results in blocking. (ii) Discharged vehicles increases as the the result of vehicles discharged from blockings and decreases as there are continuous outflow (iii) Slow vehicles increases as Free vehicles and discharged vehicles slowed because of blockings and then decreases because of blocking, discharging and outflow(iv) Blocked vehicles decreases as there are discharging and outflow. As a result all vehicles go out of the road as time increases.

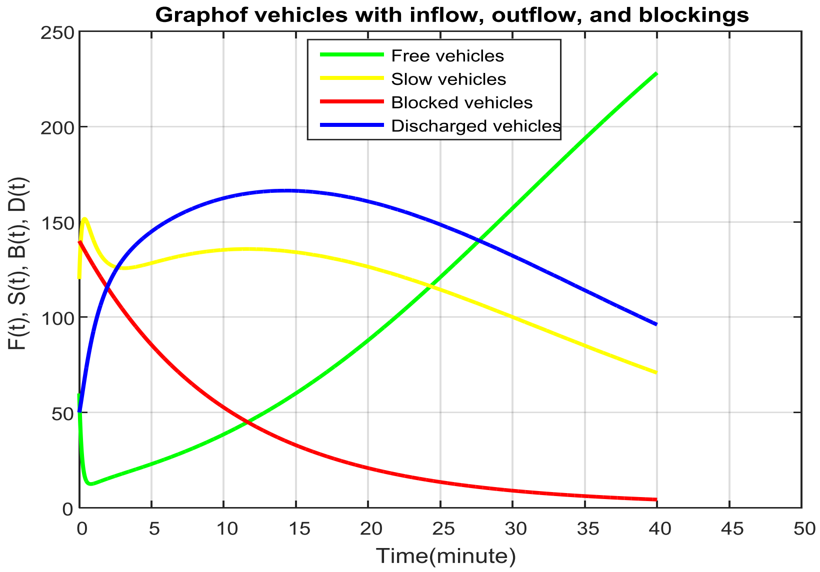

Figure 12 Dynamics of vehicles in the presence of outflow and Blockings but no inflow

The simulated graph of the model equations (2) – (5) shown in Figure 11 is obtained by using the parametric values given in Table 4 together with $\mathrm{F} = 60,\ \mathrm{S} = 120,\ \mathrm{B} = 140,\ \mathrm{D} = 50,\ \tau = 40,\ \mu_d = 0.1$. It can be interpreted that (i) Free vehicles decrease because of crowdedness of the road results in blocking. (ii) Discharged vehicles increases as the the result of vehicles discharged from blockings and decreases as there are continuous outflow (iii) Slow vehicles increases as Free vehicles and discharged vehicles slowed because of blockings and then decreases because of blocking, discharging and outflow (iv) Blocked vehicles decreases as there are discharging and outflow. As a result more Free vehicles move freely on the road as time increases. Here, we can observe that after five minute there are more discharging than slowing and more slowing than stoping. Further, the number of blocked vehicles decreases rapidily.

V. Sensitivity Analysis

Sensitivity analysis is used to determine how “sensitive” a model is towards changes in the values of parameters of the model so as to reflect it in the dynamics of the populations. It is used to discover those parameters that have a high impact on $R_0$and should be targeted by intervention strategies. More precisely, sensitivity indices allow to measure the relative change in a variable when parameter changes.

If the sensitivity index is negative, then the relationship between that parameter and R_0 is inversely proportional. In this case, the modulus of the sensitivity index will be taken in to consideration so that the size of the effect of changing that parameter can be deduced.

On the other hand, a positive sensitivity index means the relationship between that parameter and $R_0$is directly proportional.

The explicit expression of $R_0$is of the model (2) – (5) is given by

$R_0 = \{\alpha\eta\tau/[\mu_d(r_1 + \mu_d)(\gamma + \eta + \mu_d)]\}$.Since,$R_0$ depends only on six parameters, we derive an analytical expression for its sensitivity to each parameter using the normalized forward sensitivity index as given by Chitnis in [12] as follows:

$$\Upsilon_{\alpha}^{R_0} = [\partial R_0/\partial\alpha] \times [\alpha/R_0] = 1$$
$$\Upsilon_{\eta}^{R_0} = [\partial R_0/\partial\eta] \times [\eta/R_0] = [(\gamma + \eta + \mu_d - \alpha\tau\eta)/(\gamma + \eta + \mu_d)]$$
$$\Upsilon_{\mu_d}^{R_0} = [\partial R_0/\partial\mu_d] \times [\mu_d/R_0] = -\{[\alpha(r_1 + \mu_d(2 + r_1 + \mu_d))]/[\mu_d(r_1 + \mu_d)(\eta + \gamma + \mu_d)]\}$$
$$\Upsilon_{\gamma}^{R_0} = [\partial R_0/\partial\gamma] \times [\gamma/R_0] = -\gamma^2/(\gamma + \eta + \mu_d)$$
$$\Upsilon_{r_1}^{R_0} = [\partial R_0/\partial r_1] \times [r_1/R_0] = -\, r_1/(r_1 + \mu_d)$$
$$\Upsilon_{\tau}^{R_0} = [\partial R_0/\partial\tau] \times [\tau/R_0] = 1$$

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Table 2 Sensitivity of R_0againstvariousparameters

Parameter	Sensitivity index
$\boldsymbol{\alpha}$	+1
$\boldsymbol{\eta}$	+0.99468
$\boldsymbol{\gamma}$	-0.41528
$\boldsymbol{\tau}$	+1
$\boldsymbol{r_1}$	-0.00990
$\boldsymbol{\mu_d}$	-1.38877

From Table 2, we obtain$\Upsilon_{\boldsymbol{\mu_d}}^{R_0} = -1.38877$, this means the sensitivity of retardation number with respect to parameter $\mu_d$which is the largestin absolute value.Shows in decreasingμ_d will cause an increaseinR_0 . Similarly, an increase in μ_d will cause a decrease in R_0, as they are inversely proportional. We also note that α and τ are positive and their sensitive index is the second largest value which shows we should also carefully assume this parameters and these parameters are directly proportional to R_0. We can arrange these parameters in the order of their magnitude from largest to the smallest as follows:r_1,γ,η, α, τ , μ_d.

The least sensitive parameter is r_1 and the most sensitive parameter is μ_d.

VI. Results and Discussion

The simmulation figures show the impact of blocking on the vehicles flow especially in capital cities where the traffic increases continuously in the morning, midday and afternoon around sunset up to the maximum possible blocking but tends to move or stop. Also it shows the behavior of vehicles flow where there is no blockings. Figure 1 shows the flow of 60 vehicles for 40 minutes where there is no inflow, outflow and blocking in the road of consideration. This shows the road is suitable for time usage of passengers and minimization of accident that may face the Vehicles.

Figure 2 shows the flow of 60 vehicles for 40 minute with inflow and outflow but no blockings. Figures 3 shows the flow of 60 vehicles with no blocking no inflow but outflow at the rate 0.1. In this figure we observed that after an hour the road considered becomes empty. Figure 4 describes 60 freely moving vehicles with inflow and outflow but no blockings. In this case it increases to the maximum carrying capacity of the road. Figures 5, 6, 7, and 8 show the flow of each compartment in the presence of inflow, outflow and blockings. Figures 9, 10, 11, 12 show the qualitative behavior of compartments with varying values of highly sensitive parameters. As the values most sensitive parameters get smaller the blocking persists.

VII. Conclusion

In this paper, modeling traffic flow using four dimensional nonlinear dynamical systemhas been formulated. Moreover, existence, positivity and boundedness of the formulated model is verified to illustrate that the model is physically meaningful and mathematically well posed. In particular, the stability analyses of the model were investigated using the basic reproduction number and Routh Hurwitz criterion. And also, the solution of the model equation is numerically supplementedandsensitivity analysis of the model is analyzed.

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