IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 4 Ser. I (Jul – Aug 2019), PP 01-07
www.iosrjournals.org

Vertex Domination of the Nilpotent Cayley Graph of the Ring $(Z_n, \oplus, \odot)$

#1 Tippaluri Nagalakshumma, #2* Levaku Madhavi and #3 Jangiti Devendra
#1: Research Scholar, Department of Applied Mathematics, Yogi Vemana University, Kadapa-516005, A.P.,
#2: Assistant Professor, Department of Applied Mathematics, Yogi Vemana University, Kadapa-516005, A.P.,*
#3: Research Scholar, Department of Applied Mathematics, Yogi Vemana University, Kadapa-516005, A.P.,

Abstract: *The properties of the nilpotent Cayley graph $G(Z_n,\ N)$ associated with the set of nilpotent elements of the residue class ring $(Z_n, \oplus, \odot)$ is studied by the authors. The vertex cover, the vertex dominating set and the related domination parameters of these graphs are determined in this paper.*
Keywords: *Nilpotent element, Symmetric set, Cayley graph, Nilpotent Cayley graph.*
AMS Subject Classification (2010): *05C25, 05C69.*

Date of Submission: 24-06-2019 Date of acceptance: 06-07-2019

I. Introduction

As early as 1850, the chess enthusiasts were confronted with the problem of finding the minimum number of queens that can be placed on a chess board so that all the squares are attacked by a queen or occupied by a queen. This problem has roots in domination theory of graphs.

In practical problems the graph theorists are interested in a subset V_0 of the vertex set V of a graph G with minimum cardinality, which is such that every edge of G is incident with some vertex in V_0. Similar concept for edges can also be thought of. An exact notion of a dominating set that is in vogue in the present literature is said to be initiated by Berge [4] and Ore [13]. This lead to an active research work on domination theory of graphs, mostly, by Allan and Laskar [1], Allan et al., [2], Cockayne and Hedetniemi [6], Haynes and Slater [8] and many others. Later Madhavi [11], Maheswari and Madhavi [10], Sujatha and Madhavi [14], Swetha et al., [15] and others have studied the vertex domination, the edge domination and other related parameters for arithmetic graphs and arithmetic Cayley graphs associated with the arithmetic functions in number theory and the residue class ring $(Z_n, \oplus, \odot)$.

In this paper the authors study the vertex domination in the nilpotent Cayley graph of the ring $(Z_n, \oplus, \odot)$. For the graph theoretic, number theoretic and algebraic notions that are used in this paper, the reader is referred to [5], [3] and [9].

II. The Nilpotent Graph $G(Z_n,\ N)$ Of The Ring $(Z_n, \oplus, \odot)$ And Its Properties

In [12] the authors have introduced a new class of arithmetic Cayley graphs, namely, the nilpotent Cayley graphs associated with the set of nilpotent elements in the residue class ring $(Z_n, \oplus, \odot), n \geq 1$, an integer. An element $\bar{a} \neq \bar{0}$, in the ring $(Z_n, \oplus, \odot)$ is called a nilpotent element, if there exists a positive integer l such that $(\bar{a})^l \neq \bar{0}$. It is an easy verification that the set N of all nilpotent elements in the ring $(Z_n, \oplus, \odot)$ is a symmetric subset of the group $(Z_n, \oplus)$. The Cayley graph $G(Z_n,\ N)$ associated with the group $(Z_n, \oplus)$ and its symmetric subset N, is the graph whose vertex set V is Z_n and the edge set $E = \{(x, y)/x, y \in Z_n$ and either $x - y \in N$ or $y - x \in N\}$ and it is called the nilpotent Cayley graph of the ring $(Z_n, \oplus, \odot)$. We state below some basic properties relating to $G(Z_n,\ N)$, whose proofs can be followed in [12].

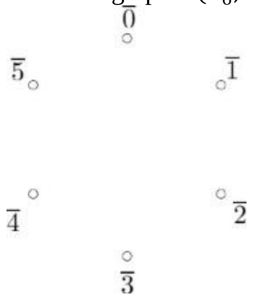
Lemma 2.1*:* (Lemma 2.7, page 3, [12]) Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$, are primes, $\alpha_i \geq 1$ and $1 \leq i \leq r$, are integers. The set N of nonzero nilpotent elements in the ring $(Z_n, \oplus, \odot)$, is given by $N = \{\bar{m} \odot \bar{1}, \bar{m} \odot \bar{2}, \ldots, \bar{m} \odot (\overline{\prod_{i=1}^{r} p_i^{\alpha_i - 1} - 1})\}$, where $m = p_1 p_2 p_3 \ldots p_r$.
Further m is the least positive integer such that $\bar{m}$ is a nilpotent element in $(Z_n, \oplus, \odot)$.

Lemma 2.2: (Corollary 2.8, page 4, [12]) The number $\mathfrak{N}(n)$ of the nilpotent elements in the ring $(Z_n, \oplus, \odot)$, where $n = \prod_{i=1}^{r} p_i^{\alpha_i}, p_1 < p_2 < \cdots < p_r$, are primes and $\alpha_i \geq 1$, $1 \leq i \leq r$, are integers, is given by $\mathfrak{N}(n) = \prod_{i=1}^{r} p_i^{\alpha_i - 1} - 1$.

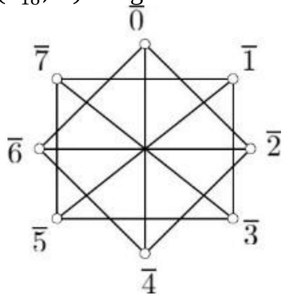
Lemma 2.3: (Lemma 2.10, page 4, [12]) The graph $G(Z_n, N)$ is $(\prod_{i=1}^{r} p_i^{\alpha_i - 1} - 1)$ −regular and the number of edges in $G(Z_n, N)$ is given by $\frac{n}{2}(\prod_{i=1}^{r} p_i^{\alpha_i - 1} - 1)$.

Lemma 2.4: (Lemma 2.11, page 5, [12]) For an integer $m = p_1 p_2 \ldots p_r$, where $p_1 < p_2 < \cdots < p_r$ are primes, the graph $G(Z_m, N)$ contains only vertices.

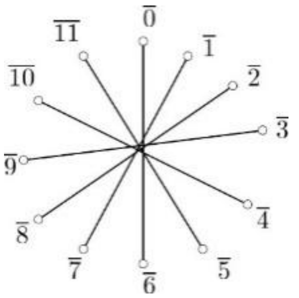
Example 2.5: The graphs $(Z_6, N), G(Z_8, N), G(Z_{12}, N)$ and $G(Z_{18}, N)$ are given below:



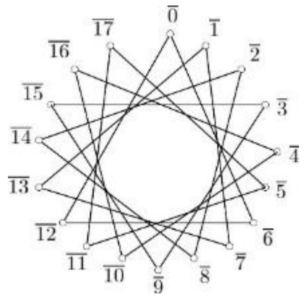
$\boldsymbol{G(Z_6, N)}$



$\boldsymbol{G(Z_8, N)}$



$\boldsymbol{G(Z_{12}, N)}$



$\boldsymbol{G(Z_{18}, N)}$

Theorem 2.6: (Theorem 3.5, page 7, [12]) If $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$ are primes, $\alpha_i \geq 1$, $1 \leq i \leq r$, are integers and $n = p_1 p_2 \ldots p_r$, then the graph $G(\mathrm{Z}_n,\ N)$ is a union of m disjoint connected components of $G(\mathrm{Z}_n,\ N)$, each of which is a complete subgraph of $G(\mathrm{Z}_n,\ N)$.

III. Vertex Cover Of The Nilpotent Cayley Graph $\boldsymbol{G(\mathbf{Z}_n,\ N)}$

This section is devoted for the discussion of the vertex cover and the vertex covering number of the nilpotent Cayley graph.

Definition 3.1: A subset S of vertices of a graph G is called a vertex cover of G, if every edge of G is incident with some vertex in S. A minimum vertex cover is one with minimum cardinality.

Definition 3.2: The cardinality of a minimum vertex cover of a graph G is called the vertex covering number and it is denoted by $\beta(G)$.

Lemma 3.3: If $n = \prod_{i=1}^{r} p_i$, where $p_1 < p_2 < \cdots < p_r$ are primes and $1 \leq i \leq r$ are integers, then the vertex cover of the graph is empty.

Proof: By the Lemma 2.4, the graph $G(\mathrm{Z}_n,\ N)$ contains only vertices. Since $G(\mathrm{Z}_n,\ N)$ has no edges, its vertex cover is empty and the vertex cover of $G(\mathrm{Z}_n,\ N)$ is empty and its vertex covering number is zero. ■

The following corollary is immediate form the Lemma 3.3.

Corollary 3.4: If p is a prime then the vertex cover of the graph $G(\mathrm{Z}_p,\ N)$ is empty and its vertex covering number is zero.

Theorem 3.5: If $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$ are primes, $\alpha_i \geq 1$, $1 \leq i \leq r$ are integers and let $m = p_1 p_2 \ldots p_r$, then the subset $V_c = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{(n-m)-1}\}$ of the vertex set V of the nilpotent graph $G(\mathrm{Z}_n,\ N)$ is a vertex cover of $G(\mathrm{Z}_n,\ N)$.

Proof: Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$ are primes, $\alpha_i \geq 1$, $1 \leq i \leq r$ are integers and let $m = p_1 p_2 \ldots p_r$. Consider the subset $V_c = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-m-1}\}$ of the vertex set V of the graph $G(\mathrm{Z}_n,\ N)$. Let (x, y) be an edge of $G(Z_n, N)$. If we show that at least one of x and y is in V_c, then V_c is a vertex cover of $G(Z_n, N)$. Suppose that $x \notin V_c$ and $y \notin V_c$. Then, $x, y \in V - V_c$, where

$$V - V_c = \{\overline{n-m},\ \overline{(n-m)+1}, \ldots, \overline{(n-m)+j}, \ldots, \overline{(n-m)+(m-2)}, \overline{(n-m)+(m-1)}\}.$$

So, $x = \overline{(n-m)+i}$ and $y = \overline{(n-m)+j}$, for some integers $i,\ j,\ 0 \leq i < j \leq m-1$, and

$$y - x = \overline{(n-m)+j} - \overline{(n-m)+i} = \overline{j-i}.$$

Now, $1 \leq i < j \leq m-1$, implies that $0 < j - i < m$. So $y - x = \overline{j-i}$, where $0 < j - i < m$. Since m is the smallest positive integer such that $\overline{m}$ is a nilpotent element in $(Z_n, \oplus, \odot)$, $j - i < m$ implies that $\overline{j-i}$ is not a nilpotent element in $(Z_n, \oplus, \odot)$. So, (x, y) is not an edge of $G(Z_n, N)$, which is a contradiction to our assumption that (x, y) is an edge of $G(Z_n, N)$. So at least one of x, or, y is in V_c and V_c is a vertex cover of $G(Z_n, N)$. ■

Lemma 3.6: If $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$ are primes, $\alpha_i \geq 1$, $1 \leq i \leq r$ are integers and let $m = p_1 p_2 \dots p_r$, then the subset $V_1 = V_c - \{\bar{l}\}$, where

$V_c = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{n-m-1}\}$

and $0 \leq l \leq (n-m) - 1$ is not a vertex cover of $G(Z_n, N)$.

Proof: Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$ are primes, $\alpha_i \geq 1$, $1 \leq i \leq r$ are integers and let $m = p_1 p_2 \dots p_r$. By the Theorem 3.5, the subset

$V_c = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{(n-m)-1}\}$

of the vertex set V is a vertex cover of $G(Z_n, N)$. For some l, where $0 \leq l \leq (n-m) - 1$, let $V_1 = V_c - \{\bar{l}\}$. Then

$$V - V_1 = \{\bar{l}, \overline{(n-m)}, \overline{(n-m)+1}, \overline{(n-m)+2}, \dots, \overline{(n-m)+l}, \dots, \overline{(n-m)+m-1}\},$$

and $\bar{l} \notin V_1$. Let $l = mq + t$, where q and t are integers such that $0 \leq t < m$ and . Since $0 \leq t < m, \overline{(n-m)+t} \in V - V_1$, so that $\overline{(n-m)+t} \notin V_1$. Now $\bar{l} - \left[\overline{(n-p_1)+t}\right] = \overline{(mq+t)} - \overline{(n-m)+t} = \overline{m(1+q)}$, since $\bar{n} = \bar{0}$. But $\overline{m(1+q)} \in N$, since $\bar{m}$ is a nilpotent element in the ring $(Z_n, \oplus, \odot)$. That is, there is an edge between, $\overline{(n-m)+t}$ and $\bar{l}$ and none of these vertices lies in V_1, showing that V_1 is not a vertex cover of $G(Z_n, N)$. ■

Lemma 3.7: If W is a subset of the vertex set V of $G(Z_n, N)$ such that $|W| < |V_c|$, then W is not a vertex cover of $G(Z_n, N)$.

Proof: In the Theorem 3.5, we have seen that $V_c = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{(n-m)-1}\}$ is a vertex cover of $G(Z_n, N)$. Let $W \subseteq V$ such that $|W| < |V_c|$. The following cases will arise.

Case (i): Let $W \subset V_c$ and $|W| < |V_c|$. Then V_c contains at least one vertex that is not in W and by the Lemma 3.6, W is not a vertex cover of $G(Z_n, N)$.

Case (ii): Let $W \cap V_c = \phi$.

Suppose $W = V - V_c$. Then the vertex set is the disjoint union of V_c and W and

$W = \{\overline{(n-m)}, \overline{(n-m)+1}, \dots, \overline{(n-m)+k}, \dots, \overline{(n-1)}\}$.

For some $l, 0 \leq l \leq n-m$, consider the vertices $x = \overline{n-lm}$ and $y = \overline{n-(l-1)m}$. By the choice of l clearly $x \in V_c$ and $y \in V_c$ so that $x \notin W$ and $y \notin W$. But

$y - x = \overline{n-(l+1)m} - \overline{(n-lm)} = \bar{m}$,

and $\bar{m}$ is a nilpotent element of $(Z_n, \oplus, \odot)$, so that there exists an edge between x and y. Further, none of x and y belong to W, so that W is not a vertex cover of $G(Z_n, N)$.

If $W \subset V - V_c$. Then

$W \subset \{\overline{(n-m)}, \overline{(n-m)+1}, \dots, \overline{(n-m)+l}, \dots, \overline{(n-1)}\}$.

So there exists $k, 0 \leq k \leq m-1$, such that $\overline{(n-m)+k} \in V - V_c$ but $\overline{(n-m)+k} \notin W$. Also $0 \leq k \leq m-1$ implies that $\bar{k} \notin V - V_c$, or, $\bar{k} \notin W$. However

$\left[\overline{(n-m)+k}\right] - \bar{k} = \overline{n-m}$,

and $\overline{n-m}$, being the inverse of $\bar{m}$ in $(Z_n, \oplus)$ is a nilpotent element in $(Z_n, \oplus, \odot)$, so that there is an edge between $(n-m)+\bar{k}$ and $\bar{k}$ with neither of these vertices in W. So W is not a vertex cover of $G(Z_n, N)$.

Case (iii): Let $|W| < |V_c|$ and $W \cap V_c = \phi$. Evidently $W \subset \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{n-1}\}$. So, let k be the largest integer such that $0 \leq k \leq n-1$ and $\bar{k} \in W$. Then $\overline{k+1} \notin W$. Consider $\overline{(n-m)+k+1} \in V$. Since $(n-m)+k+1 > k$, it follows that $\overline{(n-m)+k+1} \notin W$. However $\left[\overline{(n-m)+k+1}\right] - \left[\overline{k+1}\right] = \overline{n-m}$, and this is a nilpotent element in $(Z_n, \oplus, \odot)$, so that there is an edge between $\overline{(n-m)+k+1}$ and $\overline{k+1}$, with neither of the vertices in W. This shows that W is not a vertex cover of the graph $G(Z_n, N)$.

From the above discussion it follows that, if $W \subset V$ and $|W| < |V_c|$ then W is not a vertex cover of $G(Z_n, N)$. ■

Theorem 3.8: If $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$ are primes, $\alpha_i \geq 1$, $1 \leq i \leq r$ are integers and let $m = p_1 p_2 \dots p_r$, then the subset $V_c = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{(n-m)-1}\}$ of the vertex set V of the nilpotent graph $G(\mathrm{Z}_n, N)$ is a minimum vertex cover of $G(\mathrm{Z}_n, N)$.

Proof: The proof follows from the Theorem 3.5 and the Lemma 3.7.

The following corollary is immediate form the Theorem 3.8.

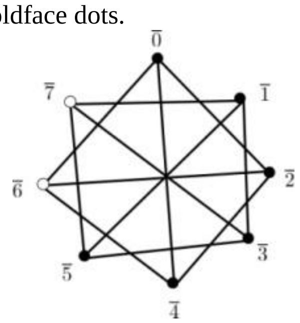
Corollary 3.9: If $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$ are primes, $\alpha_i \geq 1$, $1 \leq i \leq r$ are integers and $m = p_1 p_2 \dots p_r$, then the vertex covering number $\beta(G(Z_n, N))$ of the graph $G(Z_n, N)$ is given by $\beta(G(Z_n, N)) = n - m$.

Corollary 3.10: Let p be a prime and $r > 1$, an integer. Then the minimum vertex cover of the graph $G(Z_{p^r}, N)$ is $V_c = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{n-p-1}\}$ and the vertex covering number the graph $G(Z_{p^r}, N) = n - p$.

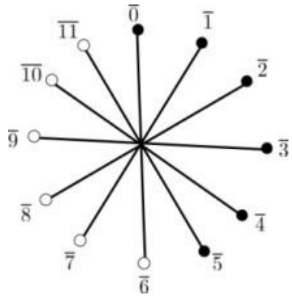
Corollary 3.11: For an integer $n = p_1 p_2 \dots p_r$, where $p_1 < p_2 < \cdots < p_r$ are primes, the vertex cover of $G(Z_n, N)$ is empty and hence vertex covering number is zero.

Corollary 3.12: For a prime p, the vertex covering number is zero.
Proof: Let p be a prime. Then the set N of nilpotent elements of the ring $(Z_p, \oplus, \odot)$ is empty, so that the vertex cover is empty and hence the vertex covering number is zero.

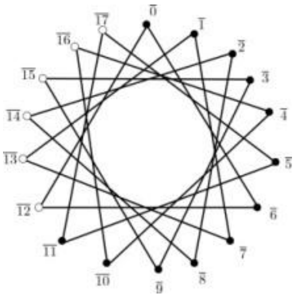
Example 3.13: Minimum vertex covers of the graphs $G(Z_8, N), G(Z_{12}, N)$ and $G(Z_{18}, N)$ are respectively given by $\{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$, $\{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ and $\{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}, \overline{8}, \overline{9}, \overline{10}, \overline{11}\}$ and these vertices are denoted by boldface dots.



$\boldsymbol{G(Z_8, N)}$



$\boldsymbol{G(Z_{12}, N)}$



$\boldsymbol{G(Z_{18}, N)}$

Remark 3.14: One can easily observe that each of subsets of the vertex set V given below, is also a minimum vertex cover of $G(Z_n, N)$.

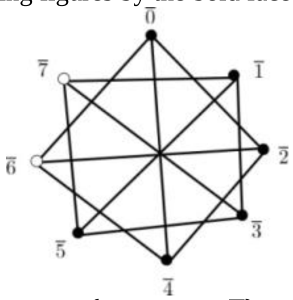
$C_0 = \{\overline{0}, \overline{m}, \overline{2m}, \dots, \overline{im}, \dots, \overline{(n-m)-1}\}$,

$C_1 = \{\overline{1}, \overline{m+1}, \overline{2m+1}, \dots, \overline{im+1}, \dots, \overline{(n-m)}\}$,

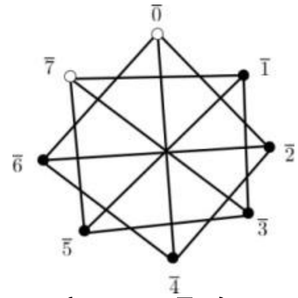
$\vdots$

$C_{m-1} = \{\overline{m-1m+m-1}, \overline{2m+m-1}, \dots, \overline{im+m-1}, \dots, \overline{n-1}\}$.

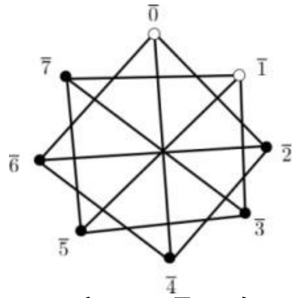
Example 3.14: The possible minimum vertex covers of the nilpotent Cayley graph $G(Z_8, N)$ are shown in the following figures by the bold face vertices.



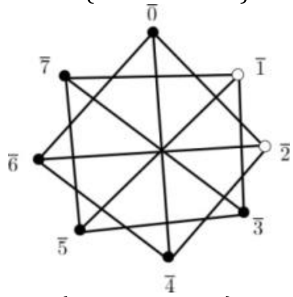
$\{\boldsymbol{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}}\}$



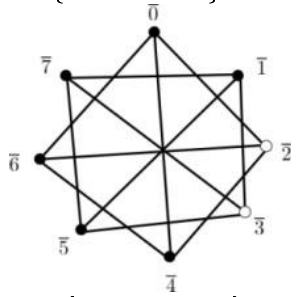
$\{\boldsymbol{\overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}}\}$



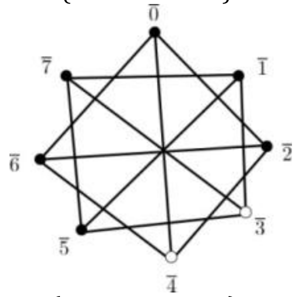
$\{\boldsymbol{\overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}}\}$



$\{\boldsymbol{\overline{3}, \overline{4}, \overline{5}, \overline{6}, 7, \overline{0}}\}$



$\{\boldsymbol{\overline{4}, \overline{5}, \overline{6}, 7, \overline{0}, \overline{1}}\}$



$\{\boldsymbol{\overline{5}, \overline{6}, 7, \overline{0}, \overline{1}, \overline{2},}\}$

IV. Vertex Domination Of The Nilpotent Cayley Graph $\boldsymbol{G(Z_n, N)}$

Another concept which is closely associated with the vertex cover of a graph is its vertex dominating set, which is defined as follows.

Definition 4.1: A subset D of the vertex V of a graph G is called a vertex dominating set of G, if each vertex of $V - D$ is adjacent to at least one vertex of D. A dominating set with minimum cardinality is called a minimum vertex dominating set of G. The domination number $\gamma(G)$ of G is the cardinality of a minimum dominating set.

In this section vertex dominating sets, minimum vertex dominating sets and the vertex dominating number are determined for the nilpotent Cayley graph $G(Z_n, N)$ of the ring $(Z_n, \oplus, \odot)$.

Lemma 4.2: Let $n = \prod_{i=1}^{r} {p_i}^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$ are primes, $\alpha_i \geq 1, 1 \leq i \leq r$ are integers. Let $m = p_1 p_2 \dots p_r$ and let $\overline{m_i} = \overline{m} \odot \overline{\iota}$, where $1 \leq i \leq \prod_{i=1}^{r} {p_i}^{\alpha_i - 1} - 1$. Then each of the subsets $D_i = \{\overline{0}, \overline{1}, \overline{2}, \dots, \overline{m_i - 1}\}$ of the vertex set V of graph $G(Z_n, N)$ is a vertex domination set of $G(Z_n, N)$.

Proof: Let $n = \prod_{i=1}^{r} {p_i}^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$ are primes, $\alpha_i \geq 1$, $1 \leq i \leq r$ are integers and let $\overline{m_i} = \overline{m} \odot \overline{\iota}$, where $m = p_1 p_2 \dots p_r$ and $1 \leq i \leq \prod_{i=1}^{r} {p_i}^{\alpha_i - 1} - 1$. We shall show that the subset $D_i = \{\overline{0}, \overline{1}, \overline{2}, \dots, \overline{m_i - 1}\}$ of vertex set V is a vertex dominating set of $G(Z_n, N)$. The vertex set

$$V = \{\overline{0}, \overline{1}, \overline{2}, \dots, \overline{m_i - 1}, \overline{m_i}, \overline{m_i + 1}, \dots, \overline{m_i + k}, , \dots, \overline{m_i + (n - m_i) - 2}, \overline{(n-1)}\}$$

of $G(Z_n, N)$ can be written as the disjoint union of D_i and $V - D_i$, where

$$V - D_i = \{\overline{m_i}, \overline{m_i + 1}, \dots, \overline{m_i + k}, , \dots, \overline{m_i + (n - m_i) - 2}, \overline{(n-1)}\}.$$

To show that D_i is a vertex dominating set of $G(Z_n, N)$, we have to show that every vertex x in $V - D_i$ is adjacent to some y in D_i.

So let $x \in V - D_i$. Then $x = \overline{m_i + k}$ for some $k, 0 \leq k \leq (n - m_i) - 1$. By the division algorithm applied to m_i and k, there exist integers t and s such that $k = m_i t + s$, and $0 \leq s < m_i$. Now $0 \leq s < m_i$, implies that $s \in D_i$. For this $s (= k - m_i t)$ we have

$$x - s = \overline{(m_i + k)} - \overline{(k - m_i k)} = \overline{m_i (k + 1)}.$$

Since $\overline{m_i}$ is a nilpotent element of $(Z_n, \oplus, \odot)$, it follows that $\overline{m_i (k + 1)}$ is also a nilpotent element of $(Z_n, \oplus, \odot)$. Hence $x - s$ is a nilpotent element of $(Z_n, \oplus, \odot)$, so that x is adjacent to s. That is, x in $V - D_i$ is adjacent to s in D_i, so that D_i is a vertex dominating set of (Z_n, N). ■

Corollary 4.3: If $n = \prod_{i=1}^{r} {p_i}^{\alpha_i}$ where $p_1, p_2, \dots, p_r$ are primes $\alpha_i \geq 1$, $1 \leq i \leq r$ are integers and $m = p_1 p_2 \dots p_r$, then the vertex dominating set $D_1 = \{\overline{0}\overline{1}, \overline{2}, \dots, \overline{m-1}\}$ is the set with minimal cardinality among the vertex dominating sets $D_i, 1 \leq i \leq \prod_{i=1}^{r} {p_i}^{\alpha_i - 1} - 1$ of $G(Z_n, N)$.

Proof: Since $m < m_i$, for $2 \leq i \leq \prod_{i=1}^{r} {p_i}^{\alpha_i - 1} - 1$, the Corollary follows form the Lemma 4.2. ■

Lemma 4.4: If $n = \prod_{i=1}^{r} {p_i}^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$ are primes, $\alpha_i \geq 1$, $1 \leq i \leq r$ are integers and let $m = p_1 p_2 p_3 \dots p_r$, then the subset $U_1 = D_1 - \{\overline{l}\}$, where $D_1 = \{\overline{0}, \overline{1}, \overline{2}, \dots, \overline{m-1}\}$ and $0 \leq l \leq m - 1$, is not a vertex dominating set of $G(Z_n, N)$.

Proof: By the Corollary 4.3, the set $D_1 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \dots, \overline{m-1}\}$ is a vertex dominating set of $G(Z_n, N)$. Let $U_1 = D_1 - \{\overline{l}\}$, for some $l, 0 \leq l \leq m - 1$. Then $\overline{l} \in V - U_1$. Let $\overline{k} \in U_1$, so that $0 \leq k \leq m - 1$ and $k \neq l$. We may assume that $k < l$. Then $0 \leq l - k \leq m - 1$, or, $l - k \leq m$. Since m is the least positive integer such that $\overline{m}$ is a nilpotent element of $(Z_n, \oplus, \odot)$, it follows that $\overline{l - k}$ is not a nilpotent element of $(Z_n, \oplus, \odot)$, so that the vertex $\overline{l}$ in $V - U_1$ is not adjacent to any vertex $\overline{k}$ in U_1. This shows that U_1 is not a vertex dominating set of $G(Z_n, N)$. ■

The following Lemma can be proved on similar lines as that of Lemma 3.7.

Lemma 4.5: If S is a subset of the vertex set of $G(Z_n, N)$ such that $|S| < |D_1|$, then S is not a vertex dominating set of $G(Z_n, N)$.

Theorem 4.6: Let $n = \prod_{i=1}^{r} {p_i}^{\alpha_i}$, where $p_1 < p_2 < \cdots, p_r, \alpha_i \geq 1, 1 \leq i \leq r$ integers and let $m = p_1 p_2 p_3 \dots p_r$. Then the vertex set $D_1 = \{\overline{0}, \overline{1}, \dots, \overline{m-1}\}$ is a minimum vertex dominating set.

Proof: The proof follows from the Lemma 4.2, Corollary 4.3, Lemma 4.4 and Lemma 4.5.

The following two corollaries follow from the Theorem 4.6.

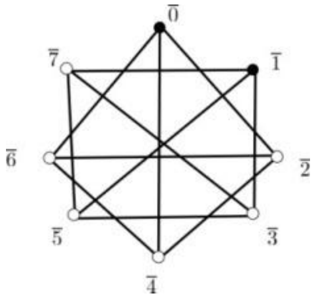
Corollary 4.7: If $n = \prod_{i=1}^{r} {p_i}^{\alpha_i}$, $p_1 < p_2 < \cdots p_r, \alpha_i \geq 1$, $1 \leq i \leq r$ are integers and $m = p_1 p_2 p_3 \dots p_r$, then the vertex domination number $\gamma(G(Z_n, N))$ of the graph $G(Z_n, N)$ is m.

Corollary 4.8: Let p be a prime and $r > 1$ an integer. Then the subset $\{\overline{0}, \overline{1}, \overline{2}, \dots, \overline{(p-1)}\}$ of the vertex set is a minimum dominating set of the graph $G(Z_{p^r}, N)$ and its vertex domination number is p.

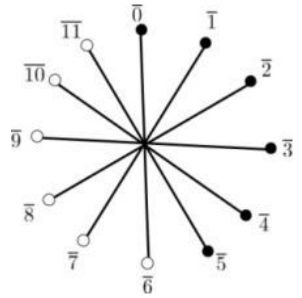
Corollary 4.9: If $n = p_1 p_2 \dots p_r$, where $p_1 < p_2 < \cdots < p_r$ are primes, then the vertex domination number of $G(Z_n, N)$ is zero.

Proof : Let $n = p_1 p_2 \dots p_r$, where $p_1 < p_2 < \cdots < p_r$ are primes, then $G(Z_n, N)$ contains only vertices and no edges. So the vertex dominating set is empty and hence the vertex domination number of $G(Z_n, N)$ is zero.

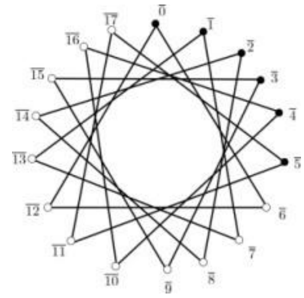
Example 4.10: The vertex dominating sets of the graphs $G(Z_8, N), G(Z_{12}, N)$ and $G(Z_{18}, N)$ are respectively given by $\{\bar{0}, \bar{1}\}$, $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$. The vertices of the respective dominating sets are exhibited by bold face dots.



$\boldsymbol{G(Z_8, N)}$



$\boldsymbol{G(Z_{12}, N)}$

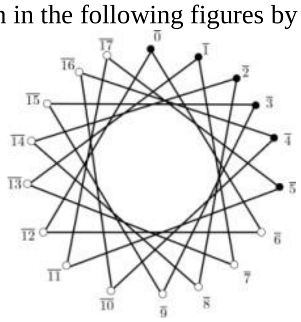


$\boldsymbol{G(Z_{18}, N)}$

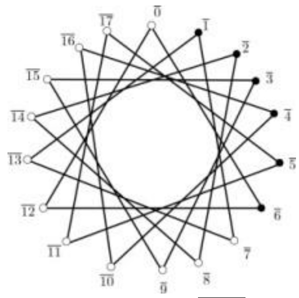
Remark 4.11: One can easily observe that each of subsets of the vertex set V given below, is also a minimum vertex cover of $G(Z_n, N)$.

$$C_0 = \{\bar{0}, \bar{1}, \bar{2}, \dots, \bar{\iota}, \dots, \overline{(m-1)}\},$$
$$C_1 = \{\bar{1}, \bar{2}, \bar{3}, \dots, \overline{i+1}, \dots, \bar{m}\},$$
$$\vdots$$
$$C_{m-1} = \{\overline{m-1}\bar{m}, \overline{m+1}, \dots, \overline{m+2}, \dots, \overline{2m+1}\}.$$

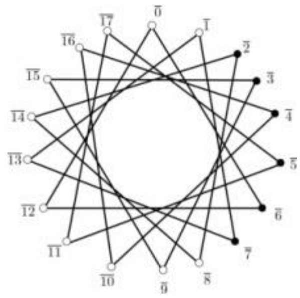
Example 4.12: The possible minimum vertex dominating sets of the nilpotent Cayley graph $G(Z_{18}, N)$ are shown in the following figures by the bold face vertices.



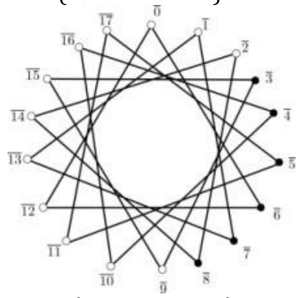
$\{\bar{\mathbf{0}}, \bar{\mathbf{1}}, \bar{\mathbf{2}}, \bar{\mathbf{3}}, \bar{\mathbf{4}}, \bar{\mathbf{5}}\}$



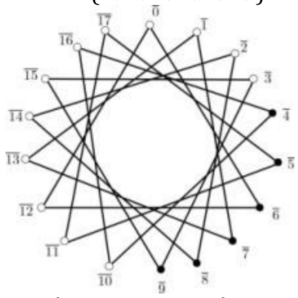
$\{\bar{\mathbf{1}}, \bar{\mathbf{2}}, \bar{\mathbf{3}}, \bar{\mathbf{4}}, \overline{\mathbf{5}, \mathbf{6}},\}$



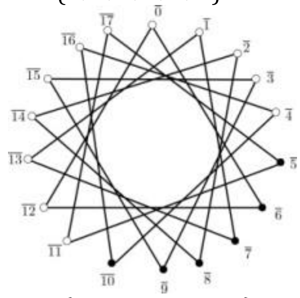
$\{\bar{\mathbf{2}}, \bar{\mathbf{3}}, \bar{\mathbf{4}}, \bar{\mathbf{5}}, \bar{\mathbf{6}}, \bar{\mathbf{7}}\}$



$\{\bar{\mathbf{3}}, \bar{\mathbf{4}}, \bar{\mathbf{5}}, \bar{\mathbf{6}}, \bar{\mathbf{7}}, \bar{\mathbf{8}}\}$



$\{\bar{\mathbf{4}}, \bar{\mathbf{5}}, \bar{\mathbf{6}}, \bar{\mathbf{7}}, \bar{\mathbf{8}}, \bar{\mathbf{9}},\}$



$\{\bar{\mathbf{5}}, \bar{\mathbf{6}}, \bar{\mathbf{7}}, \bar{\mathbf{8}}, \bar{\mathbf{9}}, \overline{\mathbf{10}},\}$

Acknowledgment

The authors express their thanks to Prof. L. Nagamuni Reddy for his suggestions during the preparation of this paper.

References

[1]. Allan, R. B, Laskar, R.: On domination and independent domination number of a Graph, Discrete Math, 23: 73-76 (1978).
[2]. Allan, R. B., Lasker, R., Hedetniemi, S.T.: A note on total domination, Discrete Math, 49: 7-13(1984).
[3]. Apostol, T. M.: *Introduction to Analytic Number Theory*, Springer International Student Edition (1989).
[4]. Berge, C.: Theory of Graphs and its Applications, Methuen, London (1962).
[5]. Bondy, J.A., Murty, U.S.R.:*Graph Theory and related topics*, Macmillan, London(1979).
[6]. Cockayne, E. J., Hedetniemi S.T.: Towards a theory of domination in graphs, Networks, 4, pp 247-261 (1977).
[7]. Dhren Kumar Basnet, Ajay Sharma and Rahul Dutta, Nilpotent Graph, arXiv: 1804 .08937 v1 [math. RA], 24 April 2018.
[8]. Haynes, T.W., Hedetniemi S.T., Slater, P.J.: Domination in graphs: Advanced topics, Marcel Dekker. Inc., (2000).
[9]. Joseph A. Gallian: *Contemporary Abstract Algebra*, Narosa publishing house, Ninth edition (2018) .

[10]. Maheswari, B., Madhavi, L.: Vertex Domination in Mangoldt Graph. Journal of APMS, 1(2):184 – 194 (2008).

[11]. Madhavi, L.: Studies on domination parameters and Enumeration of cycles in some arithmetic graphs, Ph.D Thesis, Sri Venkateswara University, Tirupati, India (2003) .

[12]. Nagalakshumma, T., Devendra, J,. Madhavi,L.: The Nilpotent Cayley Graph of the Residue Class Ring $(Z_n, \oplus, \odot)$, Journal of Computer and Mathematical Sciences, vol.10(6),1244-1252 June 2019.

[13]. Ore, O.: Theory of Graphs, Amer, Math. Soc. Colloq., publ. 38, Providence, pp 206 (1962).

[14]. Sujatha, K., Madhavi, L.: Vertex Domination In Divisor Cayley Graph, **International Journal of Computational Science, Mathematics and Engineering,** Special Issue on Computational Science, Mathematics and Biology, IJCSME- SCSMB-16,pp162 – 164, ISSN-2349-8439 (2016).

[15]. Swetha, A., Shobhalatha, G., Madhavi, L.: Domination Parameters on Graphs of the Ring Zn, **International Journal of Computational Science, Mathematics and Engineering,** Special Issue on Computational Science, Mathematics and Biology, IJCSME-SCSMB-16, pp187 – 195, ISSN- 2349-8439 (2016).

Tippaluri Nagalakshumma. " Vertex Domination of the Nilpotent Cayley Graph of the Ring (Z_n,⊕,⊙)." IOSR Journal of Mathematics (IOSR-JM) 15.4 (2019): 01-07.