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Comparism of Quassi-Seidel, Jacobi and Conjugate Gradient Methods for convergent and Speed Using Matlab for Linear System of equations.

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Abstract: *The term "iterative method" refers to a wide range of techniques which use successive approximations to obtain more accurate solutions. In this research an attempt to solve systems of linear equations of the form AX=b, where A is a known square and positive definite matrix. we are going to compare three iterative methods for solving linear system of equations namely (Jacobi, Gauss-Seidel, Conjugate Gradient). To achieve the required solutions more quickly we shall demonstrate algorithms for each of these methods. Then using Matlab language these algorithms are transformed and then used as iterative methods for solving these linear systems of linear equations. Moreover we compare the results and outputs of the various methods of solutions of a numerical example.*
The result of this research shows that conjugate gradient method is more accurate than the other methods and to also note that the CGM method is a non-stationary method. These methods are recommended for similar situations which are arise in many important settings such as finite differences, finite element methods for solving partial differential equations and structural and circuit analysis .
Key words*: Convergent, conjugate, gradient, symmetric, positive definite and successive*

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I. Introduction

Systems of linear equations are associated with many problems in Engineering and Sciences as well as with applications of Mathematics to Social Sciences and quantitative study of Business, Statistics and Economics Problems.

After the invent of access to computers, it is possible and practically easy for us to solve large set of simultaneous linear algebraic equations. To appreciate the decision of physical problems, it is sometimes appropriate to use algorithms which converge rapidly in solving these problems [8].

The processes such as weather forecasting, image processing and simulation to predict aerodynamics performance involved very large se of simultaneous equations by Numerical Methods and time is an important factor for practical application of the results. For large set of linear equations, iterative methods are preferable and are unaffected by round off errors to a large extent [3]. The well known classical numerical iterative methods are the Jacobi and Gauss- Seidel Methods.

The rate of convergent for the two methods can be accelerated by using successive relaxation (SR) technique [7]. The speed of convergent depends on the relaxation factor (ω) with a necessary condition for the convergence $(0 \le \omega \le 1)$ and SR technique is very much sensitive to relaxation factor [1].

The first is to get nxn linear system of equations, solve the three iterative methods (Jacobi, Gaussi-Seidel and Conjugate Gradient methods) by Matlab, and compare the time taken and the rate of convergent at that time.

II. Literature Review

According to [2], linear system of $Ax = b$ is non- square system of linear equations. The iteration method for solvimg the non-square matrix system of equations in the form of Fourier Metzkin was discovered by [6]. Jacobi Method is used to solve large sparse symmetric matrices [2, 3, 4]. In solving the system of linear equations when large parameters are involved which was discovered by [2,3,7]. The studied the method of Jacobi for convergent and discovered that the terminal problems have the efficient hierararchical iteration and an efficient algorithm can reduce the run-time by speeding up the convergence with accurate estimation [5, 6, 7, 11].

III. Method

There are three methods that are to be used; Jacobi, Gauss-Seidel and Conjugate Gradient Methods.
The Jacobi Method in linear algebra for determining the solutions of square systems of linear equations was improved by [6, 7]. The solution is one of the stationary iterative points where the number of iterations is equal to the number of variables. Usually the Jacobi Method is based on solving variable x_i of the vector of variables.

$$X^T = (x_1, x_2, x_3, \quad x_n)$$

The resulting method is easy to compute and implement, but the convergence with respect to the iteration parameter, ω can be evaluated as below.
Consider a square system of n linear equations in n variables.

$Ax = b$ where $A = (a_{i,j})$ *and* $i, j = 1, 2, 3, ..., n$ (1)

The column matrix of unknown to be determined X, $X^t = (x_1, x_2, x_3, ..., x_n)$

and the column matrix of known constants b is $b^t = (b_1, b_2, b_3, \quad, b_n)$.

The system of linear equations can be written $(D+R)X = b$ (2)

where $A = A + R, \quad D = (a_{i,i}) \quad i = 1, 2, 3, ..., n$ is the diagonal matrix D of A and R=L+U

where L and U are strictly lower and upper matrix of A.

Therefore, if the inverse D^{-1} exists and the equation (2) above can be written as $X = D^{-1}(b + RX)$ (3)

The Jacobi Method is an iterative technique based on solving the lef t hand side if the equation for X using previous value of X on the right hand side. Hence, equation (3) can be rewritten as iterative form after k,

$X^{(k)} = D^{-1}(b - RX^{(k-1)}, \quad k = 1, 2, 3, ...$ (4)

Rewriting (4) we get, $x_i^{\,k} = \frac{1}{a_{ii}}(b_i \sum_{j=1}^{n} a_{ij} x^k{}_{k-1} \quad , k = 1, 2, \ and \ \ i = 1, 2, 3, \ldots n$

This becomes $X^k = TX^{(k-1)} + C, \quad k = 1, 2, 3, \ ..., \quad T = D^{-1}(-L - U) \ \ and \ \ C = D^{-1}b$

The initial system is given by

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + \ldots \quad + a_{3n}x_n = b_3$$
$$.$$
$$.$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots \quad + a_{nn}x_n = b_n$$

The coefficient matrix has no zeros on its main diagonal and to solve,

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{21}}(b_2 - a_{21}x_1 - a_{23}x_3 - .. - a_{2n}x_n)$$

$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{nn}x_n)$$

Jacobi Method Using Matlab

```
%% Jacobi Method
%%    Solution of x in Ax=b using Jacobi Method
```

$\%* initialize"A"b initial guass x^1 * \%\%A = [5-230, -391-2, -1-71, 43-57] b = [-1230, 5] x = [0000]^T$

$n = size\ (x,1);$

$normVal = Inf;$

$\%\%$

$\%*Tolerence\ formmethod*$

$tol = 1e-5; itr = 0;$

$\%\%$

$while\ normVal tol$

$xold = x;$

$for\ i = 1:n$

$sigma = 0$

$for\ j = 1:n$

$if\ \ j = i$

$sigma = sigma + A(i, j) \bullet x(j)$

$x(i) = (\ ^{1}\!/_{A(i,\ j)} \bullet (b(i) - sigma):$

end

$itr = itr + 1$

$normVAl = abs(xold - x)$

end

$\%\%$

$fprinf(Solutionof\ the system is: \% f \% f \% f \% f diterations, x, itr):$

Qaussi Seidel Method

The Qaussi-Seidel Method is similar to the Jacobi Method except that it uses updated values as soon as they are available. In general, if the Jacobi Method converges but the Qaussi-Seidel Method will converge faster.

Consider $Ax = b$ and decompose A into a lower L, and upper triangular component U. Moreover, if D is the diagonal component of A and L is strictly lower component of A, then

$A = L_* + U \quad where \quad L_* = L + D$

Therefore, the given system of linear equation can be written as

$(L_* + U)X = b$ using this , we have

$L_* + U = b - UX$

The Qausi-Seidel Method is an iteration technique that solves the left hand side of the equation for X and using the previous values of X on the right hand side , we have

$X^{-k} = L_*^{-1}(b - UX^{k-1}), \quad k = 1, 2, \ldots.$ Provided that L_*^{-1} exists. Substituting for L,

$X^k = (D + L)^{-1}(b) - (D + L)^{-1}UX^{(k-1)}, \quad k = 1, 2, \ldots$

$T = -((D + L)^{-1})U \quad and \quad C = (D + L)^{-1}b$

Therefore, $X^k = TX^{k-1} + C, \quad K = 1, 2, \ldots$

However, by taking advantage of the triangular forms of D, L and U and as in Jacobi , the elements of X_*^k can be computed sequentially by forward substitution to get

$$X_*^{\ k} = \frac{1}{a_{ii}}(b_i - \sum_{j>i} a_{ij} x_j^{\ (k-1)} - \sum_{j<i} a_{ij} x_j^{\ (k-1)}),\ i = 1,2,3,...n \quad and \quad k = 1,2,3,$$

The computation for each element cannot be done parallel , so the values at iteration depends on the order of the equation.

Gaussi-Seidel Method Code

```
%% Gaussi-Seidel Method
%% Solution of x in AX=b
% Initialize 'A' 'B' 'x'
%%
A= [5-230; -391-2; 2-1-71; 43-57]
B= [-1230,5]
X= [0000]
N= size (x, 1);
normVal=Inf;
%%
% *Tolerence formethod*
Tol=le-5; itr=0;
%% algorithm for gausii-Seidel Method
%%
While normVal; tol
x_0ld = x;
For i=1 : n
Sigma=0
For j=1 : i-1
sigma = sigma + A(i, j)* x( j) :
end
for j = i +1:n
sigma = sigma+ A(i, j)* x_0ld( j) :
end
x(i) = ( 1/A(i, j) )* (b(i) – sigma);
end
itr = itr +1
normVal = norm(x_0ld – x) :
end
%%
fprint f ('Sulotionof the system :% f % f % f % f in %diterations', x, itr);
```

The conjugate Gradient Method

This is used to solve also system of equations Ax=b for the vector x where the known n by n matrix A is symmetric $(A^T = A)$ and positive definite $(xTAx > 0 \quad for\ all\ non-zero\ vector\ x \in R^n)$.

The two non-zero vectors u and v are conjugate with respect to A if $U^T AV = 0$. Since A is symmetric and positive , the left hand side defines an inner product

$$\langle U,V \rangle_A = \langle AU,V \rangle = \langle U,A^T V \rangle = \langle U,AV \rangle = U^T AV$$

Suppose that (P_k) is a sequence of n mutually conjugate directions. Then the (P_k) form a basis of R^n so that Ax=b is in the basis

$$X_n = \sum_{i=1}^{n} a_i P_i \qquad and \qquad b = AX_* = \sum_{i=1}^{n} a_i AP_i$$

The coefficients are given by $P_k{}^T b = P_k{}^T AX_* = \sum_{i=1}^{n} a_i P_k{}^T AP_i = a_k P_k{}^T AP_k$ since $\forall\ i \neq k,\ P_i, P_k$ are mutually conjugate.

$$a_k = \frac{P_k{}^T b}{P_k{}^T AP_k} = \frac{\langle P_k, b\rangle}{\langle P_k, P_k\rangle} = \frac{\langle P_k, b\rangle}{|P_k|^2 A}$$

Conjugate Gradient Method Code

```
function x = conjgrad(A, b, tol):

% CONJGRAD(A,b) attempts to solve the system of linear equations A*
X = B
% for
X.The N]by]N cofficient matrix A must have lenth N
%
% X = CONJGRAD( A, B < TOL) specifies the tolerence of the method
% defualtis 1e]10
%
  %
n = 6000;
m = 8000;
A = radn(n, m);
A = A* A;
b = randn(n,1)
tic.x = conjugrad(A,b);toe
norm(A* x]b)%
%
tol = 1e]10
end
x = b;
r = b]A* x;
if norm(r);tol
return
end
y =]r;
z = A* y;
s = y * z;
t = (r * y) / s
x = x + t * z;
for k = 1:numel(b);
r = r]t * z
if (norm(r);tol)
return
end
B = (r * z) / s :
y =]r + B * y;
z = A* y;
s = y * z
t = (r * y)/s;
x = x + t * y;
end
```

end

end

Results and Analysis
The results are obtained from the three methods with different iterations.

Method	x_1	x_2	x_3	x_4	x_5	No of iteration
Jacobi	7.8957	0.5406	0.4229	0.0736	0.0106	91
Gaussi-Seidel	7.8945	0.5316	0.4239	0.0745	0.0107	31
CGM	7.8975	0.54326	0.4249	0.0737	0.07105	5

From the table it is clear that the conjugate Gradient Method perform better than the two others and it the fastest in which the systems converge also faster. It consumes minimal power and reliable.

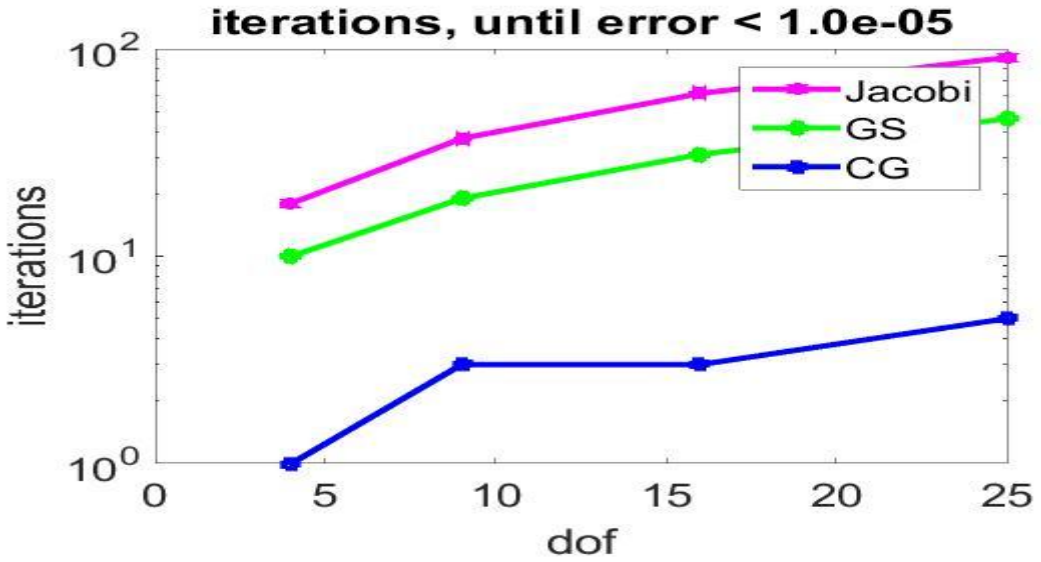

Fig 1

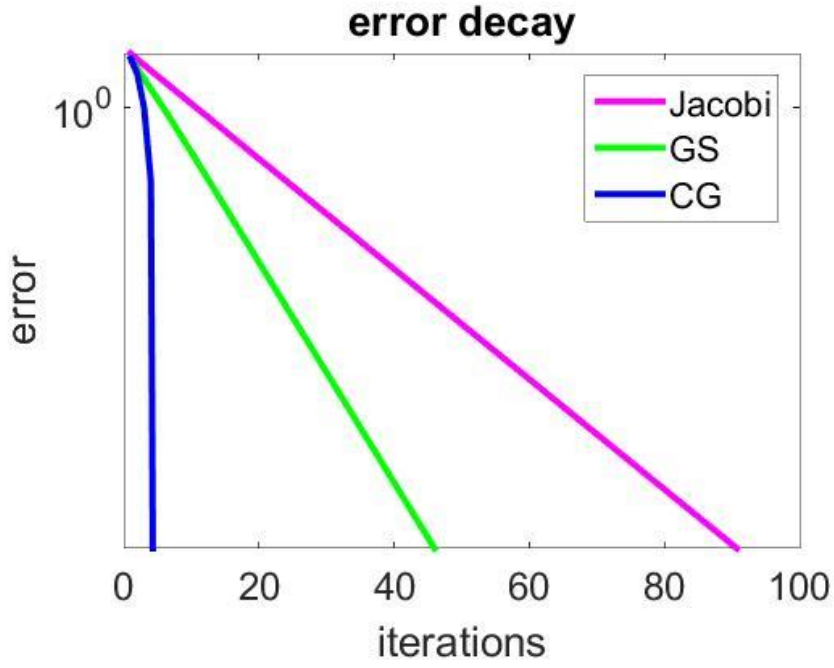

Fig 2

Analysis of The Results

In Fig 1 above it is shown that the Conjugate Gradient Method converges faster than the other two methods and is more accurate. It also takes few numbers of iterations before reaching the final solution.

In Fig 2 above it is shown that the errors in Conjugate Gradient Method is fewer than the other two methods and also takes few numbers of iterations to get to its final solution.

IV. Conclusion

The three methods were studied and analysed that Conjugate Gradient Method converges faster and uses few number of iterations to reach it final solutions. This also shows that the errors encounter during computation is lower with Conjugate Gradient Method than the other two methods. Hence, the Conjugate Gradient Method is more reliable and accurate than others. So this method is considered the best among them.

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