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Mathematical study of two phase pulmonary blood flow in arterioles during tuberculosis

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Abstract: *The aim of this paper is to study of a mathematical model for two phase pulmonary blood flow in arterioles. Here blood has been represented by non-Newtonian fluid obeying Herschel Bulkley fluid. We have collected clinical data of the patient in case of TB. The problem is solved by using numerical techniques with help of boundary conditions and results are displayed graphically for hematocrit and pressure drop. The graphical presentation for particular value is much closer to clinical observation.*

Keyword: *Non Newtonian fluid, pressure drop, Clinical data.*

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I. Introduction

1.1 Blood composition:

Blood consists of plasma with red blood cells, white blood cells, and platelets in suspension. The primary function of RBC is to transport oxygen, and carbon dioxide. Plasma is comprised of 93% water and 3% particles.WBC is an important part of the immune system and platelets are a vital component of blood clotting mechanism. RBC comprises more than 99% of all blood cells [1].

1.2 Description of arterioles:

An arteriole is a small diameter blood vessel in the microcirculation that extends and branches out from an artery and leads to capillarity[2].

Arterioles have muscular walls and the primary site of vascular resistance. The greatest change in blood pressure and blood flow occurs at the transition of arterioles to capillaries. The decreased velocity of flow in the capillaries increases the blood pressure, due to Bernoulli's principal. The induced gas and nutrients to move from blood to the cells, due to the lower osmotic pressure outside the capillary. The opposite process occurs when the blood leaves the capillary and enters the veinules where the blood pressure drops due to an increase in flow rate. Arterioles receive autonomic nervous system innervations and respond to various circulatory hormones in order to regulate their diameter. Retinal vessels lack a functional sympathetic innervations[3].

1.3 Pulmonary circulation:

The pulmonary circulation has the function of delivering a layer of blood about .01 mm in thickness to this very large surface. It is spread out over this tremendous area and taken up again in an average period of 6 seconds. During those 6 seconds. The blood remains in the pulmonary capillary from one tenth to one hundredth of a second. It is in the short time that the respiratory exchange takes place [4].

1.4 Structure and function of lungs:

The lungs are the main organ of the lower respiratory tract and responsible for the exchange of oxygen, which is necessary for cellular processes. For carbon dioxide a cellular waste product. Atmospheric
Air which contains mostly nitrogen and oxygen is inhaled through the upper respiratory system into the lungs. The oxygen passes into the circulation and carbon dioxide is exhaled out from the lungs through the mouth and nose along with the nitrogen and other gases the body does not use. This is the process known as breathing. Oxygen is necessary for cellular processes, but the gas exchange that occurs in the lungs is also important for maintaining blood pH. The exchange occurs due to differences in the pressure in the lungs, atmosphere and blood. Exhalation occurs passively, whereas inhalation requires energy. [5]

Gray's anatomy of the human body, 20^{th} edition 1918. [6] .The lungs main function is to help oxygen from the air, we breathe enter the red blood cells in the blood. Then RBC carry oxygen around the body to be used in the cells found in our body. The lungs also help the body to get rid of carbon di oxide, when we breathe out; there are a number of other jobs carried out by the lungs that include-

- Changing the PH of blood by increasing or decreasing the amount of carbon- di- oxide in the body.
- Filtering out small gas bubbles that may occur in the blood stream.
- Converting a chemical in the blood called angiotensin I to angiotensin II. These chemicals are important in the control of bloodpressure.[7]

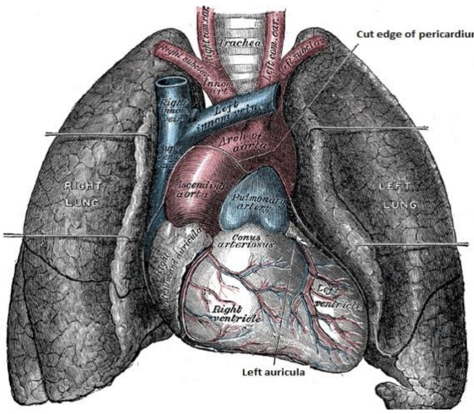


Figure 1

1.5 Lungs disease (T.B.):

Tuberculosis (TB) is an airborne disease caused by mycobacterium tuberculosis that usually affacts the lungs leading to severe coughing, fever and chest pain.[8] it is primarily transmitted by the resipiratory route, individuals with active disease may infect others if the airborne particles they produce when they talk, cough or sing are inhaled by others. More recently, a highly visible world health organization effort to promote a unified global control strategy, tuberculosis remains a leading cause of infectious mortality. During tuberculosis approximately two millions deaths each year.[9]

II. Real Model

2.1 Choice of frame of reference:

We have to select a frame of reference for mathematical modeling of the state of moving blood. Keeping in view the difficulty and generality of the problem of the blood flow. We select generalized three dimensional orthogonal curvilinear coordinate system, briefly prescribed as E3 called as 3-dim Euclidean space. We interpret the quantities related to blood flow in tensorial form which is comparatively more realistic, the biophysical laws thus expressed fully hold good in any coordinate system, which is compulsion for the truthfulness of the law. [10]

Now let the coordinate axes be OX^i, where O is origin. And let X^i be the co-ordinate of any point P in space, where i=1, 2, 3. The mathematical description of the state if moving blood is affected by means of functions, which give the distribution of the blood velocity $v^k = v^k(X^i, t)$ where k=1, 2, 3 and of any two thermodynamic quantities pertaining to the blood. For instance the pressure $p = p(X^i, t)$ and the density$\rho = \rho(X^i, t)$. As is well known, all the thermodynamic quantities are determined by the values any two of them. Together with the equate of state. Hence if we are given five quantities namely the three components of velocityv^k, the pressure p and densityρ. The state of moving blood is completely determined. All these quantities are function of the co-ordinates X^i, i=1, 2, 3 and of time t. we emphasize that $v^k(X^i, t)$ is the velocity of the blood at a given point X^i in space and at a given t. i.e. it refers to fixed point in space and not to fixed particles of the blood. [11]

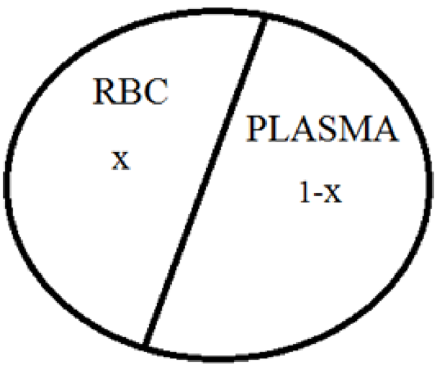


Figure 2

Blood is a mixed fluid; mainly there are two phases in blood. The first phase is red blood cells and other is plasma. The blood cells are enclosed with a semi permeable membrane whose density is greater than that of plasma these blood cells are uniformly distributed in plasma. Thus blood can be considered as a homogenous mixture of two phases [12].

III. Basic Bio-Fluid Equation For Two Phase Blood Flow

3.1 Equation of continuity for two phase blood flow:

The flow of blood is affected by presence of blood cells. This effect is directly proportional to the volume occupied by blood cells [13]. Let the volume portion covered by blood cells in unit volume be X, this X is replaced by H/100, where H is hematocrit the volume percentage of blood cells. Then the volume portion covered by the plasma will be $1-\mathrm{X}$, then the mass ratio of blood cells to plasma is:

$$r = \frac{X\rho_c}{(1-X)\rho_p} \quad \text{--} \quad (1)$$

Where ρ_c and ρ_p are densities of blood cells and blood plasma respectively. Usually this mass ratio is not a constant; even then this may be supposed to constant in present context (1986).The both phase of blood, i.e. blood cells and plasma move with the common velocity. Hence equation of continuity for two phases according to the principle of conservation of mass [13].

$$\frac{\partial(X\rho_c)}{\partial t} + (X\rho_c v^i)_{,i} = 0 \quad \text{--} \quad (2)$$

And $$\frac{\partial(1-X)\ \rho_p}{\partial z} + \{(1-X)\ \rho_p\ v^i\}_{,i} = 0 \quad \text{------------------------------} \quad (3)$$

Where v is common velocity of two phase blood cells and plasma.
If we define the uniform density of the blood ρ_m as follow:

$$\frac{1+r}{\rho_m} = \frac{r}{\rho_c} + \frac{1}{\rho_p} \quad \text{-------------------------------} \quad (4)$$

Then equation (2) and (3) implies that

$$\frac{\partial\rho_m}{\partial t} + \left(\rho_m v^i\right)_{,i} = 0 \quad \text{---} \quad (5)$$

3.2 Equation of motion for blood flow:

The hydro dynamical pressure between the two phases of blood can be supposed to be uniform because the both phases (red blood cells and plasma) are always is in equilibrium state in blood.

Taking viscosity coefficient of blood cells to be η_c and applying the principle of conservation of momentum in pulmonary circulatory system. We get the equation of motion for the two phases of blood cells as follows-

$$X\rho_c \frac{\partial v^i}{\partial t} + (X\rho_c v^j)v^i_{,j} = -XPg^{ij} + X\eta_c\left(g^{jk}v^i_{,k}\right)_{,j} \quad \text{--} \quad (6)$$

Similarly taking the viscosity coefficient of plasma, equation of motion for plasma will be as follows:

$$(1-X)\rho_p \frac{\partial v^i}{\partial t} + \{(1-X)\rho_p v^i\}v^i_{,j} = -(1-X)P_j g^{ij} + (-X)\eta_c\left(g^{jk}v^i_{,k}\right)_{,j} \quad \text{--------------------------------} \quad (7)$$

From equation (4), (6) and (7), then the equation of motion for two phase blood flow is:

$$\rho_m \frac{\partial v^i}{\partial t} + (\rho_m v^j)v^i_{,j} = -Pg^{ij} + \eta_m\left(g^{jk}v^i_{,k}\right)_{,j} \quad \text{----------------------------------} \quad (8)$$

Where $\eta_m = X\eta_c + (1-X)\ \eta_p$ is the viscosity coefficient of blood as mixture of two phases.

3.3 Choice of parameters:

blood is non Newtonian fluids, then using the constitutive equation for fluids

$\tau = \eta e^n$n

If n=1 then he nature of fluid is Newtonian and if n≠1 then the nature of fluid is non Newtonian. where τ is denoted by stress , e is denoted by strain rate this constitutive equation is called Herschel Bulkley non Newtonian law and n is denoted by the parameter, these equation uses equation of motion. In present study there are five parameters are used but three parameters are frequently used namely velocity, pressure P and density ρ. [14]

3.4 Choice of constitutive equation:
We have using in two phase blood flow through arterioles and whose constitutive equation is as follows-
$T^{'} = \eta_m e^n + T_p \quad (T^{'} \geq T_p)$
Where T_p is the yield stress.
When strain rate $e = 0 \left(T^{'} < T_p\right)$, a core region is formed which flows just like a plug.
We get,
$p\pi r_p^2 = T_p 2\pi r_p$
$r_p = 2\frac{T_p}{p}$ -- (9)

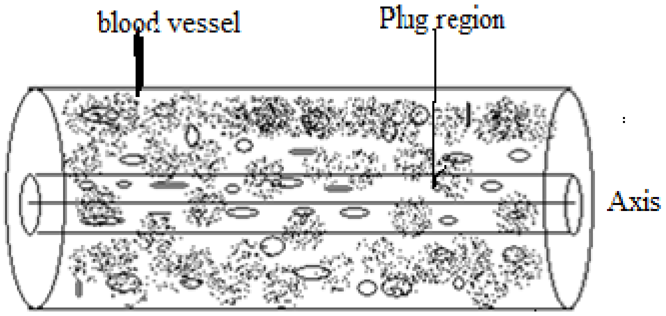


Figure 3. Herschel- Bulkley blood flow

The constitutive equation for test part of the blood vessels is
$T^{'} = \eta_{me^n} + T_p$
$T^{'} - T_p = \eta_m e^n = T_e$
Here T_e = effective stress
Whose generalized form will be as follows:-
$T^{ij} = -pg^{ij} + T_e^{ij}$
Where $T_e^{ij} = \eta_m (e^{ij})^n$ while $e^{ij} = g^{jk} v_k^i$
Where, the symbols have their usual meanings.
Now we describe the basic equation for Herschel Bulkley blood flow as follows:

Equation of continuity:

$$\frac{1}{\sqrt{g\sqrt{(gv^i),i}}} = 0$$

Equation of motion:
$\rho_m \frac{dv^i}{dt} + \rho_m v^j v^i = -T^{ij} e_j$ --- (10)

Since the blood vessels are cylindrical; the above governing equations have to be transformed into cylindrical co-ordinates.

$$x^1 = r, x^2 = \theta, x^3 = z$$

Matrix of metric tensor in cylindrical co-ordinates is $\left[g_{ij}\right]$ and matrix of conjugate metric tensor is $[g^{ij}]$ whereas the chritoffel's symbols of second kind are as follows:
$\begin{Bmatrix} 1 \\ 2 \quad 2 \end{Bmatrix} = -r, \begin{Bmatrix} 1 \\ 2 \quad 2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \quad 2 \end{Bmatrix} = \frac{1}{r}$ Remaining others is zero.

The governing tensorial equations can be transformed into cylindrical forms which are follows
The equation of continuity:-
$\frac{\partial v}{\partial z} = 0$
The equation of motion:-

R-component: $-\frac{\partial P}{\partial z} = 0$

Z-component: $0 = -\frac{\partial P}{\partial z} + \frac{\eta_m}{r}\left[r\left(\frac{\partial v_z}{\partial r}\right)^n\right]$ -- (11)

Here, this fact has been taken in view that the blood flow is axially symmetric in arterioles concerned, i.e. $v_\theta = 0$ and v_r ,v_z and p do not depend upon θ.

We get $v_z = v(r)$ and $P = P(z)$ and $0 = -\frac{dP}{dz} + \frac{\eta_m}{r}\left[r\left(\frac{dv}{dr}\right)^n\right]$

Since pressure gradient: - $-\frac{dp}{dz} = P$

$$r\left(\frac{dv}{dz}\right)^n = -\frac{Pr^2}{2\eta_m} + A$$

We apply boundary conditions at $r = 0, v = v_0 \ then\ A = 0$.

$$-\frac{dv}{dr} = \left(\frac{Pr}{2\eta_m}\right)^{\frac{1}{n}}$$

$$\frac{dv}{dr} = -\left(\frac{\frac{1}{2}Pr - \frac{1}{2}Pr_p}{\eta_m}\right)^{\frac{1}{n}}$$

$$\frac{dv}{dr} = -\left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}}\left(r - r_p\right)^{\frac{1}{n}} \quad \text{---} \quad (12)$$

Integrating above equation under the no slip boundary condition $v = 0$,at $r = R$ so as we get

$$V = \frac{n}{n+1}\left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}}\left[\left(R - r_p\right)^{\frac{1}{n}+1} - \left(r - r_p\right)^{\frac{1}{n}+1}\right] \quad \text{---} \quad (13)$$

Above formula for velocity of blood flow in arterioles veinules and veins.

Putting $r = r_p$ to get the velocity v_p of plug flow as follows:

$$v_p = \frac{n}{n+1}\left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}}\left(R - r_p\right)^{\frac{1}{n}+1} \quad \text{---} \quad (14)$$

IV. Analysis (Solution)

Observations: - hematocrit vs. blood pressure by dr.

Patient name: - Radha Tiwari

Diagnosis: - Tuberculosis

Date	BP(mmhg)	BP (Pascal)	HB	Hematocrit
28/9/15	100/60	13332.2/7999.32	13.2	39.6
29/9/15	90/60	11998.98/7999.32	13.5	40.5
30/09/15	104/66	13865.28/8799.12	11.4	34.2
1/10/2015	112/72	14931.84/9599.04	11.7	35.1
2/10/2015	125/90	16665.0/11998.8	10.9	32.7

Table 1

The flow flux of two phased blood flow in arterioles:-

$$Q = \int_0^{r_p} 2\pi\, rV_p\, dr + \int_{r_p}^{R} 2\pi rV dr$$

$$= \int_0^{r_p} 2\pi r \frac{n}{n+1}\left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}}\left(R - r_p\right)^{\frac{1}{n}+1} dr + \int_0^{r_p} 2\pi r\left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}}\left[\left(R - r_p\right)^{\frac{1}{n}+1} - \left(r - r_p\right)^{\frac{1}{n}+1}\right] dr$$

$$= \frac{2\pi n}{n+1}\left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}}\left(R - r_p\right)^{\frac{1}{n}+1}\left[\frac{r^2}{2}\right]_0^{r_p} + \frac{2\pi n}{(n+1)}\left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}}\left[\frac{r^2}{2}\left(R - r_p\right)^{\frac{1}{n}+1} - \frac{r\left(r - r_p\right)^{\frac{1}{n}}}{\frac{1}{n}+2} + \frac{\left(r - r_p\right)^{\frac{1}{n}+3}}{\left(\frac{1}{n}+2\right)\left(\frac{1}{n}+3\right)}\right]_{r_p}^{R}$$

$$Q = \frac{\pi n}{n+1}\left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}} R^{\frac{1}{n}+3}\left[\frac{r_p^2}{R^2}\left(1 - \frac{r_p^2}{R}\right)^{\frac{1}{n}+1} + \left(1 + \frac{r_p}{R}\right)\left(1 - \frac{r_p}{R}\right)^{\frac{1}{n}+2} - \frac{2\left(1 - \frac{r_p}{R}\right)^{\frac{1}{n}+2}}{\left(\frac{1}{n}+2\right)} + \frac{2\left(1 - \frac{r_p}{R}\right)^{\left(\frac{1}{n}+3\right)}}{\left(\frac{1}{n}+2\right)\left(\frac{1}{n}+3\right)}\right]$$

Here $Q = 425\ ml/min$ = 0.0071 l/sec

R = 1, r_p = 1/3 [5]

η_m = 0.027 $pascal\ sec$
η_p = 0.0013 $pascal\ sec$ [5]

Average systolic pressure = 14158.66 Pascal
Average diastolic pressure = 9279.12 Pascal
P_i = pressure on arterioles = $\frac{D+S}{2}$ = 11718.89 Pascal
P_f = pressure on veins = $\frac{\frac{D+S}{2}+D}{3}$ = 6999.337 Pascal

Length of arterioles $(z_f - z_i)$ = 0.00095 meter [15]
Now

$$\eta_m = \eta_c X + \eta_p(1 - X)$$

When $X = \frac{H}{100} = \frac{34.2}{100} = 0.342$
0.027 = η_c(0.342) + 0.0013(1-0.342)
η_c = 0.07645
Again,
$\eta_m = \eta_c X + \eta_p(1 - X)$
η_m = 0.0007515H+ 0.0013

Now,
$Q = \pi\left(\frac{2p}{6\eta_m}\right)^{\frac{1}{n}}\frac{2}{27}\left[\frac{26n^3+33n^2+9n}{6n^3+11n^2+6n+1}\right]$

$\frac{27Q}{2\pi} = \left(\frac{P}{3\eta_m}\right)^{\frac{1}{n}}\left(\frac{26n^3+33n^2+9n}{6n^3+11n^2+6n+1}\right)$

Let $A = \frac{26n^3+33n^2+9n}{6n^3+11n^2+6n+1}$

$$\frac{P}{3\eta_m} = \left(\frac{27Q}{2\pi A}\right)^n$$

$$P = \left(\frac{27Q}{2\pi A}\right)^n \times 3\eta_m$$

$$P = -\frac{dP}{dz}$$

Implies that $dp = -Pdz$

$$\int_{p_i}^{p_f} dP = -\int_{z_i}^{z_f}\left(\frac{27Q}{2\pi A}\right)^n \times 3\eta_m \,\mathrm{dZ}$$

Where
Pressure drop = P_i-P_f
Length of arterioles = $z_f - z_i$

$$P_f - P_i = \left(\frac{27Q}{2\pi A}\right)^n \times 3\eta_m(z_f - z_i)$$

By substituting all values, we get n = -3.71911
Again,

$\frac{27Q}{2\pi} = \left(\frac{\Delta p}{\Delta z}\frac{1}{3\eta_m}\right)^{\frac{1}{n}}\frac{26n^3+33n^2+9n}{6n^3+11n^2+6n+1}$

$$\left[\frac{27Q}{2\pi} \times \frac{6n^3 + 11n^2 + 6n + 1}{26n^3 + 33n^2 + 9n}\right]^n = \frac{\Delta p}{\Delta z}\frac{1}{3\eta_m}$$

$$\Delta P = \Delta z.\,3\eta_m\left[\frac{27Q}{2\pi} \times \frac{6n^3 + 11n^2 + 6n + 1}{26n^3 + 33n^2 + 9n}\right]^n$$

Substituting the value of η_m, Δz, Q and n,
We get

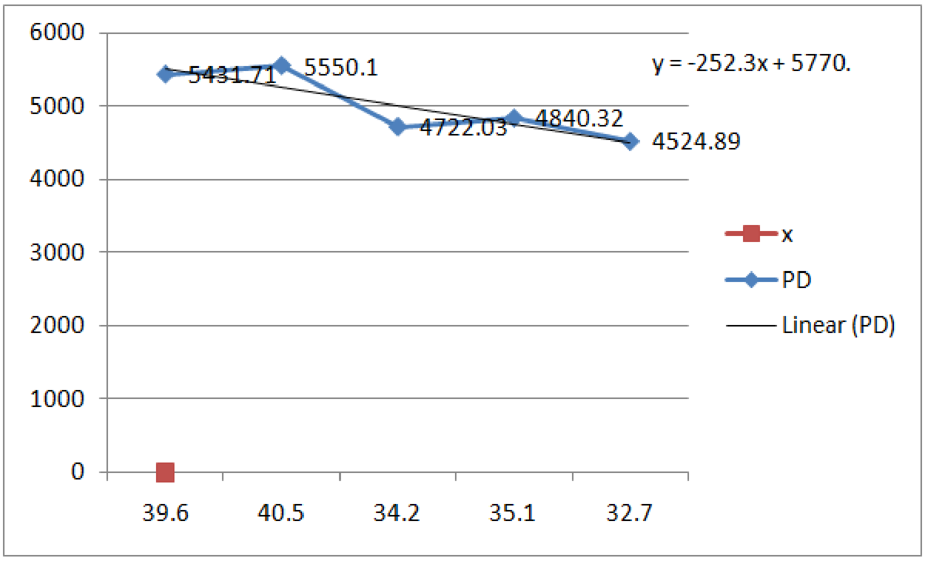
$\Delta P = \Delta z.3\eta_m\left(\frac{27Q}{2\pi} \times 0.2638245368\right)^{-3.71911}$
$\Delta P = 3\eta_m \times 58293.89234$
$\Delta P = 3(0.0007515H + 0.0013) \times 58293.89234$
$\Delta p = 131.4235803H + 227.3461801$

V. Result and Discussion

We get values of pressure drop if hematocrit known, by using above equation.

Hematocrit	39.6	40.5	34.2	35.1	32.7
Pressure drop	5431.71996	5550.001182	4722.032626	484 0.313848	4524.897256

Table 2.



Graph 1

VI. Conclusion

In bio physical interpretation we have taken the clinical data regarding with blood pressure and hematocrit of TB patient and there is formed a linear relation. Y = -252.3x + 5770

By using two phase non Newtonian model Herschel Bulkley and draw the graph pressure drop and hematocrit and trend of the graph shows the relation just similar to two phase relation.

$\eta_m = \eta_c X + \eta_p (1 - X)$ Where $X = \frac{H}{100}$.

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References

[1]. J.T.Ottsen and et al (2006): Applied Mathematical Models in Human Physiology, Biomath-Group, Department of Mathematics and Physics, Roskilde University, Denmark , PP. 26.
[2]. Maton, Anthea, J Hopkins and et al (1993): Human Biology and Health. Englewood cliffs, New Jersy: prentice Hall, ISBN: 0-13-981176-1.
[3]. Riva, CE Grunewald, JE and et al. (1986): Auto regulation of human retinal blood flow of an investigation with laser Doppler velocimetry , Invest Opthalmol Vis. Science 27, PP. 1706-1712.
[4]. W.F. Hamilton: The physiology of the pulmonary circulation, the journal of Allergy, PP.397-410.
[5]. Neha Trivedi and et al.(2013): Mathematical modeling in two phases pulmonary blood flow through arterioles in lungs with special reference chronic obstructive pulmonary disease(COPD), Asian Journal of Science and Technology, Vol- 4, Issue. 5, PP. 27-31.
[6]. Gray's anatomy of the human body, 20th edition 1918.
[7]. Weinman J. and et al. (2009): How accurate is patient's anatomical knowledge: A cross sectional, questionnaire on six patient groups and a general public sample, BMC fam pract.
[8]. Nicole Fogel (2015): Tuberculosis- A disease without boundaries, Available from http://intl.elsevierhealth.com/journals/tube, PP. 527-531.
[9]. Caroline Colijn, Ted Cohen, Megan Murray (2006): Mathematical Models of Tuberculosis: Accomplishment and Future Challenges, PP.1-26.
[10]. U.C. De, Absos Ali Sheikh and Joydeep Sengupta (2008): Tensor Calculus (II edition), Alpha Science International Ltd, Oxford, U.K.
[11]. Om Prakash and et.al (2015): A Non- Newtonian mathematical model for two phase hepatic mean blood flow using Herschel Bulkley law in hepatic arterioles with special reference to hepatitis B, International Journal of Innovative Research in Science Engineering and Technology, Vol. 4 Issue 8 PP.7476-7486.
[12]. Neelam Bajpai and et al (2014): A Non- Newtonian mathematical model on the two phase renal mean blood flow in renal arterioles with special reference to kidney infection, International Journal of Engineering and Innovative Technology, Vol. 4, Issue 5, PP.104-111.
[13]. J.N. Kapur and R.C.Gupta (1963): Power law fluid flow in the inlet length of a circular pipe; The Mathematics Seminar III, PP. 55-67.

[14]. Harish Chandra and et al (2013): A mathematical model on two phase renal diastolic blood flow in arterioles with special reference to diabetes, , International Journal of Scientific and Engineering Research, Vol. 4, Issue 1, PP 01-09.
[15]. Mary P. Wiedeman, (1963): Dimension of blood vessels from distributing artry to collecting vein, Circulation Research, Journal of the American Heart Association, Vol. 12, PP.375-378 Online ISSN: 1524-4571.

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