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Comparison between the Rayleigh Ritz Method (RR) and Other Numerical Methods for Solving Second Order Boundary Value Problems

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Abstract: *This paper introduces the Rayleigh-Ritz method (RR) with different basis function and comparing this method with other numerical methods for solving second order boundary value problems to describe how this method is achieving the high accuracy, using linear basis function, quadratic, cubic hermit, cubic b-spline and polynomial functions and different step size h to show how the choice of the trail function effect on reducing the errors and this is illustrated by using the cubic b-spline which is the best for RR . The models described in this paper were implemented through a prototype software developed by the authors in a Mathematica environment.*

Keywords: *Rayleigh-Ritz, quadratic interpolation, Cubic Hermite, Cubic b-spline, Finite Element Method, boundary value problems, Finite Difference Method, Least Squares Method, Collocation Method, and Galerkin Method.*

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I. Introduction

Finite Element Method (FEM) is the most powerful technique for numerical treatment, widely used in engineering and applied science such as (structural analysis, structural mechanics and fluid mechanics). FEM is based on Variational method combined with analytical function. The finite-difference approach replaces the continuous operation of differentiation with the discrete operation of finite differences. The Rayleigh-Ritz method is a Variational technique. The boundary-value problem is first reformulated as a problem of choosing, from the set of all sufficiently differentiable functions satisfying the boundary conditions, the function to minimize a certain integral [1], [20].

The finite-difference method for boundary value problems is more flexible in generalization the boundary value problems in higher space dimensions, it is best suited for problems in which the domain is relatively simple, such as a rectangular domain. We now consider an alternative approach that, in higher dimensions which is more easily applied for problems geometrical complicated domains. This method is known as the Rayleigh-Ritz Method [11].

In (1908), Ritz laid out his famous method for determining frequencies and mode shapes, choosing multiple admissible displacement functions, and minimizing a functional involving both potential and kinetic energies, then he demonstrated it in detail in 1909 for the completely free square plate.

Here at (1911), Rayleigh wrote a paper congratulating Ritz on his work, but stating that he had used Ritz's method in many places in his book and in another publication. Subsequently, hundreds of research articles and many books have appeared which use the method, some calling it the “Ritz method” and others the “Rayleigh–Ritz method”, although Rayleigh solved a few problems which involved minimization of a frequency, these solutions were not used for the straightforward, direct method presented by Ritz but also used by others, the method is presented in Burden, Richard L .and Douglas Fairs book's called Numerical Analysis [1], Chad Magers approach least square method to this method [4].

After that, Luay S. Al-Ansari, Calculating Static Deflection And Natural Frequency of Stepped Cantilever Beam Using Modified Rayleigh Method, [15], Surashmi Bhattacharyya and Arun Kumar solved three parameters eigenvalue problems [20], then Ch.Zhang and others resolvant sampling Rayleigh-Ritz method for large- scale nonlinear eigenvalue problems and by rational interpolation approach and resolvant sampling based [3], Nabanita Datta based the approach of characterizing the vertical vibration of non-uniform hull girder [17], Nicolae Danet introduced a paper "solving two boundary value problem with Mathcad" [18], Lun Liu and others with him made studies on global analytical mode for a three-axis attitude stabilized spacecraft [14], the method still competitive so, Gang Bi wrote paper with the name "Generalized Stress Field In Granular Soils

Heap With Rayleigh Ritz Method" [10] and Giorgio Gnecco, On the Curse of Dimensionality in the Ritz Method, [11], D. Gallistl studied the stability for the Rayleigh-Ritz method for eigenvalue [5], Ivo Senjanović, Neven Alujević, Ivan Ćatipović, Damjan Čakmak, Nikola Vladimir, Vibration analysis of rotating toroidal shell by the Rayleigh-Ritz method and Fourier series [12] and Yajuvindra Kumar, A Rayleigh–Ritz Method For Navier–Stokes Flow Through Curved Ducts, [21].

II. The Mathematical Formulation

This method can be applied to a Euler Bernoulli beam with arbitrarily varying mass and stiffness distributions and it has been effective in computing the eigenvalues of self-adjoint problems, in this section, The Rayleigh Ritz method process is presented as follow:
Using the linear boundary value problem: [1], [13]

$$-\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\}+q(x)y(x)=f(x) \quad ,0<x<1 \tag{1}$$

With boundary conditions y (0) = y (1) = 0. Multiplying by u(x) "test function" and then integrating over the domain [0,1]. Then, minimizing I[u], where y(x)=u(x)

$$I[u]=\int_0^1 \mathrm{p(x)}[u'(x)]^2+q(x)[u(x)]^2\,\mathrm{dx}-2\int_0^1 u(x)\,\mathrm{f(x)}dx=0 \tag{2}$$

To find an approximation of I[u], restricted to a subspace of c_0^2 [0,1] by $y(x)=u(x)=\sum_{i=0}^{n} c_i\phi_i(x)$ With B.C $\phi_i(0)=\phi_i(1)=0$ to achieve minimization.

$$\frac{\partial I}{\partial c_i}=\sum_{i=0}^{n}\int_0^1[\,p(x)\,\mathrm{c}_i\,\phi_i'(x)\phi_j'(x)+q(x)c_i\phi_i\phi_j\,]dx-\int_0^1 f(x)\phi_i(x)\,\mathrm{dx}=0 \tag{3}$$

Then

$$\sum_{i=0}^{n}\int_0^1 \mathrm{c}_i[\,p(x)\phi_i'(x)\phi_j'(x)+q(x)\phi_i\phi_j\,]dx=\int_0^1 f(x)\phi_i(x)\,\mathrm{dx} \tag{4}$$

This system can be written in the matrix – vector from Ac = b; Where c is a vector of the unknown coefficients $c_1, c_2, \ldots, c_n$, A=(a_{ij}) and b=(b_i)
So:

$$a_{ij}=\int_0^1[\,p(x)\phi_i'(x)\phi_j'(x)+q(x)\phi_i\phi_j\,]dx \tag{5}$$

$$b_i=\int_0^1 f(x)\phi_i(x)dx \tag{6}$$

The last step is to use trial function $\phi_1,\phi_2,\ldots\ldots,\phi_n$, divide the interval [0,1] where x_o=0, x_{n+1}= 1, subinterval[x_{i-1},x_i] , with step size h.

III. Different Basis Functions

In this section, different basis functions were introduced to describe the effect of using it.

1. Piecewise Linear Function

Using equation (7) as a basis function

$$\phi_i(x)=\begin{cases}0 & 0\le x\le x_{i-1}\\ \frac{1}{h}(x-x_{i-1}) & x_{i-1}\le x\le x_i\\ \frac{1}{h}(x_{i+1}-x) & x_i\le x\le x_{i+1}\\ 0 & x_{i+1}\le x\le 1\end{cases} \tag{7}$$

$\phi_i(x)$ must satisfy the boundary condition.

$$\phi_i'(x) = \begin{cases} 0 & 0 \le x \le x_{i-1} \\ \dfrac{1}{h} & x_{i-1} \le x \le x_i \\ \dfrac{-1}{h} & x_i \le x \le x_{i+1} \\ 0 & x_{i+1} \le x \le 1 \end{cases} \tag{8}$$

$\phi_i(x)\ \phi_j(x) = 0$ and $\phi_i'(x)\ \phi_j'(x) = 0$.

2. Piecewise Quadratic Function

Using the Concept of the Lagrange interpolation to construct quadratic function to implement Rayleigh Ritz method:

$$\phi_i(x) = \begin{cases} 0 & 0 \le x \le x_{i-1} \\ \dfrac{(x - x_{i-1})(x_{i+1} - x)}{h^2} & x_{i-1} \le x \le x_i \\ \dfrac{(x_{i+1} - x)^2}{h^2} & x_i \le x \le x_{i+1} \\ 0 & x_{i+1} \le x \le 1 \end{cases} \tag{9}$$

$$\phi_i'(x) = \begin{cases} 0 & 0 \le x \le x_{i-1} \\ \dfrac{(x_{i+1} - x) - (x - x_{i-1})}{h^2} & x_{i-1} \le x \le x_i \\ \dfrac{-2(x_{i+1} - x)}{h^2} & x_i \le x \le x_{i+1} \\ 0 & x_{i+1} \le x \le 1 \end{cases} \tag{10}$$

3. Piecewise Cubic Hermite Function

Using the cubic hermite function to implement RR ([2], [16], [19])

$$\phi_i(x) = \begin{cases} 0 & 0 \le x \le x_{i-1} \\ \dfrac{-2(x - x_{i-1})^3}{h^3} + \dfrac{3(x - x_{i-1})^2}{h^2} & x_{i-1} \le x \le x_i \\ \dfrac{2(x - x_i)^3}{h^3} - \dfrac{3(x - x_i)^2}{h^2} + 1 & x_i \le x \le x_{i+1} \\ 0 & x_{i+1} \le x \le 1 \end{cases} \tag{11}$$

$$\phi_i'(x) = \begin{cases} 0 & 0 \le x \le x_{i-1} \\ \dfrac{-6(x - x_{i-1})^2}{h^3} + \dfrac{6(x - x_{i-1})}{h^2} & x_{i-1} \le x \le x_i \\ \dfrac{6(x - x_i)^2}{h^3} - \dfrac{6(x - x_i)}{h^2} & x_i \le x \le x_{i+1} \\ 0 & x_{i+1} \le x \le 1 \end{cases} \tag{12}$$

4. Cubic spline (b-spline)

Using the cubic spline function to implement RR ([2], [6], [8], [9], [16], [19])

$$s(x)=\begin{cases}0 & x\le -2\\ \frac{1}{4}(2+x)^3 & -2\le x\le -1\\ \frac{1}{4}[(2+x)^3-4(1+x)^3] & -1\le x\le 0\\ \frac{1}{4}[(2-x)^3-4(1-x)^3] & 0\le x\le 1\\ \frac{1}{4}(2-x)^3 & 1\le x\le 2\\ 0 & 2<x\end{cases} \tag{13}$$

To construct the basis function ϕ_i in $c^2{}_0[0,1]$, first partition [0,1] by choosing positive integer n

$$\phi_i(x)=\begin{cases}s(\frac{x}{h})-4s(\frac{x+h}{h}) & for\ i=0\\ s(\frac{x-h}{h})-s(\frac{x+h}{h}) & for\ i=1\\ s(\frac{x-ih}{h}) & for\ 2\le i\le n-1\\ s(\frac{x-nh}{h})-s(\frac{x-(n+2)h}{h}) & for\ i=n\\ s(\frac{x-(n+1)h}{h})-4s(\frac{x-(n+2)h}{h}) & for\ i=n+1\end{cases} \tag{14}$$

IV. Polynomial Functions

These polynomials are defied on the interval $0<x<1$by the formula:

4.1 Second Degree Polynomial

Considering the form:

$$u(x)=cx(1-x) \tag{15}$$

Implement RR method

$$I=\int_0^1[p(x)[u'(x)]^2+q(x)[u(x)]^2]dx-2\int_0^1 f(x)u(x)\,\mathrm{dx}=0 \tag{16}$$

Achieve minimization

$$\frac{dI}{dc}=0 \tag{17}$$

4.2 Third degree Polynomial

Considering the form:

$$u(x)=-(c_2+c_3)x^3+c_2x^2+c_3x \tag{18}$$

Implement RR method as equation (27), then achieve minimization

$$\frac{dI}{dc_2}=0,\frac{dI}{dc_3}=0 \tag{19}$$

4.3 Fourth degree Polynomial

Considering the form:

$$u(x)=-(c_2+c_3+\mathrm{c}_4)x^3+c_2x^3+c_3x^2+c_4x \tag{20}$$

Implement RR method as equation (27), then achieve minimization

$$\frac{dI}{dc_2}=0,\frac{dI}{dc_3}=0,\frac{dI}{dc_4}=0 \tag{21}$$

V. Applications

$$-\frac{d^2u}{dx^2}+\pi^2u-2\pi^2\sin\pi x=0 \qquad ,0<x<1 \qquad (22)$$

$$u(0)=0, u(1)=0$$

$P(x) = 1$, $q(x) = \pi^2$, $f(x) = 2\pi^2 \sin(\pi x)$, for $h = 0.1$

1. Piecewise Linear Function

Implement RR using the basis function as in equations (7), (8).
Finding a, b as in equations (5) and (6), then solving the linear system to find the coefficients c_i.

$$a=\begin{bmatrix} +20.658 & -9.83551 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9.83551 & +20.658 & -9.83551 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -9.83551 & +20.658 & -9.83551 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -9.83551 & +20.658 & -9.83551 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9.83551 & +20.658 & -9.83551 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -9.83551 & +20.658 & -9.83551 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9.83551 & +20.658 & -9.83551 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -9.83551 & +20.658 & -9.83551 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -9.83551 & +20.658 \end{bmatrix}, \quad b=\begin{bmatrix} 0.604975 \\ 1.15073 \\ 1.58384 \\ 1.86192 \\ 1.95774 \\ 1.86192 \\ 1.58384 \\ 1.15073 \\ 0.604975 \end{bmatrix} \qquad c=(A)^{-1}*b$$

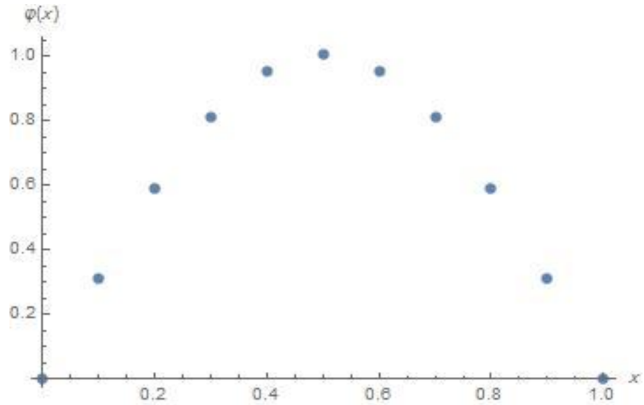

Fig 1. Final shape of the approximation linear function for h = 0.1

2. Piecewise Quadratic Function

Implement RR using the basis function as in equations (9), (10), then the same steps as the above section.

3. Piecewise Cubic Hermite Function

Implement RR using the basis function as in equations (11), (12), then the same steps as the above section.

Table 1. Comparison between the RR with linear basis, quadratic, cubic hermite function and the exact solution

i	xi	Exact	Linear	Error 1	Quadratic	Error 2	Cubic Hermite	Error 3
0	0	0	0	0	0	0	0	0
1	0.1	0.309016	0.310287	0.001271	0.255143	0.053873	0.281955	0.027061
2	0.2	0.587785	0.590200	0.002415	0.490931	0.096854	0.536310	0.051475
3	0.3	0.809016	0.812341	0.003325	0.682549	0.126467	0.738167	0.070849
4	0.4	0.951056	0.954964	0.003908	0.809793	0.141263	0.867767	0.083289
5	0.5	1.000000	1.004110	0.004110	0.858944	0.141056	0.912425	0.087575
6	0.6	0.951056	0.954964	0.003908	0.824016	0.12704	0.867767	0.083289
7	0.7	0.809016	0.812341	0.003325	0.707252	0.101764	0.738167	0.070849
8	0.8	0.587785	0.590200	0.002415	0.518818	0.068967	0.536310	0.051475
9	0.9	0.309016	0.310286	0.001270	0.275714	0.033302	0.281955	0.027061
10	1	0	0	0	0	0	0	0

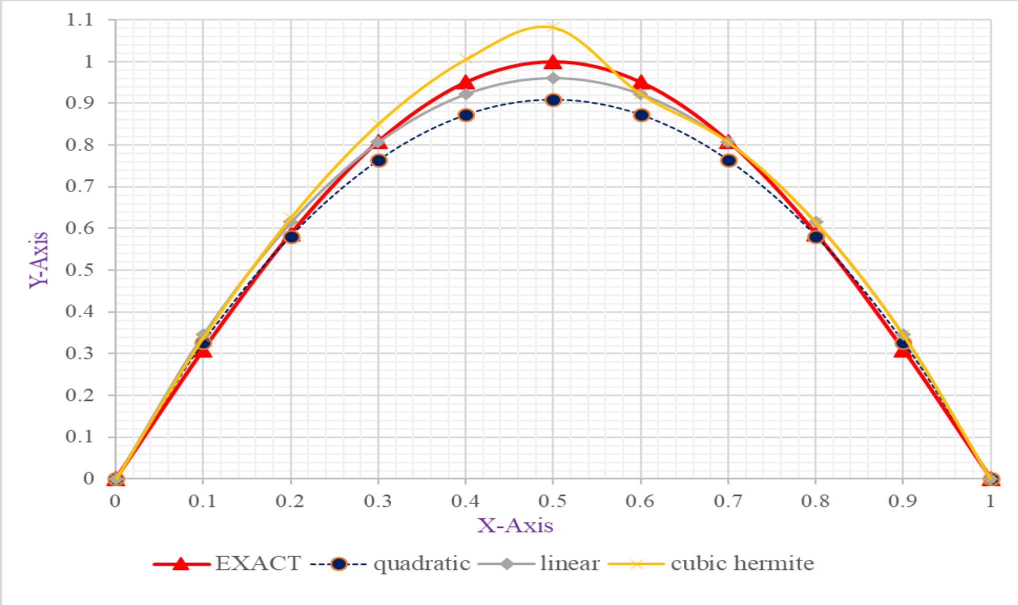

Fig 2. Comparison between the RR with linear basis, quadratic, cubic Hermite function and the exact solution for h= 0.1

4. Cubic Spline (b-Spline)

Implement RR using b spline function as in equations (19) and construct the basis function as equation (20). For h=0.25, n=3, constructing equations (21) to (25), then the same steps as the above section but less iteration than above.

$$a = \begin{vmatrix} 6.54635 & 4.07647 & -1.37224 & -0.73898 \\ 10.32909 & 0.26081 & -1.66782 & -0.73898 \\ 0.26081 & 8.66126 & 0.26081 & -1.37222 \\ -1.66782 & 0.26081 & 10.32909 & 4.07647 \\ -0.73898 & -1.37222 & 4.07647 & 6.54635 \end{vmatrix}$$

$$b = \begin{vmatrix} 1.08035 \\ 4.72025 \\ 6.67544 \\ 4.72025 \\ 1.08035 \end{vmatrix}$$

Table 2. Comparison between the exact and the cubic spline function

i	**Xi = i h**	**c_i**	**U(x)**	**U(exact)**	**E = $\lvert u_{exact} - u_{rr} \rvert$**
0	0	0.00060266	0	0	0
1	0.25	0.52243908	0.70745256	0.70710678	0.00034578
2	0.50	0.7394512	1.0006708	1	0.0006708
3	0.75	0.52243908	0.70745256	0.70710678	0.00034578
4	1	0.00060266	0	0	0

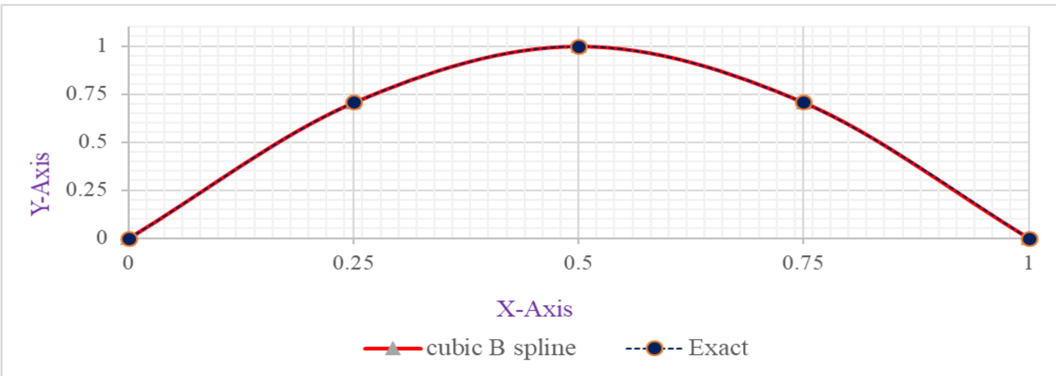

Fig 3. Shape of the comparison between the exact and the cubic spline function for h = 0.25

Table 3. Comparison between the exact each approach function for h = 0.25

i	Xi = i h	Linear	Quadratic	Cubic Hermite	b-spline	U(exact)
0	0	0	0	0	0	0
1	0.25	0.725156	0.578647	0.657384	0.70745256	0.70710678
2	0.50	1.02552	0.868012	0.929682	1.0006708	1
3	0.75	0.725156	0.648907	0.657384	0.70745256	0.70710678
4	1	0	0	0	0	0

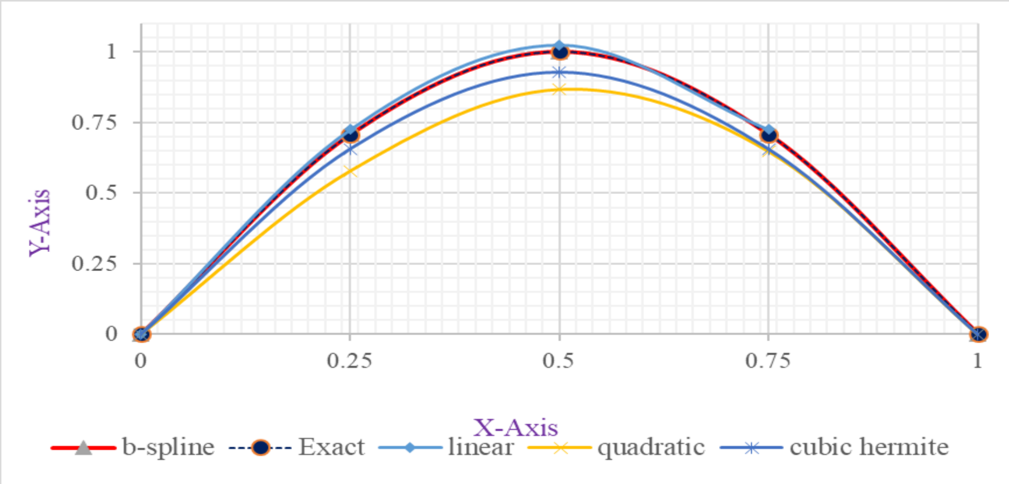

Fig 4. Comparison between the exact each approach function for h = 0.25

5. Polynomials Function

5.1 Second Degree

As explained in equations (15), (16), (17)

$$u(x) = \frac{240}{\pi\left(10+\pi^2\right)}(x-x^2)$$

$$I = \int_0^1 [[u'(x)]^2 + \pi^2[u(x)]^2]dx - 4\pi^2 \int_0^1 Sin\pi x u(x)\,\mathrm{dx}$$

5.2 Third Degree

As explained in equations (18) and (19)

$$\frac{dI}{dc_2} = 0, \frac{dI}{dc_3} = 0$$

$$c_2 = \frac{-240}{\pi\left(10+\pi^2\right)}, c_3 = \frac{240}{\pi\left(10+\pi^2\right)} \Rightarrow c_1 = 0$$

$$u(x) = -(-\frac{240}{\pi(10+\pi^2)} + \frac{240}{\pi(10+\pi^2)})x^3 - \frac{240}{\pi(10+\pi^2)}x^2 + \frac{240}{\pi(10+\pi^2)}x$$

As seen the first term of x^3 will be zero, so the values of the table will be as above.

5.3 Fourth Degree Polynomial

As explained in equations (20) and (21)

$$\frac{dI}{dc_2}=0,\frac{dI}{dc_3}=0,\frac{dI}{dc_4}=0$$

$$c_2=\frac{(20160\ (-1680+17\pi^4)}{\pi^3\left(1008+112\pi^2+\pi^4\right)}=-7.07169\ ,c_3=0.423321\Rightarrow c_4=3.11253$$

$$c_1=-(-7.07169+0.423321+3.11253)=3.535839$$

$$u(x)=-(\frac{(20160\ (-1680+17\pi^4)}{\pi^3\left(1008+112\pi^2+\pi^4\right)}-\frac{6720(-3024+31\pi^4)}{\pi^3(1008+112\pi^2+\pi^4)}+\frac{3360(-3024+31\pi^4)}{\pi^3(1008+112\pi^2+\pi^4)})x^4+$$

$$\frac{(20160\ (-1680+17\pi^4)}{\pi^3\left(1008+112\pi^2+\pi^4\right)}x^3-\frac{6720(-3024+31\pi^4)}{\pi^3(1008+112\pi^2+\pi^4)}x^2+\frac{3360(-3024+31\pi^4)}{\pi^3(1008+112\pi^2+\pi^4)}x$$

The values of the five-degree polynomial made the same as fourth degree. When increasing the power of polynomial, the error decreases and the five-degree polynomial achieve the best and least errors.

Table 4. Comparison between the exact and the polynomial functions

i	Xi = i h	2nd and 3rd Polynomial	E	4th and 5th Polynomial	Exact	E
0	0	0	0	0	0	0
1	0.10	0.346031	0.037015	0.308768	0.309016	0.000248
2	0.20	0.615166	0.027381	0.588522	0.587785	0.000737
3	0.30	0.807405	0.001611	0.809561	0.809016	0.000545
4	0.40	0.922749	0.028307	0.950671	0.951056	0.000385
5	0.50	0.961196	0.038804	0.999122	1.000000	0.000878
6	0.60	0.922749	0.028307	0.950671	0.951056	0.000385
7	0.70	0.807405	0.001611	0.809561	0.809016	0.000545
8	0.80	0.615166	0.027381	0.588522	0.587785	0.000737
9	0.90	0.346031	0.037015	0.308768	0.309016	0.000248
10	1	0	0	0	0	0

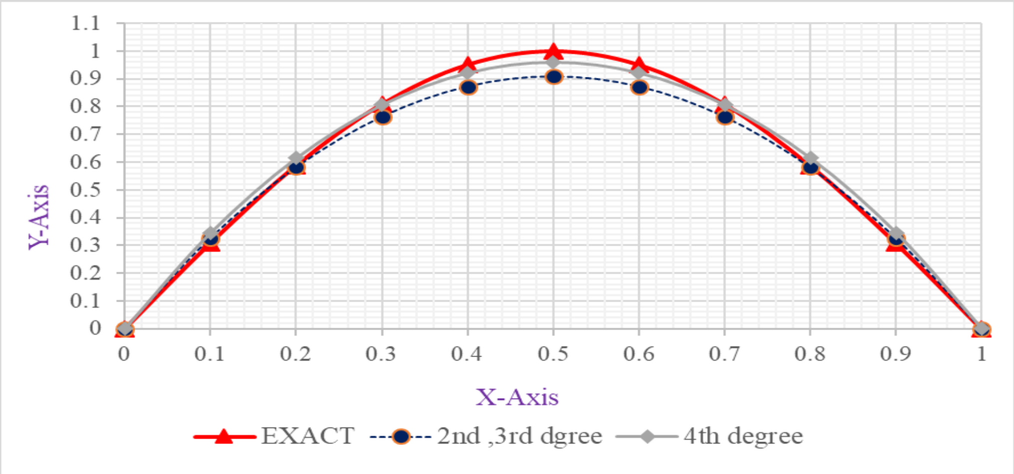

Fig 5. Comparison between the exact and the polynomial functions

6. Galerkin Method (GM)

Solving the example (22), considering the form:

$$y(x) = -(a_2 + a_3)x + a_2 x^2 + a_3 x^3$$

$$\mathrm{R(x)} = (2 + \pi^2 x - \pi^2 x^2)a_2 + (6x + \pi^2 x - \pi^2 x^3)a_3 + 2\pi^2 \sin \pi x$$

$$N_1(x) = x - x^2 \qquad N_2(x) = x^2 - x^3$$

$$a_2 = \frac{-240}{\pi(10+\pi^2)} \qquad a_3 = 0 \qquad a_1 = \frac{240}{\pi(10+\pi^2)}$$

$$y(x) = \frac{240}{\pi(10+\pi^2)} x - \frac{240}{\pi(10+\pi^2)} x^2$$

7. Collocation Method (CM)

$$y(x) = -(a_2 + a_3)x + a_2 x^2 + a_3 x^3$$

$$(2 + \pi^2 x - \pi^2 x^2)a_2 + (6x + \pi^2 x - \pi^2 x^3)a_3 = -2\pi^2 \sin \pi x$$

There are three unknowns, need four points as collocation points, choose x = 0.25, x = 0.5, use B.C X = 0, X = 1

For x = 0.25

$$(2 + \pi^2(0.25) - \pi^2(0.25)^2)a_2 + (6(0.25) + \pi^2(0.25) - \pi^2(0.25)^3)a_3 = -2\pi^2 \sin \pi(0.25)$$

$$3.85a_2 + 3.8131a_3 = -13.9577$$

For x = 0.5

$$(2 + \pi^2(0.5) - \pi^2(0.5)^2)a_2 + (6(0.5) + \pi^2(0.5) - \pi^2(0.5)^3)a_3 = -2\pi^2 \sin \pi(0.5)$$

$$4.4674a_2 + 6.7011a_3 = -19.7392$$

Solve the two equations to get unknowns, then $a_2 = -2.2577$, $a_{3=}$ -1.3809 so a_1=3.6386

$$y(x) = 3.6386x - 2.2577x^2 - 1.3809x^3$$

8. Least Squares Method (LSM)

$$y(x) = -(a_2 + a_3)x + a_2 x^2 + a_3 x^3$$

$$\mathrm{E(x)} = (-2 + 6x - \pi^2 x^2 - \pi^2 x^3)a_2 + (6x + \pi^2 x - \pi^2 x^3)a_1 - 2\pi^2 \sin \pi x$$

$$\mathrm{F(X)} = [\mathrm{E(x)}]^2 = (-2 + 6x - \pi^2 x^2 - \pi^2 x^3)a_2 + (6x + \pi^2 x - \pi^2 x^3)a_1 - 2\pi^2 \sin \pi x$$

Integrating and differentiate with unknowns

$$\int_0^1 \frac{\partial F}{\partial a_1} = 0 \qquad \int_0^1 \frac{\partial F}{\partial a_2} = 0$$

Solving using Mathematica program to get a_1, a_2, a_3

$$a_1 = \frac{480\pi}{120 + 20\pi^2 + \pi^4} \qquad a_2 = \frac{-480\pi}{120 + 20\pi^2 + \pi^4} \qquad a_3 = 0$$

$$y(x) = \frac{480\pi}{120 + 20\pi^2 + \pi^4} x - \frac{480\pi}{120 + 20\pi^2 + \pi^4} x^2$$

9. Finite Difference Method (FDM)

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$$-\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + \pi^2 y_i = 2\pi^2 \sin(\pi x)$$

Table 5. Comparison between the exact and other methods

i	Xi = i h	LSM	GM	CM	FDM	RR	Exact
0	0	0	0	0	0	0	0
1	0.10	0.327185	0.346031	0.3399021	0.310289	0.310287	0.309016
2	0.20	0.581663	0.615166	0.626364	0.590204	0.590200	0.587785
3	0.30	0.763432	0.807405	0.85110	0.812347	0.812341	0.809016
4	0.40	0.872494	0.922749	1.0058	0.954971	0.954964	0.951056
5	0.50	0.908848	0.961196	1.0822	1.00412	1.004110	1.000000
6	0.60	0.872494	0.922749	0.922749	0.954971	0.954964	0.951056
7	0.70	0.763432	0.807405	0.807405	0.812347	0.812341	0.809016
8	0.80	0.581663	0.615166	0.615166	0.590204	0.590200	0.587785
9	0.90	0.327185	0.346031	0.346031	0.310289	0.310286	0.309016
10	1	0	0	0	0	0	0

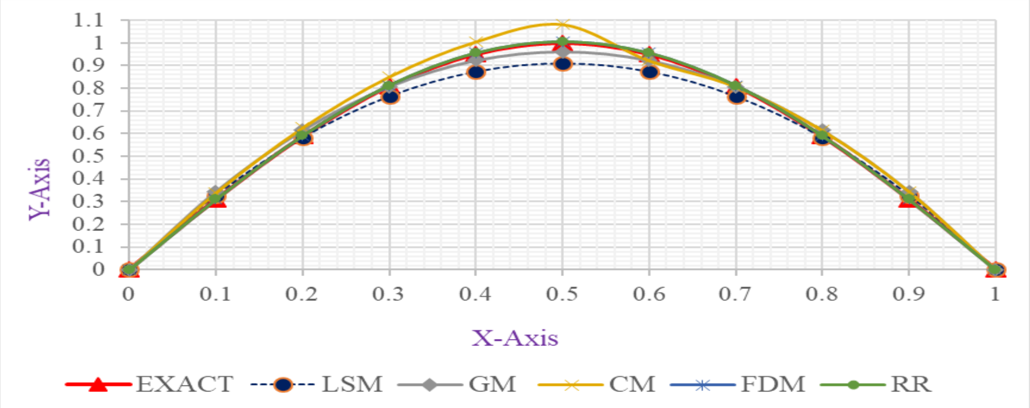

Fig 6. Comparison between the exact and other method

VI. Error Analysis and Numerical Results

Table 5. Comparison between the errors of each approach of RR method for h = 0.25

RR linear	Quadratic	Cubic Hermite	b-spline
0	0	0	0
0.018049	0.12845978	0.04972278	0.000346
0.02552	0.131988	0.070318	0.000671
0.018049	0.05819978	0.04972278	0.000346
0	0	0	0

Table 6. Comparison between the errors of each method

Poly4,5	Poly3	b-spline	Quadratic	Cubic Hermit	RR	Galerkin	CM	LSM	FDM
0	0	0	0	0	0	0	0	0	0
0.000248	0.037015	0.00000055	0.053873	0.027061	0.001271	0.037015	0.037015	0.018169	0.001273
0.000737	0.027381	0.00000024	0.096854	0.051475	0.002415	0.027381	0.027381	0.006122	0.002419
0.000545	0.001611	0.00000012	0.126467	0.070849	0.003325	0.001611	0.001611	0.045584	0.003331
0.000385	0.028307	0.00000015	0.141263	0.083289	0.003908	0.028307	0.028307	0.078562	0.003915
0.000878	0.038804	0.00000020	0.141056	0.087575	0.004110	0.038804	0.038804	0.091152	0.00412
0.000385	0.028307	0.00000061	0.12704	0.083289	0.003908	0.028307	0.028307	0.078562	0.003915
0.000545	0.001611	0.00000074	0.101764	0.070849	0.003325	0.001611	0.001611	0.045584	0.003331
0.000737	0.027381	0.00000165	0.068967	0.051475	0.002415	0.027381	0.027381	0.006122	0.002419
0.000248	0.037015	0.00000111	0.033302	0.027061	0.001270	0.037015	0.037015	0.018169	0.001273
0	0	0	0	0	0	0	0	0	0

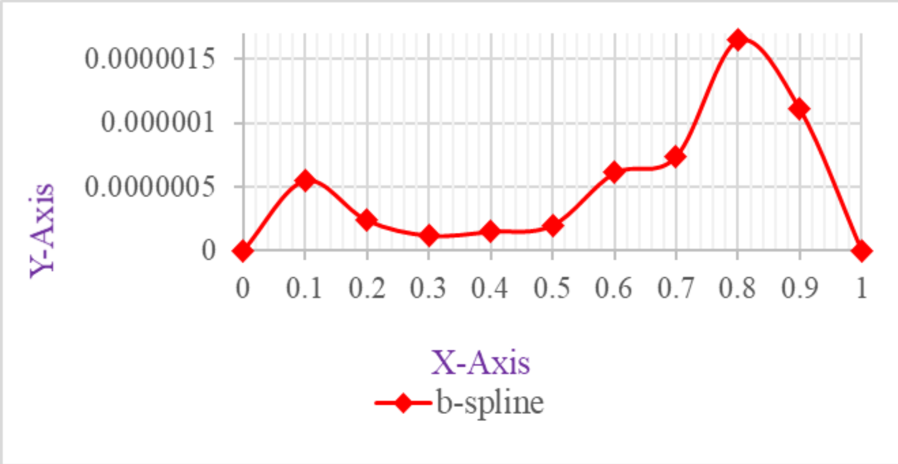

Fig 7. Analysis of the errors of the cubic (b-spline)

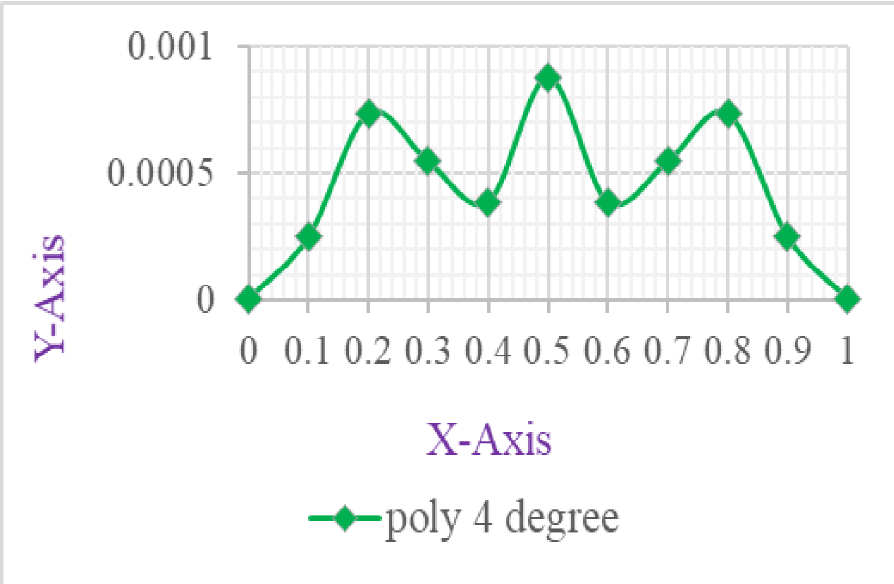

Fig 8. Analysis of the errors of the fourth degree polynomial

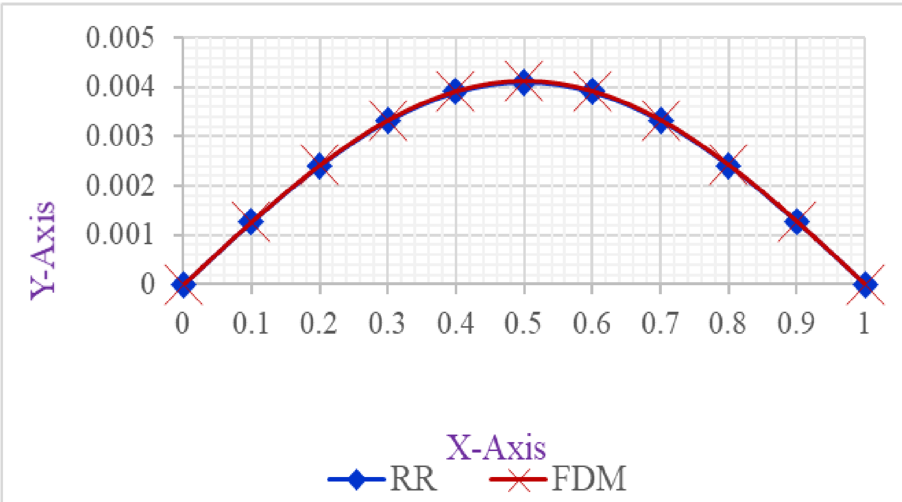

Fig 9. Analysis of the errors of the Rayleigh Ritz and FDM

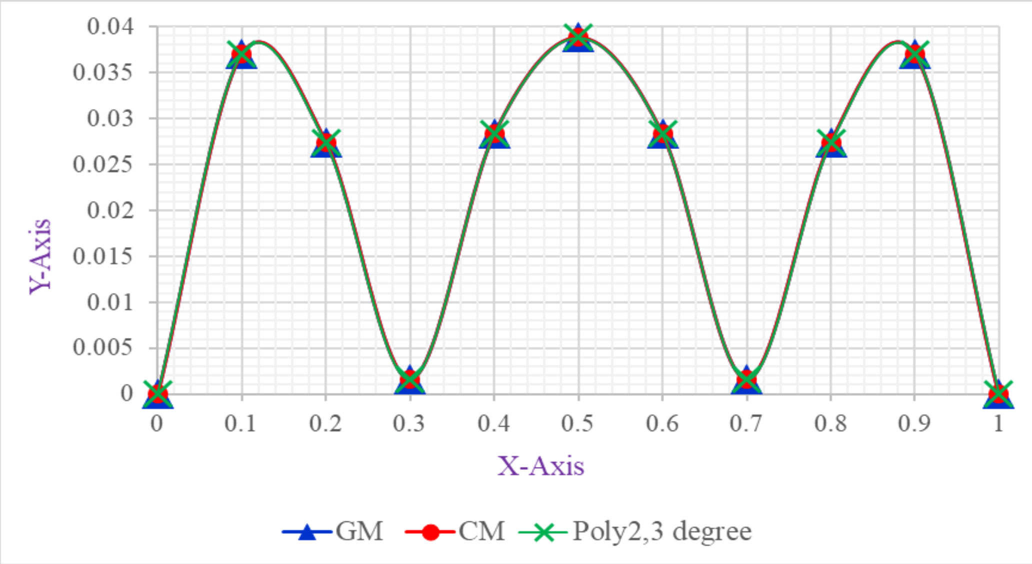

Fig 10. Analysis of the errors of GM, CM and the second and third degree

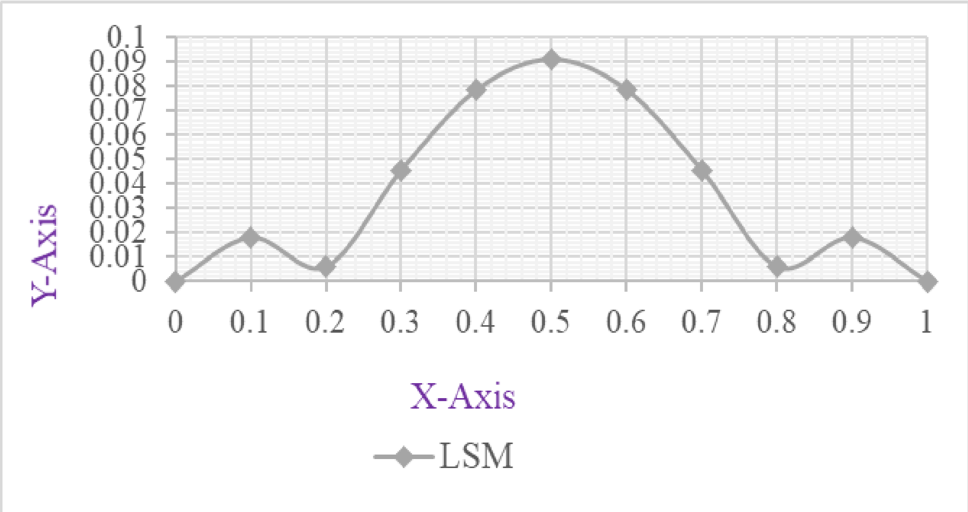

Fig 11. Analysis of the errors of LSM

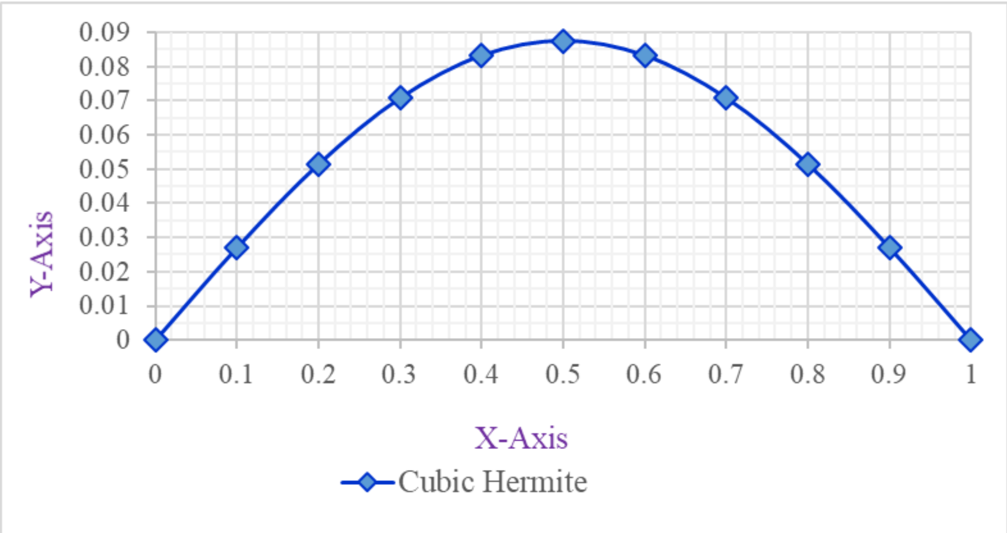

Fig 12. Analysis of the errors of the Cubic hermite function

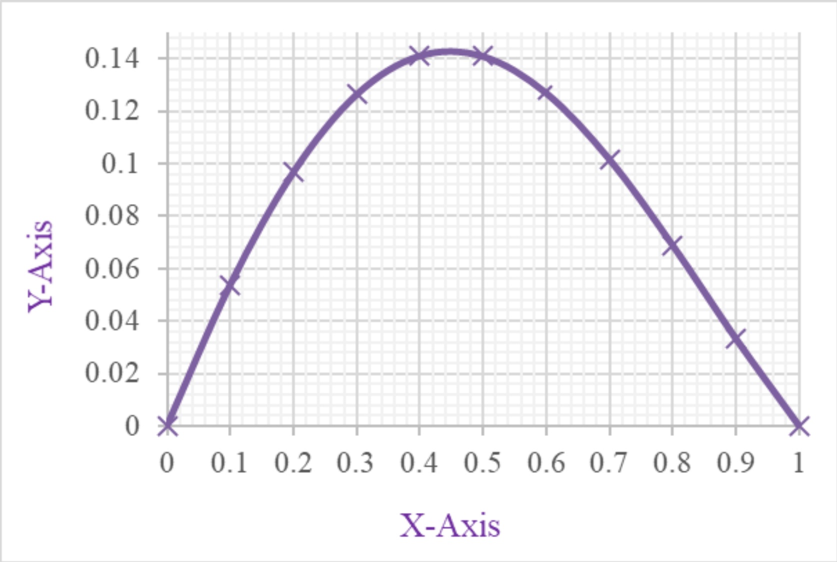

Fig 13. Analysis of the errors of quadratic function

V.1. It's clear from table (6), figure (7) that the most accurate numerical method used is the Rayleigh Ritz method (using b- spline approach)
V.2. Using the 4th degree polynomial produces good result, but less than accuracy than the b- spline approach as illustrated from figure (8).
V.3. Using the 5th degree polynomial produces the same result as the 4th degree polynomial,
V.4. The finite difference method produces the same results as RR (linear basis) compared to the fifth decimal figure (9), LSM is less accurate than them figure (10).
V.5. From figure (12) and (13), it is proven that using RR (quadratic Lagrange, cubic Hermite approach) introduces accurate result, but not more good than the b- spline approach.
V.6. It's illustrated from table (5) and (6) how the decreasing of the value of h effect on improving the error and decreasing it.

VII. Conclusion

As illustrated in the above numerical example, the Rayleigh Ritz method is considered one of the best Numerical method for solving boundary value problems, the accuracy of this method depend on the choice of the trail function, the use of b-spline function is the best, the 4th polynomial degree is desirable, the use of cubic hermit and quadratic hasn't reduced the error, by comparing other methods, it is found that FDM, LSM achieve good accuracy but the RR achieve the best.

References

[1]. R. L. Burden, Numerical Analysis (Ninth Edition, 2005).
[2]. C. D. Boor, A practical Guide to Splines (1978).
[3]. C. Zheng, J. Xiao, S. Meng, and C. Zhang, Resolvent sampling based Rayleigh-Ritz method for large-scale nonlinear eigenvalue problems, Comput. Methods Appl. Mech. Engrg, 2016.
[4]. C. Magers, Least Squares Approach to the Rayleigh-Ritz Method (Mississippi State University).
[5]. D. Gallistl and P. Huber, On the stability of the Rayleigh-Ritz method for eigenvalue, INS Preprint No. 1527, 2017.
[6]. D. J. Fyfe, The use of cubic splines in the solution of two-point boundary value problems, The Computer Journal, 12(2), 1969, 188–192.
[7]. D. Mortari, Least-Squares Solution of Linear Differential Equations, Mathematics, 5(48), 2017.
[8]. F. A. Abd El-Salam, On a Parametric Spline Function, IOSR Journal of Mathematics (IOSR-JM) Volume 9, Issue (2), 2013, 19-22.
[9]. F. A. Abd El-Salam, A parametric spline method for second-order singularly perturbed boundary-value problem, IOSR Journal of Mathematics (IOSR-JM) Volume 9, Issue (3), 2013, 1-5.
[10]. G. Bi, Generalized Stress Field in Granular Soils Heap with Rayleigh-Ritz Method, Journal of Rock Mechanics and Geotechnical Engineering, 9(1), 2017, 135-149.
[11]. G. Gnecco, On the Curse of Dimensionality in the Ritz Method, Journal of Optimization Theory and Applications, 168(2), 2016, 488–509.
[12]. I. Senjanović, N. Alujević, I. Ćatipović, D. Čakmak, and N. Vladimir, Vibration analysis of rotating toroidal shell by the Rayleigh-Ritz method and Fourier series, Engineering Structures, 173, 2018, 870–891.
[13]. J. Lambers, Lecture 27 Notes (Mat 461/561, Spring Semester, 2009).
[14]. L. Lun and X. Tan, Studies on global analytical mode for a three-axis attitude stabilized space craft by the Rayleigh-Ritz method, Archive of Applied Mechanics 86(12), 2016.
[15]. S. Al-Ansari, Calculating Static Deflection and Natural Frequency of Stepped Cantilever Beam Using Modified Rayleigh Method, International Journal of Mechanical and Production Engineering Research and Development, 3(4), 2013, 2249-6890.
[16]. M. H. Schultz, Spline Analysis (Prentice-Hall, Inc., Englewood Cliffs, N.J 1973).

[17]. N. Datta and J. Thekinens, A Rayleigh-Ritz based approach to characterize the vertical vibration of non-uniform hull girder, Ocean Engineering, 125, 2016, 113-123.

[18]. N. Dăneţ, Solving A Two-Point Boundary-Value Problem Using the Rayleigh-Ritz Method with Mathcad, Proceedings of Mathematics and Educational Symposium of the Depart. of Math. and Comp. Science, 2nd Edition, 2016, 39 – 44.

[19]. P.M. Prenter, Splines and Variational Methods (Wiley classics edition 1989).

[20]. S. Bhattacharyya and A. K. Baruah, On Rayleigh-Ritz Method in Three-Parameter Eigenvalue Problems, International Journal of Computer Applications, 86(3), 2014, 0975 – 8887.

[21]. Y. Kumar, The Rayleigh–Ritz method for linear dynamic, static and buckling behavior of beams, shells and plates: A literature review, Journal of Vibration and Control, 24(7), 2018, 1205-1227.

Shimaa Holail. "Comparison between the Rayleigh Ritz Method (RR) and Other Numerical Methods for Solving Second Order Boundary Value Problems." IOSR Journal of Mathematics (IOSR-JM) 15.3 (2019): 47-60.