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Characterization Theorem for Commutative Lattice Ordered Ring.

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Abstract: *Several people presented solutions to the Birkhoff's problem "Develop a common abstraction which includes Boolean algebras (rings) and lattice ordered groups as special cases". Many common abstractions namely dually residuated lattice ordered semi groups, lattice ordered groups, DRℓ - groups, lattice ordered rings are presented in [6], [4], [3] and [2] respectively.*
The objective of this paper is to introduce Characterization Theorem for commutative lattice ordered ring or commutative ℓ-ring which is an abstraction between Boolean algebra and lattice ordered group.

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I. Preliminaries

In this section are listed, a number of definitions and results which are made use of throughout the paper. The symbols $\leq$, $\nleq$, $+$, $.$, $-$, $\vee$, $\wedge$ and $*$ will denote inclusion, non-inclusion, sum, product, difference, join (least upper bound), meet (greatest lower bound) and symmetric difference in a lattice L or commutative ℓ-ring R (whenever they are defined)

To start with

Definition 1.1

A Boolean algebra is a non-empty set B with two binary operations $\vee$, $\wedge$ and an unary operation $'$ defined on it and satisfy the following.

1. $(B, \vee , \wedge)$ is a lattice
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in B$
3. B has least element 0 and greatest element 1
4. For each $a \in B$ there exists $a' \in B$ such that $a \vee a' = 1$ and $a \wedge a' = 0$

That is a Boolean algebra B is a distributive complemented lattice.

Definition 1.2

A ring R is called a Boolean ring if $a^2 = a$ for all $a \in R$

Definition 1.3

A Boolean ring R is called Boolean ring with identity if there exists $1 \in R$ such that $1 \cdot a = a \cdot 1 = a$, for all $a \in R$.

Theorem 1.1

The following systems are equivalent

1. Boolean Algebra
2. Boolean ring with identity

Definition 1.4

A non – empty set G is called lattice ordered group or ℓ - group if

(i) $(G, +)$ is a group
(ii) $(G, \vee , \wedge)$ is a lattice
(iii) $a + x \vee y + b = (a + x + b) \vee (a + y + b)$
$a + x \wedge y + b = (a + x + b) \wedge (a + y + b)$

for all $a, b, x, y \in G$.

Definition 1.5

A non – empty set R is called lattice ordered ring or ℓ - ring if

(i) $(R, +, \cdot)$ is a ring
(ii) $(R, \vee , \wedge)$ is a lattice

(iii) $a + x \vee y + b = (a + x + b) \vee (a + y + b)$
$a + x \wedge y + b = (a + x + b) \wedge (a + y + b)$

for all $a, b, x, y \in R$

(iv) $a (x \vee y) b = (a x b) \vee (a y b)$
$a (x \wedge y) b = (a x b) \wedge (a y b)$

for all $a, b, x, y \in R$ and $a \geq 0, b \geq 0$

Definition 1.6

A non-empty set B is called a Browerian Algebra if and only if

i) $(B, \leq)$ is a lattice

ii) B has a least element

iii) To each $a, b \in B$, there exists $x = a- b \in B$ such that $b \vee x \geq a$

II. Definition And Examples

In this section two equivalent definitions for commutative lattice ordered ring are introduced and established that commutative lattice ordered ring is an abstraction between Boolean algebra and lattice ordered group.

Definition 2.1

A non-empty set R is called commutative lattice-ordered ring if it has two binary operations $+, \cdot$ and a binary relation $\leq$ defined on it and satisfy the following

(i) $(R, +, \cdot)$ is a commutative ring

(ii) $(R, \leq)$ is a lattice.

(iii) $a \leq b \Rightarrow a + c \leq b + c$ for all $a, b, c \in R$.

(iv) $a \leq b, 0 \leq c \Rightarrow a c \leq b c$ for all $a, b, c \in R$.

Definition 2.2

A non-empty set R is called a commutative lattice-ordered ring if it has four binary operations $+, ., \vee$ and $\wedge$ defined on it and satisfy the following

(i) $(R, +, \cdot)$ is a commutative ring

(ii) $(R, \vee, \wedge)$ is a lattice

(iii) $(a + c) \vee (b + c) = (a \vee b) + c$

$(a + c) \wedge (b + c) = (a \wedge b) + c$

for all $a, b, c \in R$

(iv) $a c \vee b c = (a \vee b) c$

$a c \wedge b c = (a \wedge b) c$ for all $a, b, c \in R$ and $c \geq 0$.

We observe that

Theorem 2.1

Two definitions of a commutative lattice-ordered ring are equivalent.

Theorem 2.2

Any commutative lattice ordered ring is a lattice ordered group.

Theorem 2.3

A Boolean ring is a commutative lattice ordered ring.

Theorem 2.4

If R is a commutative ℓ-ring and $ab = a \wedge b$ for all $a, b \in R$ then R is a Boolean ring.

Theorem 2.5:

Any Boolean algebra is a commutative lattice-ordered ring.

Theorem 2.6:

Any commutative ℓ - ring need not be a Boolean algebra.

Proof: By an example.

Let $Q[\lambda]$ denote the ring of polynomials over the ordered field Q. Then $Q[\lambda]$ is a commutative lattice ordered ring but not a Boolean algebra.

If R is a commutative ℓ-ring then we have

Property 1: $[(a - b) \vee 0] + b = a \vee b$, for all $a, b \in R$

Property 2: $a \leq b \Rightarrow a - c \leq b - c$ and $c - b \leq c - a$, for all $a, b, c \in R$.

Property 3: $(a \vee b) - c = (a - c) \vee (b - c)$ for all $a, b, c \in R$.

Property 4: $a - (b \vee c) = (a - b) \wedge (a - c)$ for all $a, b, c \in R$.

Property 5: $a - (b \wedge c) = (a - b) \vee (a - c)$ for all a, b, c $\in$ R.

Property 6: $(b \wedge c) - a = (b - a) \wedge (c - a)$ for all a, b, c $\in$ R.

Property 7: $a \geq b \Rightarrow (a - b) + b = a$, for all a, b $\in$ R.

Property 8: $a \vee b + a \wedge b = a + b$, for all a, b $\in$ R

Property 9: $[(a - b) \vee 0] + a \wedge b = a$ for all a, b $\in$ R

Property10: $(a \vee b) - (a \wedge b) = (a - b) \vee (b - a)$ for all a, b $\in$ R.

Property 11: $a - (b - c) \leq (a - b) + c$

$(a + b) - c \leq (a - c) + b$ for all a, b, c $\in$ R.

Property 12: $a \wedge b = 0$ and $a \wedge c = 0$ then $a \wedge (b + c) = 0$

$a \vee b = 0$ and $a \vee c = 0$ then $a \vee (b + c) = 0$ for all a, b, c $\in$ R.

Theorem2.7:

Any commutative ℓ-ring is a distributive lattice.

III. Characterization Theorem

In this section to find the relation between commutative ℓ-ring and Browerian algebra and further established the characterization theorem for Commutative lattice-ordered ring

Clearly we have

Theorem3.1

If R is a commutative ℓ-ring and $a + b = a \vee b$ to each a, b $\in$ R, there exists a least element x $\in$ R such that $b \vee x = b + x \geq a$ then R is a Browerian algebra.

Theorem 3.2

Any commutative ℓ-ring R is a direct product of a Browerian Algebra B and an ℓ-ring S if and only if

i) $(a + b) - (c + c) \geq (a - c) + (b - c)$

and

ii) $(ma + nb) - (a + b) \geq (ma - a) + (nb - b)$

for all a, b, c $\in$ R and any two positive integers m, n.

Proof:

First Part:

Assume that (i) $(a + b) - (c + c) \geq (a - c) + (b + c)$ ------------------ (1)

and

(ii) $(ma + nb) - (a + b) \geq (ma - a) + (nb - b)$--------------- (2)

for all a, b, c $\in$ R and any two positive integers m.n.

To prove B is a Browerian Algebra,

S is a ℓ-ring

and $R = B \times S$

Let a, b, c $\in$ R be arbitary

$\Rightarrow (a + b) - c \leq (a - c) + b$, by property 11

$\Rightarrow ((a + b) - c) - c \leq ((a - c) + b) - c$

$= (a - c) + b - c$

$\Rightarrow (a + b) - (c + c) \leq (a - c) + (b - c)$ ------------------ (3)

From (1) and (3) we get $(a + b) - (c + c) = (a - c) + (b - c)$ -------------------- (4)

Also $(ma + nb) - a \leq (ma - a) + nb$, by property 11

$\Rightarrow [(ma + nb) - a] - b \leq [(ma - a) + nb] - b$

$\Rightarrow (ma + nb) - (a + b) \leq (ma - a) + (nb - b)$ -------------------------- (5)

From (2) and (5) we get

$(ma + nb) - (a + b) = (ma - a) + (nb - b)$ ---------------------------- (6)

Take $B = \{a \in R / a + a - a = 0\}$

$S = \{a \in R / a + a - a = a\}$

Claim 1: B is a Browerian Algebra.

i) B is closed with respect to $\vee$ and $\wedge$.

First we claim that if a $\in$ B then $a + a = a$

For let a $\in$ B be arbitrary.

$\Rightarrow a + a - a = 0 \leq 0$

$\Rightarrow [(a + a) - a] + a \leq 0 + a$

$\Rightarrow a + a \leq a$

Also $0 = (a + a) – a \leq (a – a) + a$, by property 11

$\Rightarrow 0 \leq a$

$\Rightarrow 0 + a \leq a + a$

$\Rightarrow a \leq a + a$

Thus a + a = a --------------- (7)

Next to claim that if for all $a, b \in B$ then $a \vee b$, $a \wedge b \in B$.

For let $a, b \in B$ be arbitrary. Then

$(a + b) + (a – b) = (a + a) – (b + b)$, by (4)

$= a – b$, by (7)

$\Rightarrow a – b \in B$

$\Rightarrow (a – b) + b \in B$

$\Rightarrow ((a – b) \vee 0) + b \in B$

$\Rightarrow a \vee b \in B$, by property (11)

Also $(a + b) – (a \vee b) = (a + b) – [(a \vee b) + (a \vee b)]$, since $a \vee b \in B$

$= [a – (a \vee b)] + [b – (a \vee b)]$, by (4)

$= [(a - a) \wedge (a - b)] + [(b - a) \wedge (b - b)]$, by property 4

$= [0 \wedge (a - b)] + [(b - a) \wedge 0]$

$= 0$

$\Rightarrow a + b = a \vee b$

Let $a, b \in B$ $\Rightarrow a + b, a \vee b \in B$

$\Rightarrow (a + b) – (a \vee b) \in B$

$\Rightarrow a \wedge b \in B$, by property 8

ii) (B, $\vee$, $\wedge$) is a lattice

Idempotent law:

Let $a \in B$ be arbitary. Then

$a \vee a = a + a = a$

$a \wedge a = (a + a) – (a \vee a)$, by property 8

$= a + a – a$

$= a$

Thus $a \vee a = a$, $a \wedge a = a$ for all $a \in B$

Commutative law:

Let $a, b \in B$ be arbitary. Then

$a \vee b = a + b$

$= b + a$

$= b \vee a$

$a \wedge b = (a + b) – (a \vee b)$

$= (b + a) – (b \vee a)$

$= b \wedge a$

Thus $a \vee b = b \vee a$, $a \wedge b = b \wedge a$ for all $a, b \in B$.

Associative law:

Let $a, b, c \in B$ be arbitrary. Then

$a \vee (b \vee c) = a + (b + c)$

$= (a + b) + c$

$= (a \vee b) \vee c$

$a \wedge (b \wedge c) = [a + (b \wedge c)] – [a \vee (b \wedge c)]$

$= [a + (b \wedge c)] – [a + (b \wedge c)]$

$= 0$

$(a \wedge b) \wedge c = [(a \wedge b) + c] – [(a \wedge b) \vee c]$

$= [(a \wedge b) + c] – [(a \wedge b) + c] = 0$

Thus $a \vee (b \vee c) = (a \vee b) \vee c$

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$ for all $a, b, c \in B$.

Absorption law:

Let $a, b \in B$ be arbitary. Then

$a \vee (a \wedge b) = a + (a \wedge b)$

$= a + [(a + b) – (a \vee b)]$

$= a + [(a + b) – (a + b)]$

$= a + 0 = a$

$a \wedge (a \vee b) = [a + (a \vee b)] – [a \vee (a \vee b)]$
$= [a + (a \vee b)] – [(a \vee a) \vee b]$
$= [a + (a \vee b)] – (a \vee b) = a$

Thus $a \vee (a \wedge b) = a \wedge (a \vee b) = a$ for all $a, b \in B$.
Hence $(B, \vee, \wedge)$ is a lattice.

iii) B has a least element

Let $a \in B$ arbitrary. Then

$0 = (a + a) – a$
$\leq (a - a) + a$, by property 1

$\Longrightarrow 0 \leq a$, for all $a \in R$

Therefore B has a least element.

iv) To each a, b $\in$ B these exists x = a- b $\in$ B such that b $\vee$ x $\geq$ a

Let $a, b \in B$ be arbitary

$\Longrightarrow x = a – b \in B$ and $b \vee x = b + x$
$= b + a – b = a \geq a$

Hence B is Browerian Algebra.

Claim 2: S is a ℓ - ring.

i) (S, +, ·) is a ring.

Closure law:

Let $a, b \in S$ be arbitary. Then

$[(a + b) + (a + b)] – (a + b) = (2a + 2b) – (a + b)$
$= (2a - a) + (2b - b)$, by (6)
$= a + b$
$\Longrightarrow a + b \in S$

Thus $a, b \in S \Longrightarrow a + b \in S$.

Clearly '+' is associative and commutative in S, since S is a subset of R.

Existence of Identity:

For let $a \in S$ be arbitary

Clearly $0 \in S$, since $0 = 0 + 0 – 0$
Then $a + 0 = 0 + a$ for all $a \in S$
Thus there exist an element $0 \in S$ such that $a + 0 = 0 + a = a$ for all $a \in S$

Existence of Inverse:

For let $a \in S$. Then

$(-a) + (-a) – (-a) = – a – a + a$
$= – a$

$\Longrightarrow -a \in S$
$\Longrightarrow a + (-a) = (-a) + a = 0$

Thus to each $a \in S$ there exist an element $–a \in S$ such that $a + (-a) = 0$

Closure law:

For let $a, b \in S$ be arbitary. Then

$a + a – a = a$
$b + b – b = b$
$\Longrightarrow ab = (a + a -a) b$
$= ab + ab – ab$
$\Longrightarrow ab \in S$

Associative law:

Clearly · is associative in S since $S \subseteq R$.

Distributive law:

Clearly distributive law hold in S since $S \subseteq R$

Therefore (S, +, ·) is a ring.

(ii) $(S, \vee, \wedge)$ is a lattice.

Let $a, b \in S$ be arbitary

$\Longrightarrow a, -b, b \in S$
$\Longrightarrow (a - b) + b \in S$
$\Longrightarrow a \vee b \in S$, by property 1

Also $a, b \in S \Longrightarrow a + b, a \vee b \in S$

$\Rightarrow (a + b) - (a \vee b) \in S$
$\Rightarrow a \wedge b \in S$, by property 8

Idempotent law:

Let $a \in S$ be arbitary. Then

$$a \vee a = (a - a) + a = a$$
$$a \wedge a = (a + a) - (a \vee a)$$
$$= (a + a) - a = a$$

Thus $a \vee a = a$, $a \wedge a = a$ for all $a \in S$.

Commutative law:

Let $a, b \in S$ be arbitary. Then

$$a \vee b = (a + b) - (a \wedge b)$$
$$= (a + b - a) \vee (a + b - b)$$
$$= b \vee a$$
$$a \wedge b = (a + b) - (a \vee b)$$
$$= (a + b - a) \wedge (a + b - b)$$
$$= b \wedge a$$

Thus $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ for all $a, b \in S$.

Associative law:

Let $a, b, c \in S$ be arbitary. Then

$$a \vee (b \vee c) = [a - (b \vee c)] + b \vee c$$
$$= a - (b \vee c) + b \vee c = a$$
$$(a \vee b) \vee c = [(a \vee b) - c] + c$$
$$= (a \vee b) - c + c$$
$$= a \vee b$$
$$= (a - b) + b = a$$
$$a \wedge (b \wedge c) = [a + (b \wedge c)] - [a \vee (b \wedge c)]$$
$$= [a + (b \wedge c)] - [(a - (b \wedge c)) + b \wedge c]$$
$$= a + (b \wedge c) - [a - (b \wedge c) + b \wedge c]$$
$$= a + (b \wedge c) - a$$
$$= b \wedge c$$
$$= (b + c) - (b \vee c)$$
$$= (b + c) - [(b - c) + c]$$
$$= b + c - b = c$$
$$(a \wedge b) \wedge c = [(a \wedge b) + c] - [(a \wedge b) \vee c]$$
$$= [(a \wedge b) + c] - [[(a \wedge b) - c] + c]$$
$$= (a \wedge b) + c - (a \wedge b) = c$$

Thus $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ for all $a, b, c \in S$.

Absorption law:

Let $a, b \in S$ be arbitary. Then

$$a \vee (a \wedge b) = [a - (a \wedge b)] + (a \wedge b) = a$$
$$a \wedge (a \vee b) = [a + (a \vee b)] - [a \vee (a \vee b)]$$
$$= [a + (a \vee b)] - [(a \vee a) \vee b]$$
$$= a + (a \vee b) - (a \vee b) = a$$

Thus $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$ for all $a, b \in S$

Hence $(S, \vee, \wedge)$ is a lattice.

Next to claim that

1. $a + x \vee y + b = (a + x + b) \vee (a + y + b)$
$a + x \wedge y + b = (a + x + b) \wedge (a + y + b)$
for all $a, b, x, y \in S$.

2. $a (x \vee y) b = (a x b) \vee (a y b)$
$a (x \wedge y) b = (a x b) \wedge (a y b)$
for all $a, b, x, y \in S$ and $a > 0$, $b > 0$.

For (1); Let $a, b, x, y \in S$ be arbitrary. Then

$$(a + x + b) \vee (a + y + b) = [(a + x + b) - (a + y + b)] + (a + y + b)$$
$$= a + x + b$$
$$(a + x \vee y + b) = [a + ((x - y) + y) +b]$$
$$= a + x + b$$
$$(a + x + b) \wedge (a + y + b) = [(a + x + b) + (a + y + b) - (a + x + b)]$$
$$= a + x + b$$

$(a + x \wedge y + b) = [a + [(x + y) - (x \vee y)] + b]$
$= a + (x + y) - (x + y - y) + b$
$= a + y + b$

Thus $(a + x + b) \vee (a + y + b) = (a + x \vee y + b)$,
$(a + x + b) \wedge (a + y + b) = (a + x \wedge y + b)$ for all $a, b, x, y \in S$.

For (2); Let $a, b, x, y \in S$ be arbitrary and $a > 0$, $b > 0$. Then

$(a x b) \vee (a y b) = (a x b - a y b) + a y b = a x b$
$a (x \vee y) b = a (x - y + y) b = a x b$
$a (x \wedge y) b = a (x + y - (x \vee y)) b$
$= a (x + y - [(x - y) + y]) b = a y b$
$(a x b) \wedge (a y b) = (a x b + a y b) - ((a x b) \vee (a y b))$
$= a x b + a y b - [(a x b - a y b) + a y b]$
$= a x b + a y b - a x b = a y b$

Thus $(a x b) \vee (a y b) = a (x \vee y) b$, $(a x b) \wedge (a y b) = a (x \wedge y) b$
for all $a, b, x, y \in S$ and $a > 0$, $b > 0$
Hence S is a ℓ-ring.

Claim 3: $R = B \times S$

It is enough to prove for any $a \in R$ can be uniquely expressed as $a = x + y$,
$y = (a + a) - a$, $x = a - [(a + a) - a] \Longrightarrow x \in B$ and $y \in S$.

Then $(y + y) - y = [(2a - a) + (2a - a)] - (2a - a)$
$= [(2a + 2a) - (a + a)] - (2a - a)$, by (6)
$= (4a - 2a) - (2a - a)$
$= 4a - 2a - a$, since $2a - a \leq a$
$= a = (a + a) - a = y$
$\Longrightarrow (y + y) - y \geq y$
Also $(y + y) - y \leq (y - y) + y$, by property 11
$= y$
$\Longrightarrow (y + y) - y \leq y$
Therefore $(y + y) - y = y$
$\Longrightarrow y \in S$
$y = (a + a) - a$
$\Longrightarrow y \leq a$
$x \geq 0 \Longrightarrow x + x \geq 0 + x = x$
$\Longrightarrow x + x \geq x$
Now $(a - y) + (a - y) = (a + a) - (y + y)$, by (4)
$= 2a - 2y$
$= 2a - 2 (2a - a)$
$= 2a - (4a - 2a)$
$\Longrightarrow x + x = 2a - (4a - 2a)$
We have $(4a - 2a) + (a - (2a - a)) = (2a - a) + (2a - a) + (a - (2a - a))$
$\geq (2a - a) + a$
$= 2a$
$\Longrightarrow (4a - 2a) + (a - (2a - a)) \geq 2a$
$\Longrightarrow 2a - (4a - 2a) \leq a - (2a - a)$
$\Longrightarrow x + x \leq x$
Therefore $x + x = x$
$\Longrightarrow x + x - x = 0$
$\Longrightarrow x \in B$
Thus if $a \in R$ then $a = x + y$ implies $x \in B$, $y \in S$

Uniqueness part:

Suppose $a = x' + y'$ where $x' \in B$ and $y' \in S$
Then $a + a = (x' + y') + (x' + y')$
$= (x' + x') + (y' + y')$
$= x' + 2y'$ since $x' \in B$
$= (x' + y') + y'$
$= a + y'$
$\Longrightarrow (a + a) - a = (a + y') - a$
$\Longrightarrow ((a + y') - a) - y' = (a + y') - (a + y') = 0$

$\Rightarrow a + y' = a + y'$
$\Rightarrow a + y' - a = y'$
$\Rightarrow (a + a) - a = y'$
$\Rightarrow y = y'$
Now $a = x' + y'$
$\Rightarrow a - y' = x' \leq x' \Rightarrow a - y' \leq x'$
$x' - (a - y') \leq (x' - a) + y'$
$= (x' - (x' + y')) + y' = 0$
$\Rightarrow x' \leq a - y'$
Hence $x' = a - y' = a - [(a + a) - a]$
$= x$
$\Rightarrow x' = x$
Hence $R = B \times S$

Second Part:

Conversely assume that a commutative ℓ - ring $R = B \times S$ where B is a Browerian algebra and S is a ℓ - ring.

To prove i) $(a + b) - (c + c) \geq (a - c) + (b - c)$
ii) $(ma + nb) - (a + b) \geq (ma - a) + (nb - b)$
for all a, b, c in R and any two positive integers m, n.
Let $a, b, c \in R$ be arbitrary.
$\Rightarrow a, b, c \in B$, since $a = a + 0$, $b = b + 0$, $c = c + 0$
$\Rightarrow a - c, b - c, a + b \in B$
$\Rightarrow (a - c) + (b - c), a + b \in B$
$\Rightarrow (a - c) + (b - c) - (a + b) \in B$ such that
$(a + b) \vee x \geq (a - c) + (b - c)$, since B is a Browerian algebra.
$\Rightarrow x = - (c + c) \in B$ such that
$(a + b) - (c + c) \geq (a - c) + (b - c)$
$\Rightarrow (a + b) - (c + c) \geq (a - c) + (b - c)$
Similarly let $a, b \in R$ be arbitrary.
$\Rightarrow a, b, \in B$
$\Rightarrow ma, nb, , a , b \in B$, since $a + a = a$, $a + a + a = a$ etc.
$\Rightarrow ma - a, nb - b, ma + nb \in B$
$\Rightarrow (ma - a) + (nb - b), ma + nb \in B$
$\Rightarrow x = [(ma - a) + (nb - b)] - (ma + nb) \in B$ such that
$(ma + nb) + x \geq (ma - a) + (nb - b)$
$\Rightarrow (ma + nb) - (a + b) \geq (ma - a) + (nb - b)$
Hence the proof.

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