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On the Weyl group of SO(2n; C)

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Abstract: *The Weyl group, W of G is defined as $N_G(T)/C_G(T)$, where $N_G(T)$ is the normalizer of T in G and $C_G(T)$ is the centralizer. This group is a finite group which can be regard as permutation groups on certain relevant sets of points in Z, see ([4],[5]). In this paper we defined a special way of denoting to write the Weyl elements for the group SO(2n; C).*

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I. Preliminaries

Consider the semisimple Lie group G = SO(2n; C) with a bilinear form b, where

$$G = SO(2n, \mathbb{C}) = \{A \in Sl(2n, \mathbb{C}) : A^t A = I\}$$

Let T be a maximal Tours in G. Fix the maximal Torus to be T = $Z_G(h)$. Choose the set of simple roots of SO(2n; C), where

$$\mathfrak{h} = \begin{pmatrix}
0 & 0 & \dots & \dots & \dots & \dots & 0 & \nu_1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \dots & \dots & \dots & \dots & \nu_2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \nu_q & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \nu_q & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & \nu_2 & \dots & \dots & \dots & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\nu_1 & 0 & \dots & \dots & \dots & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \nu_{n-1} & 0 & 0 \\
0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 & 0 & \dots & 0 & -\nu_{n-1} & 0 & 0 & 0 \\
0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \nu_n \\
0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -\nu_n & 0
\end{pmatrix}$$

The simple roots of (SO_{2n}; C) with respect to the above h are

$$\nu_j - \nu_{1+j}, \qquad 1 \leq j \leq q-1,$$

$$\nu_j - i\nu_{1+j}, \qquad q \leq j \leq n-1$$

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Recall that the Weyl group is defined to be W = N(T)=T , then W is a finite group which is isomorphic to a group of permutations. In the next sections we will explain how can we write the permutation group which isomorphic to the Weyl group.

In [3], Lakshmibai defined the Weyl group of SO(2n; C) to be,

$$W(G) := \{w = (a_1, a_2, ..., a_{2n}) \in S_{2n} : a_i = 2n + 1 - a_{2n+1-i}, 1 \leq i \leq 2n\} \quad (1)$$

where $\#\{i, 1 \leq i \leq n : a_i > n\}$ is even.

Example 1.1. Consider the semisimple group SO(4; C), then the Weyl group will be

$$W(SO(4, \mathbb{C})) = \{(1234), (2143), (3412), (4321)\}$$

According to the above definition $(a_1, a_2, ...a_{2n}) \in W(G)$ will be known, once $(a_1, a_2, ...a_n)$ is known. That means we can use only permutations of n-entries.

To make it easy to understand, first, I will rewrite the definition (1) to be,

$$W(G) := \{w = (a_1, a_2, ..., a_{2n}) \in S_{2n} : a_{2n+1-i} = -a_i, 1 \leq i \leq n\} \quad (2)$$

where $\#\{i, 1 \leq i \leq n : a_i < 0\}$ is even.

Example 1.2. Return back to the example of SO(4; C), the Weyl group become

$$W(SO(4, \mathbb{C})) = \{(12-2-1), (21-1-2), (-2-112), (-1-221)\}$$

Again, $(a_1, a_2, ...a_{2n}) \in W(G)$ will be known, once $(a_1, a_2, ...a_n)$ is known. So we can only write $(a_1, a_2, ...a_n)$.

II. Modern style of the Weyl group elements

In this section we will introduce a new style of writing the permutions which defined the Weyl elements. First, we can de ne the Weyl group to be:

$$W(G) := \{w = (a_1, a_2, ..., a_n) \in S_n : -n \leq a_i \leq n, 1 \leq i \leq n\} \quad (3)$$

where $\#\{i, 1 \leq i \leq n : a_i < 0\}$ is even and only one of a_i or $-a_i$ appear for all i.

To understand this notation in term of reflections, recall that the set of simple roots of SO(2n; C) are given by

$$\nu_j - \nu_{1+j}, \quad 1 \leq j \leq q-1,$$
$$\nu_j - i\nu_{1+j}, \quad q \leq j \leq n-1$$

Let us denote by $\{\theta_i, 1 \leq i \leq n\}$, the simple reflection in $W(G)$, namely,
$\theta_i =$ reflection with respect to $\nu_j - \nu_{1+j}, 1 \leq j \leq q-1, \nu_j - i\nu_{1+j}, q \leq j \leq n-1$, and
$\theta_n =$ reflection with respect to $\nu_{n-1} + i\nu_n$.
Then we have,

$$\theta_i = \begin{cases} S_i & 1 \leq i \leq n-1 \\ -S_{n-1} & i = n \end{cases}$$

where S_i denote the transposition $(i, i+1)$, $1 \leq i \leq n-1$, and $-S_{n-1}$ denote the transposition $-(n-1, n)$.

In other words, if we have a permutation $(a_1, a_2, ..., a_n)$, then the fundamental reflections $_i$ interchange the coordinates of this permutation in the following way

$$\theta_i : (a_1, ..., a_i, a_{i+1}, ..., a_n) \longrightarrow (a_1, ..., a_{i+1}, a_i, ..., a_n)$$

And

$$\theta_i : (a_1, ..., a_{n-1}, a_n) \longrightarrow (a_1, ..., -a_n, -a_{n-1})$$

Then we conclude that the Weyl group of SO(2n; C) acts by permuting the coordinates and multiplying the coordinates by signs where only an even number of signs flips is allowed.

In general, the Weyl group is the semi-direct product $S_n \ltimes \mathbb{Z}_2^{n-1}$ of symmetric group and normal elementary abelian 2-subgroup.

Example 2.1. Consider the semisimple group SO(6; C), then the Weyl group will be

$$W(SO(6, \mathbb{C})) = \{(123), (213), (312), (321), (132), (231), (-1-23), (-2-13), (-3-12), (-3-21), (-1-32), (-2-31), (1-2-3), (2-1-3), (3-1-2), (3-2-1), (1-3-2), (2-3-1), (-12-3), (-21-3), (-31-2), (-32-1), (-13-2), (-23-1)\}.$$

III. Action of the Weyl group on an orthogonal basis

Define (r, s)-basis to be $(r_1, ..., r_n, s_n, ..., s_1)$ where $b(r_i, r_j) = b(s_i, s_j) = 0$ and $b(r_i, s_j) = \delta_{ij}$ for all i, j. Recall that a flag

$$F = (V_1 \subset V_2 \subset ... \subset V_{2n})$$

called a maximally b-isotropic flag if $V_i \subset V_i^{\perp}$ and $V_{2n-i}^{\perp} \subset V_{2n-i}$ for all $1 \leq i \leq n$.

If we have a maximally isotropic flag associated to some permutation of the basis r_i , s_i, then in the first n positions of the flag the above condition hold: if r_i appears, then s_i does not and vice versa. Now the full flag is determined by the first n positions: regard it as a permutation of the r_i and if s_i occurs instead of an r_i, we mark that spot with a minus sign such that the number of minus signs should be even.

Example 3.1.

$$\textit{Consider The basis } (\nu_1, \nu_2, u_2, u_1). \textit{ Choose } w = (-2\ \ -1), \textit{ then}$$

$$w \cdot (\nu_1, \nu_2, u_2, u_1) \longrightarrow (u_2, u_1, \nu_1, \nu_2).$$

IV. The Length Of The Weyl Elements

Recall that the length of an element w in a Weyl group W , denoted by l(w), is the smallest number k so that w is a product of k reflections by simple roots, see ([4],[5]). As indicated in the previous, we defined a special way of writing for the Weyl elements, so let us state how can we compute the length of elements $w \in S_n \ltimes \mathbb{Z}_2^n$ relative to our notation for the Weyl group elements.

Lemma 4.1. *Fix $w \in S_n \ltimes \mathbb{Z}_2^n$, construct an $\tilde{w} \in S_n \ltimes \mathbb{Z}_2^n$ by the following algorithm:*

1. Start from left to right in w, using simple reflections, place all positive numbers in w step by step in a sequence of n empty boxes beginning from the first one in $\tilde{w}$, in the same order as they appeared in w.
2. From left to right in w replace a negative number with its absolute value in the n empty boxes starting from right to left.

If $\tilde{w} = (\tilde{w}(1), \tilde{w}(2), ..., \tilde{w}(n))$, then define $L(\tilde{w}) = \frac{n^2-n}{2}$ – number of $\{\tilde{w}(i) : i < k$ and $\tilde{w}(i) < \tilde{w}(k)\}$, and if we have m negative signs in w in the following positions $j_1, j_2, ..., j_m$, then define $f(w) = \sum_{i=1}^{m}[(n - j_i)]$.It follows that the length of w is

$$l(w) = L(\tilde{w}) + f(w) + m$$

Proof. The length of the permutation $w \in S_n \ltimes \mathbb{Z}_2^n$ is the minimal number of simple reflection which define w. To compute this note that we have two kind of reflections , the first $n-1$ reflections are the simple reflections in S_n and the last one is the reflection which flips the sign. This means that if we want to flip the sign of $w(j)$, we should move $w(j)$ from its position to the last position and then flip its sign and then return it back to its position. More precisely, consider the word $w_0 = (123...n)$ and let w be the word where $w_{j_i}, w_{j_2}, ..., w_{j_m}$ in the positions $j_1 < j_2 < ... < j_m$, $1 \leq j_i \leq n$, are negatives. To construct w from w_0, we first flip the signs for the numbers $w_{j_i}, w_{j_2}, ..., w_{j_m}$. For this purpose we move each number ,starting from w_{j_m}, to the last position and then flip the sign of it. Then we apply the simple reflection to the positive numbers to put them in the same order as they appear in the word w. In this way the sum of all these movements and flips is exactly $L(\tilde{w}) + m$. The last step is to move each w_{j_i} to its original position in w and denote the total number of these movements by $f(w)$. It then follows that $l(w) = L(\tilde{w}) + f(w) + m$. □

Example 4.2. *Let* $w = ((-12)5(-34) \in S_5 \ltimes \mathbb{Z}_2^5$, *then by following the above remark we have*

$$(-12)5(-34) \Longrightarrow 254(-1-3) \Longrightarrow 254(31)$$

so $w = (-12)5(-34)$ *and* $\tilde{w} = 254(31)$, *then* $L(\tilde{w}) = 7$ *and* $f(w) = 5$.
Hence $l(w) = 7 + 5 + 2 = 14$.

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