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On the Approximation of Quadratic Objective functions in Unconstrained Nonlinear Optimization Programs

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Abstract: *Practical optimization problems frequently involve nonlinear behaviors that must be taken into account. Approaches to nonlinear optimization problems often utilize approximation of complicated functions by simpler ones which are easier to calculate, and which show the relations between the variables more clearly. In this paper, we obtained an approximate optimal solution to a convex quadratic objective function by quadratic approximation. This approach was applied to a real-world numerical example to obtain the optimum which is the same as that from the conventional method.*

Keywords: *Objective function approximation, Optimal solution, Quadratic approximation, Nonlinear optimization programs.*

Contribution/Originality: *This study contributes to the existing literature on the use of quadratic approximation method. It explicitly provides the values of the parameters in the given quadratic model. Basically, the first time applied to obtain the optimum value to a real-world optimization problem.*

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I. Introduction

1.1 Problem Definition

Practical optimization Programs frequently involve nonlinear behavior that must be taken into account. There are different types of Nonlinear Optimization Programs (NOPs), depending on the characteristics of the objective function $f(x)$ and the constraints $g_i(x)$ functions. Different algorithms are used for the different types of NOPs. NOPs arise continuously in a wide range of fields and this creates the need for effective methods of solving them. Approaches to NOPs often utilize approximations.

The fundamental problem of approximation theory is to resolve a possibly complicated function called the 'target function' by simpler easier to compute function called the 'Approximant'. Let $f^*(x)$ be the approximant and $f(x)$ the target function. If $f(x)$ is continuous and differentiable k times, $k = 1, 2, \ldots, n$, then as k increases the error in approximation reduces. This means the approximating function, $f^*(x)$ (Approximant) becomes closer to the function $f(x)$ being approximated (Target function). That is, for $\varepsilon > 0$ and very small, we have the error: $\left| f^*(x) - f(x) \right| \leq \varepsilon, \quad or \quad \underset{k \to n}{Lim}\, \varepsilon_k \leq \varepsilon,\ k = 1, 2, \ldots, n.$

Nonlinear models are inherently much more difficult to optimize especially as it is hard to distinguish a local optimum from a global one. We show that, this is only possible when the approximating function is assumed convex. In this work, we consider any unconstrained convex quadratic objective function and device an approximation approach to obtain the approximate optimal solution of such an objective function. The main idea behind the approach is to compare the result obtained to that from calculus and the conventional method. The beauty lies in the smooth convergence to the optimum.

1.2 Previous Research

Approximation theory has been applied to address various problems in the physical sciences, see for example [1], [2], [3], [4], [5], [6], [7] and the references contained therein, and has continued to be a subject of particular attention for a number of researchers. [8] provided some theoretical estimates for the approximation error required for the learning theory in an interpolation space of the couple $(\mathbb{B}, \mathbb{H})$. [9] reviewed the approximation error theory and investigated the interplay between the parameters in optical diffusion tomography. [10] estimated several kinetic parameters using seven iterative, nonlinear estimation methodologies, [11] considered positive approximation on continuous multi-functions. [12] used the Cesaro sub-method to approximate Lipschitz class of functions to obtain sharper estimates (Approximants), [13] approximated a new special error function with its CDF in terms of both Chebyshev polynomials and the error function. [14] obtained the similarity for the best approximation degree of functions in weighted space and the best approximation degree in the same space by trigonometric polynomials, and [15] obtained the degree of signals (functions) belonging to the class $Lip(\alpha, p, w)$ by the summability method. For more on the applications of general approximation theory see [16], [17], [18].

Interests have also been centered on the approximations of nonlinear functions. [19] and [20] developed a nonlinear approximations of the equation of motion with respect to problems with mixed boundary conditions, and to the growth of cosmic perturbations respectively. [21] investigated the $L_p -$ error of approximation to a function by obtaining a linear combination of n exponentials on the d – dimensional torus. Both [22] and [23] studied nonlinear n-term approximations in piecewise differentiable polynomials in $\Re^2$ from hierarchical sequences of stable local basis and those generated by multilevel nested triangulations respectively. [24] focuses on nonlinear approximation in L_p from regular piecewise polynomials in $\Re^2$. For general nonlinear approximation see [25].

The aspect of objective function approximation was treated by [26] and [27]. [26] computed the CM- and S-estimates by localizing the global minimum of an objective function with an inequality constraint. [31] presented new methods for solving NOPs that will address the difficulties associated with handling nonlinear constrained and nonlinear objective functions. The methods obtained good solutions by using Lagrange-Multiplier-based formulation to handle nonlinear constraints, and by using a new trace-based global search method to overcome local minima in the objective functions. [32] proposed a new method to obtain the fuzzy optimal solution of the NOPs with linear constraints.

All these are the vast applications of approximation theory. However, none of these works have specifically considered the optimality of a convex quadratic objective function. The approximation of such a function is the focus of this study. The approach is verifiable by a numerical example.

II. Nonlinear Optimization Programs (NOPS)

Nonlinear programming is a technique that deals with optimization programs where the objective functions or constraints, or both involve nonlinear mathematical functions. In one general form, the Nonlinear Programming Programs (NPPs) or NOPs is to find $X = (x_1, x_2, \ldots, x_n)^T$ such as to:

$$\left.\begin{array}{l} Optimize\ f(X) \\ Subject\ to: g_i(X)\ =,\ \leq,\ or\ \geq\ b_i,\quad i = 1,2,\ldots,m \\ \qquad\qquad X \geq 0 \end{array}\right\} \qquad (1)$$

Where $f : \Re^n \to \Re$ is a continuous (smooth) and differentiable real-valued objective function of the vector $X \in \Re^n$. $X = (x_1, x_2, \ldots, x_n)^T$ contains all the decision variables of the NOPs/NPPs. $g_i(X)$ are constraint functions of the vector $X \in \Re^n$.

Some design programs do not have constraints, or the constraints are negligible. Unconstrained NOPs/NPPs have many practical applications. The general form of the unconstrained NOPs/NPPs (Free Mathematical Programs) is given by:

$$\left.\begin{array}{l} Optimize\ f(X) \\ Subject\ to: X \in \Re^n \end{array}\right\} \qquad (2)$$

Where X, f are already defined in (1). If (2) is single-variable, the search for the optimum is conducted over the entire $n-$ dimensional (Euclidean) space or infinite interval $(-\infty, \infty)$. If the search is restricted to a finite subinterval $[a,b]$, then the program becomes:

$$\left.\begin{array}{l} Optimize\ f\left(X\right) \\ Subject\ to: a \le X \le b,\ \ a,b \in \Re \end{array}\right\} \tag{3}$$

Nonlinear Programming algorithms generally are unable to distinguish between a local optima and a global optima. In practice, one usually does not know whether a particular objective function $f\left(X\right)$ is unimodal over a specified interval. When a search procedure is applied in such a situation, there is no assurance it will uncover the desired global optimum. In practice, one usually does not know whether a particular objective function $f\left(X\right)$ is unimodal over a specified interval. When a search procedure is applied in such a situation, there is no assurance it will uncover the desired global optimum on a search interval except when $f\left(X\right)$ is unimodal on the interval. Exceptions include programs that have convex or concave objective functions.

III. Optimality of Nonlinear Optimization Programs/Nonlinear Programming Programs

Since the goal of optimization is to locate and find the optimal solution, the notion and theory of Optimality is essential. To find the optimal solution of (2), we find the extremum (local and global) points. An extremum of a function $f\left(X\right)$ defines either the maximum or minimum of the function $f\left(X\right)$. How do we know and find the extremum point X^* of a given function $f:\Re^n \to \Re$?

Mathematically, a feasible solution $X^* \in \Re^n$ is a globally optimal solution of $f\left(X\right)$ if

$$\left.\begin{array}{l} f\left(X^*\right) \le f\left(X\right) \Rightarrow\ global\ \min imum \\ f\left(X^*\right) \ge f\left(X\right) \Rightarrow\ global\ \max imum \end{array}\right\}\ \ \forall X \in \Re^n \tag{4}$$

A feasible solution $X^* \in \Re^n$ is a locally optimal solution of $f\left(X\right)$ if there exists a $\varepsilon > 0$ very small, such that: $\left\|X - X^*\right\| < \varepsilon$ and $\left\|X^* - X\right\| < \varepsilon$ respectively satisfies:

$$\left.\begin{array}{l} f\left(X^*\right) \le f\left(X\right) \Rightarrow\ local\ \min imum \\ f\left(X^*\right) \ge f\left(X\right) \Rightarrow\ local\ \max imum \end{array}\right\}\ \ \forall X \in \Re^n \cap B_\varepsilon\left(X^*\right) \tag{5}$$

Where $B_\varepsilon\left(X\right) = \left\{X \middle| \left\|X - X^*\right\| < \varepsilon\right\}$

It is desired to know the conditions under which any local optima is guaranteed to be a global optima. If a NPP has no constraints, the objective function being Concave (Convex), it guarantees that a local maximum (minimum) is a global maximum (minimum). Given that, for each pair of values of X, say $x_1, x_2 \in I\left(x_1 < x_2\right)$ and $\lambda \in [0,1]$, then a function of single variable $f\left(X\right)$ on an interval I (finite or infinite) is:

Convex if $\quad f\left[\lambda x_1 + (1-\lambda)x_2\right] \le \lambda f\left(x_1\right) + (1-\lambda) f\left(x_2\right)$

Concave if $\quad f\left[\lambda x_1 + (1-\lambda)x_2\right] \ge \lambda f\left(x_1\right) + (1-\lambda) f\left(x_2\right)$

Then the first order necessary condition is, assume $f\left(X\right)$ is convex (concave) and differentiable, then the local minimum(maximum) X^* is a global minimize (maximize) of $f\left(X\right)$, if and only if $\nabla f\left(X^*\right) = 0$. That is the minimization (maximization) of $f\left(X\right)$ is equivalent to the solution of the equation $\nabla f\left(X^*\right) = 0$. Also, the second order sufficient condition is that, if $f\left(X\right)$ is twice differentiable, then $f\left(X\right)$ is:

Convex $\quad \Leftrightarrow \quad \nabla^2 f\left(X\right) \ge 0, \quad \forall X \in \Re^n$

Concave $\quad \Leftrightarrow \quad \nabla^2 f\left(X\right) \le 0, \quad \forall X \in \Re^n$

IV. Objective Function Approximation

$$Let\ f(x) = \delta_0 + \delta_1 x + \delta_2 x^2 \ be\ approximated\ by\ f^{\bullet}(x) = \beta_0 + \beta_1 x + \beta_2 x^2 \tag{6a}$$

$$such\,that \quad \beta_0 = \delta_0,\ \beta_1 = \delta_1 \ and \ \beta_2 = \delta_2$$

$$then\ \frac{df^*}{dx} = \beta_1 + 2\beta_2 x = 0 \quad \Rightarrow \quad x_{opt} \quad = \quad \frac{-\beta_1}{2\beta_2} \tag{6b}$$

$$Let\ x_1, x_2\ and\ x_3\ be\ points\ on\ f(x)$$
$$so\ that \quad f(x_1) = f^*(x_1),\ f(x_2) = f^*(x_2)\ and\ f(x_3) = f^*(x_3)$$

Then,

$$\beta_0 + \beta_1 x_1 + \beta_2 x_1^2 \ = \ f(x_1) \qquad (*)$$
$$\beta_0 + \beta_1 x_2 + \beta_2 x_2^2 = f(x_2) \qquad (**)$$
$$\beta_0 + \beta_1 x_3 + \beta_2 x_3^2 = f(x_3) \qquad (***)$$

$$Solving\ for\ \beta_0, \beta_1\ and\ \beta_2,\ we\ obtained:$$

$$\beta_0 = \frac{\begin{bmatrix} f_1 & x_1 & x^2{}_1 \\ f_2 & x_2 & x^2{}_2 \\ f_3 & x_3 & x^2{}_3 \end{bmatrix}}{\begin{bmatrix} 1 & x_1 & x^2{}_1 \\ 1 & x_2 & x^2{}_2 \\ 1 & x_3 & x^2{}_3 \end{bmatrix}},\ \beta_1 = \frac{\begin{bmatrix} 1 & f_1 & x^2{}_1 \\ 1 & f_2 & x^2{}_2 \\ 1 & f_3 & x^2{}_3 \end{bmatrix}}{\begin{bmatrix} 1 & x_1 & x^2{}_1 \\ 1 & x_2 & x^2{}_2 \\ 1 & x_3 & x^2{}_3 \end{bmatrix}}\ and\ \beta_2 = \frac{\begin{bmatrix} 1 & x_1 & f_1 \\ 1 & x_2 & f_2 \\ 1 & x_3 & f_3 \end{bmatrix}}{\begin{bmatrix} 1 & x_1 & x^2{}_1 \\ 1 & x_2 & x^2{}_2 \\ 1 & x_3 & x^2{}_3 \end{bmatrix}} \tag{7}$$

$\Rightarrow$

$$\beta_0 = \frac{f_1\left(x_3^2 x_2 - x_3 x_2^2\right) - x_1\left(x_3^2 f_2 - f_3 x_2^2\right) + x_1^2\left(x_3 f_2 - f_3 x_2\right)}{1\left(x^2{}_3 x_2 - x_3 x^2{}_2\right) - x_1\left(x^2{}_3.1 - 1.x^2{}_2\right) + x^2{}_1\left(x_3.1 - 1.x_2\right)}$$
$$= \frac{f_1\left(x_3^2 x_2 - x_3 x_2^2\right) - x_1\left(x_3^2 f_2 - f_3 x_2^2\right) + x_1^2\left(x_3 f_2 - f_3 x_2\right)}{\left(x_3^2 x_2 - x_3 x_2^2\right) - x_1\left(x_3^2 - x^2{}_2\right) + x_1^2\left(x_3 - x_2\right)} \tag{8}$$

$$\beta_1 = \frac{1\left(x^2{}_3 f_2 - f_3 x^2{}_2\right) - f_1\left(x^2{}_3.1 - 1.x^2{}_2\right) + x^2{}_1\left(f_3.1 - 1.f_2\right)}{1\left(x^2{}_3 x_2 - x_3 x^2{}_2\right) - x_1\left(x^2{}_3.1 - 1.x^2{}_2\right) + x^2{}_1\left(x_3.1 - 1.x_2\right)}$$
$$= \frac{\left(x^2{}_3 f_2 - f_3 x^2{}_2\right) - f_1\left(x^2{}_3 - x^2{}_2\right) + x^2{}_1\left(f_3.- f_2\right)}{\left(x^2{}_3 x_2 - x_3 x^2{}_2\right) - x_1\left(x^2{}_3 - x^2{}_2\right) + x^2{}_1\left(x_3 - x_2\right)} \tag{9}$$

$$\beta_2 = \frac{1\left(f_3 x_2 - x_3 f_2\right) - x_1\left(f_3.1 - 1.f_2\right) + f_1\left(x_3.1 - 1.x_2\right)}{1\left(x^2{}_3 x_2 - x_3 x^2{}_2\right) - x_1\left(x^2{}_3.1 - 1.x^2{}_2\right) + x^2{}_1\left(x_3.1 - 1.x_2\right)}$$

$$= \frac{\left(f_3x_2 - x_3f_2\right) - x_1\left(f_3 - f_2\right) + f_1\left(x_3 - x_2\right)}{\left(x^2{}_3x_2 - x_3x^2{}_2\right) - x_1\left(x^2{}_3 - x^2{}_2\right) + x^2{}_1\left(x_3 - x_2\right)} \quad \textbf{(10)}$$

Plugging (9) *and* (10) int*o* (6), *we obtain*:

$$x^* = x_{opt} = \frac{-\beta_1}{2\beta_2} = \frac{-\left[\dfrac{\left(x^2{}_3f_2 - f_3x^2{}_2\right) - f_1\left(x^2{}_3 - x^2{}_2\right) + x^2{}_1\left(f_3 - f_2\right)}{\left(x^2{}_3x_2 - x_3x^2{}_2\right) - x_1\left(x^2{}_3 - x^2{}_2\right) + x^2{}_1\left(x_3 - x_2\right)}\right]}{2\left[\dfrac{\left(f_3x_2 - x_3f_2\right) - x_1\left(f_3 - f_2\right) + f_1\left(x_3 - x_2\right)}{\left(x^2{}_3x_2 - x_3x^2{}_2\right) - x_1\left(x^2{}_3 - x^2{}_2\right) + x^2{}_1\left(x_3 - x_2\right)}\right]}$$

$$= \frac{-x_3{}^2f_2 + f_3x_2{}^2 + f_1\left(x_3{}^2 - x_2{}^2\right) - x_1^2\left(f_3 - f_2\right)}{x_3{}^2x_2 - x_3x_2{}^2 - x_1\left(x_3{}^2 - x_2{}^2\right) + x_1^2\left(x_3 - x_2\right)} \times \frac{x_3{}^2x_2 - x_3x_2{}^2 - x_1\left(x_3{}^2 - x_2{}^2\right) + x_1^2\left(x_3 - x_2\right)}{2\left[f_3x_2 - f_2x_3 - x_1\left(f_3 - f_2\right) + f_1\left(x_3 - x_2\right)\right]}$$

$$\Rightarrow \quad \frac{-x_3{}^2f_2 + f_3x^2{}_2 - f_1\left(x_3{}^2 - x_2{}^2\right) - x_1^2\left(f_3 - f_2\right)}{2\left[f_3x_2 - x_3f_2 - x_1\left(f_3 - f_2\right) + f_1\left(x_3 - x_2\right)\right]} \quad (11)$$

Now, simplifying (11) *by letting* $x_1 = x_2 - \lambda$ *and* $x_3 = x_2 + \lambda$

$$x_{Opt} = \frac{-\left(x_2 + \lambda\right)^2 f_2 + f_3x_2{}^2 + f_2\left(\left(x_2 + \lambda\right)^2 - x_2{}^2\right) - \left(x_2 - \lambda\right)^2\left(f_3 - f_2\right)}{2\left[f_3x_2 - f_2\left(x_2 + \lambda\right) - \left(x_2 - \lambda\right)\left(f_3 - f_2\right) + f_1\left(x_2 + \lambda - x_2\right)\right]}$$

$$x_{Opt} = \frac{-2x_2\lambda f_2 + f_1 2x_2\lambda + f_1\lambda^2 + 2x_2\lambda f_3 - 2x_2\lambda f_2 - f_3\lambda^2}{2\left[-\lambda f_2 + \lambda f_3 - \lambda f_2 + \lambda f_1\right]}$$

$$x_{Opt} = \frac{\lambda f_1\left(2x_2 + \lambda\right) - 4x_2\lambda f_2 + \lambda f_3\left(2x_2 - \lambda\right)}{2\lambda\left(f_1 - 2f_2 + f_3\right)}$$

$$\Rightarrow \quad X^* = x_{Opt} = \frac{f_1\left(\lambda + 2x_2\right) - 4f_2x_2 + f_3\left(2x_2 - \lambda\right)}{2\left(f_1 - 2f_2 + f_3\right)} \bullet \quad (12)$$

V. Numerical Example

The number of bacteria in a refrigerated food is given by:

$N(T) = 20T^2 - 20T + 120$, for $-2 \le T \le 14$ (13)

and where T is the temperature of the food in Celsius. At what temperature will the number of bacteria be minimal?

5.1. Conventional Approach

5.1.1. By Calculus

From (13) $\dfrac{dN(T)}{dT} = 40T - 20$. But the minimum is at:

$$\frac{dN(T)}{dT} = 0 \quad \Rightarrow \quad 40T - 20 = 0 \quad \Rightarrow \quad T^* = \frac{1}{2}c^0 \qquad (14)$$

5.1.2. Approximation

Let (13) be approximated by

$$N^\bullet(T) = \beta_2 T^2 + \beta_1 T + \beta_0 \qquad (15)$$

$$\Rightarrow \quad \frac{dN^\bullet(T)}{dT} = 2\beta_2 T + \beta_1 \;\; and \;\; \frac{d^2N^\bullet(T)}{dT^2} = 2\beta_2 \qquad (16).$$

But the minimum temperature occurs at:

$$\frac{dN^\bullet(T)}{dT} = 0 \quad \Rightarrow \quad 2\beta_2 T + \beta_1 = 0 \quad \Rightarrow \quad T^* = \frac{-\beta_1}{2\beta_2} \qquad (17)$$

$$From \;\; (13) \quad \frac{dN(T)}{dT} = 40T - 20 \quad and \quad \frac{d^2N(T)}{dT^2} = 40 \qquad (18)$$

$$But \quad N(T) = N^\bullet(T), \quad \frac{dN(T)}{dT} = \frac{dN^\bullet(T)}{dT} \quad and \quad \frac{d^2N(T)}{dT^2} = \frac{d^2N^\bullet(T)}{dT^2}$$

$$Therefore\ at\ T_1 = 2 \in [-2,14],\ we\ have:$$

$$N(2) = N^\bullet(2) \quad \Rightarrow \quad 20(2)^2 - 20(2) + 120 = \beta_2(2)^2 + \beta_1(2) + \beta_0$$

$$\Rightarrow \quad 160 = 4\beta_2 + 2\beta_1 + \beta_0 \qquad (19)$$

$$Also, \quad \frac{dN(2)}{dT} = \frac{dN^\bullet(2)}{dT} \quad \Rightarrow \quad 40(2) - 20 = 2\beta_2(2) + \beta_1$$

$$\Rightarrow \quad 60 = 4\beta_2 + \beta_1 \qquad (20)$$

$$Again, \quad \frac{d^2N(2)}{dT^2} = \frac{d^2N^\bullet(2)}{dT^2} \quad \Rightarrow \quad 40 = 2\beta_2 \quad \Rightarrow \quad \beta_2 = 20 \qquad (21)$$

$$Hence\,(20)\ \ becomes, \quad 60 = 4(20) + \beta_1 \quad \Rightarrow \quad \beta_1 = -20 \qquad (22)$$

$$(19)\ \ becomes, \quad 160 = 4(20) + 2(-20) + \beta_0 \quad \Rightarrow \quad \beta_0 = 120 \qquad (23)$$

$$\Rightarrow \quad N^\bullet(T) = 20T^2 - 20T + 120 \qquad (24)$$

$$and\ \ from\,(17), \quad T^* = \frac{-\beta_1}{2\beta_2} = \frac{-(-20)}{2(20)} = \frac{1}{2}c^0 \qquad (25)$$

5.1.3. Our Approach

From $(6a)$ and (13), letting $f_i = N_i$ *so that* $x_i = T_i,\ i = 1,2,3.$

$$Then\ for\ T_1 = -1 \Rightarrow N_1 = 160, T_2 = 4 \Rightarrow N_2 = 360\ and\ T_3 = 9 \Rightarrow N_3 = 1560, \quad \forall\, T \in [-2,14]\ we\ have:$$

$$\beta_0 = \frac{N_1\left(T_3^2T_2 - T_3T_2^2\right) - T_1\left(T_3^2N_2 - N_3T_2^2\right) + T_1^2\left(T_3N_2 - N_3T_2\right)}{T_3^2T_2 - T_3T_2^2 - T_1\left(T_3^2 - T_2^2\right) + T_1^2\left(T_3 - T_2\right)}$$

$$\beta_0 = \frac{160\left(9^2 \cdot 4 - 9 \cdot 4^2\right) - (-1)\left(9^2 \cdot 360 - 1560 \cdot 4^2\right) + (-1)^2\left(9 \cdot 360 - 1560 \cdot 4\right)}{9^2 \cdot 4 - 9 \cdot 4^2 - (-1)\left(9^2 - 4^2\right) + (-1)^2(9-4)}$$

$$\beta_0 = \frac{160 \cdot 180 + 4200 - 3000}{324 + 70 - 144} = \frac{30000}{250} = 120 \tag{26}$$

$$\beta_1 = \frac{T_3^2N_2 - N_3T_2^2 - N_1\left(T_3^2 - T_2^2\right) + T_1^2\left(N_3 - N_2\right)}{T_3^2T_2 - T_3T_2^2 - T_1\left(T_3^2 - T_2^2\right) + T_1^2\left(T_3 - T_2\right)}$$

$$\beta_1 = \frac{9^2 \cdot 360 - 1560 \cdot 4^2 - 160\left(9^2 - 4^2\right) + (-1)^2(1560 - 360)}{9^2 \cdot 4 - 9 \cdot 4^2 - (-1)\left(9^2 - 4^2\right) + (-1)^2(9-4)}$$

$$\beta_1 = \frac{4200 + 1200 - 10400}{324 + 70 - 144} = \frac{-5000}{250} = -20 \tag{27}$$

$$\beta_2 = \frac{N_3T_2 - T_3N_2 - T_1\left(N_3 - N_2\right) + N_1\left(T_3 - T_2\right)}{T_3^2T_2 - T_3T_2^2 - T_1\left(T_3^2 - T_2^2\right) + T_1^2\left(T_3 - T_2\right)}$$

$$\beta_2 = \frac{1560 \cdot 4 - 9 \cdot 360 - (-1)(1560 - 360) + 160(9-4)}{9^2 \cdot 4 - 9 \cdot 4^2 - (-1)\left(9^2 - 4^2\right) + (-1)^2(9-4)}$$

$$\beta_2 = \frac{3000 + 1200 + 800}{324 + 70 - 144} = \frac{5000}{250} = 20 \tag{28}$$

$$From\,(11),\quad T^* = t_{Min} = \frac{-\beta_1}{2\beta_2} = \frac{-T_3^2N_2 + N_3T_2^2 + N_1\left(T_3^2 - T_2^2\right) - T_1^2\left(N_3 - N_2\right)}{2\left[N_3T_2 - N_2T_3 - T_1\left(N_3 - N_2\right) + N_1\left(T_3 - T_2\right)\right]}$$

$$T^* = t_{Min} = \frac{-9^2\cdot 360 + 1560\cdot 4^2 + 160\left(9^2 - 4^2\right) - (-1)^2\left(1560 - 360\right)}{2\left[1560\cdot 4 - 360\cdot 9 - (-1)(1560 - 360) + 160(9 - 4)\right]}$$

$$T^* = t_{Min} = \frac{-29160 + 24960 + 160\cdot 65 - 1(1200)}{2\left[6240 - 3240 + 1200 + 800\right]} = \frac{35360 - 30360}{2\left[8240 - 3240\right]} = \frac{500}{1000} = \frac{1}{2}c^0 \qquad (29)$$

$$Given\ that\ T_1 = T_2 - \lambda,\ and\ T_3 = T_2 + \lambda,\ for\ \lambda = 5\ (Simplifying\ Parameter),\ we\ have:$$

$$T^* = t_{Min} = \frac{N_1\left(\lambda + 2T_2\right) - 4N_2T_2 + N_3\left(2T_2 - \lambda\right)}{2\left[N_1 - 2N_2 + N_3\right]}$$

$$T^* = t_{Min} = \frac{160\left(5 + 2\cdot 4\right) - 4\cdot 360\cdot 4 + 1560\left(2\cdot 4 - 5\right)}{2\left[160 - 2\cdot 360 + 1560\right]} = \frac{2080 - 5760 + 4680}{2\left[1000\right]} = \frac{1000}{2000} = \frac{1}{2}c^0 \qquad (30)$$

VI. Conclusion

Nonlinear models are inherently much more difficult to optimize especially as it is hard to distinguish a local optimum from a global optimum. We showed that this is only possible when the approximating function is convex. In this paper, we have been able to obtain the approximate optimal solution of an unconstrained convex quadratic objective function by quadratic approximation. The optimal solution to a real-life numerical problem is obtained first by calculus, second by the usual conventional approximation method, and thirdly by this study's approach. Comparing the parameters of the convex quadratic model $\beta_0,\ \beta_1\ and\ \beta_2$ in equations: (21) and (28), $(22)\ and\ (27)$, and $(23)\ and\ (26)$, and also the optimal (minimum) solution in the conventional $(14)\ and\ (25)$, and $(29)\ and\ (30)$ based on our approach shows they are the same. However, this programs can also be looked upon under policy constraints as further study.

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