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Stability Analysis of A General SEIR Epidemic Model with Homogenous Transmission Function and Treatment Rate

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Abstract: *In this paper, we consider a SEIR epidemic model with homogenous transmission function and treatment. Found the basic reproduction number* R_0 *and equilibrium points namely disease-free equilibrium and endemic equilibrium. The global stability of the disease free equilibrium and endemic equilibrium is proved using Lyapunov function and Poincare-Bendixson theorem plus Dulac's criterion respectively and also study the sociological and psychological effect on the infected population. We gave some numerical result to analyze our model with actual model.*

MSC 2010: *34D20, 37B25, 93A30*

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I. Introduction

Many authors have studied the epidemic models. Most of the authors are curious in the conception of the incidence rate, i.e. communicable rate of affected individuals through their contacts with contaminated individuals [2, 4, 7]. Several authors employ the bilinear incidence rate $\beta S I,$ where β is the transmission coefficient, S and I are the number of susceptible and infected individuals in the population respectively. Laarabi[1] describe the second and third incidence rate of the form $\frac{\beta S I}{1+\alpha_1 S}$, $\frac{\beta S I}{1+\alpha_2 I}$ respectively where the positive constant is α_1 epidemic control strems the effects of intensity factor(refer to α_1) and α_2 is the positive constant. Due to the bunch the contaminate individual or due to the protection measures by the affected individuals, the number of effective contacts in this incidence rate between contaminate and affected individuals may saturate at high communicating level. Pathak [5] considered the transmission rate $\varphi = \frac{KSI}{1+\alpha S+\beta I}$ which displayed a saturation effect accounting for the fact that the number of contacts in individual reaches some maximal value done to spatial or social distribution of the population. Besides the rate and nature of incidence, treatment plays an important role the spread of disease. Wang [8] proposed a treatment function.

$$T(I) = \begin{cases} rI, \ if \ 0 \le I \le I_0 \\ k, if \ I > I_0 \end{cases}$$

where $k = rI_o$

II. Mathematical model

Sharma et al. [6] considered an general $SEIR$ epidemic model with a saturated incidence rate

$$\begin{aligned}
\frac{dS}{dt} &= A - \frac{\beta S I}{1+\alpha I} - \mu S \\
\frac{dE}{dt} &= \frac{\beta S I}{1+\alpha I} - (\varepsilon + \mu) E \\
\frac{dI}{dt} &= \varepsilon E - (\mu + r + d) I - T(I) \\
\frac{dR}{dt} &= r I - \mu R + T(I)
\end{aligned} \tag{2.1}$$

The Proposed Model

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{\beta S I}{1+aS+bI} - \mu S \\
\frac{dE}{dt} &= \frac{\beta S I}{1+aS+bI} - (\gamma + \mu) E \\
\frac{dI}{dt} &= \gamma E - (\delta + \mu) I - T(I) \\
\frac{dR}{dt} &= \delta I - \mu R + T(I)
\end{aligned} \tag{2.2}$$

Parameter	description
Λ	recruitment rate
γ	rate of developing infectivity
δ	recovery rate
μ	natural death rate

$\frac{\beta S I}{1+aS+bI}$ transmission rate, where a and b are the parameters which measure the effects of sociology, psychological or other mechanisms.

The first four equations of system (2.2) do not contain R, we need to study the following system

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{\beta S I}{1+aS+bI} - \mu S \\
\frac{dE}{dt} &= \frac{\beta S I}{1+aS+bI} - (\gamma + \mu) E \\
\frac{dI}{dt} &= \gamma E - (\delta + \mu + r) I - T(I)
\end{aligned} \tag{2.3}$$

It follows from system (2.3)

$\frac{d}{dt}(S+E+I)=\Lambda-\mu(S+E+I)-(\delta+r)I-T(I)\le\Lambda-\mu(S+E+I)$ Then

$\limsup_{t\to\infty}(S+E+I)\le\frac{\Lambda}{\mu}$. So the feasible region for system (2.3) is

$$\Omega=\left\{(S,E,I):S+E+I\le\frac{\Lambda}{\mu},S>0,\ E\ge0,\ I\ge0\right\}$$

The region Ω is positively invariant with respect to system (2)

Equilibria The system (1) has always the disease-free equilibrium $P_0(S_0,E_0,I_0,R_0)=\left(\frac{\Lambda}{\mu},0,0,0\right)$ and endemic equilibrium $P_*=(S_*,E_*,I_*,R_*)$.

$$S_*=\frac{(\mu+\gamma)(\delta+\mu+r)[1+\beta I]}{\beta\gamma-a(\delta+\mu)(\delta+\mu+r)}$$

$$E_*=\frac{(\delta+\mu+r)I}{\varepsilon}$$

$$I_*=\frac{\gamma\left[\Lambda\beta\gamma-(\mu+\gamma)(\mu+\delta+r)(\Lambda a+\mu)\right]}{(\mu+\gamma)(\mu+\delta+r)\left[\beta\mu\gamma+\beta\gamma-a(\mu+\gamma)(\mu+\delta+r)\right]}$$

$$R_*=\frac{(\delta+r)I}{\mu}$$

Reproduction Number

To find the reproduction number using next generation method

Let

$$X=(E,I,S)$$

$$\frac{dX}{dt}=F(X)-V(X)$$

Where

$$F=\begin{bmatrix}0&\frac{\beta\Lambda}{\mu+a\Lambda}&0\\0&0&0\\0&0&0\end{bmatrix},\quad V=\begin{bmatrix}-(\gamma+\mu)&0&0\\\gamma&-(\delta+\mu+r)&0\\0&0&-\mu\end{bmatrix}$$

$$V^{-1}=\frac{1}{\mu(\gamma+\mu)(\delta+\mu+r)}\begin{bmatrix}(\gamma+\mu)&0&0\\-\gamma&(\delta+\mu+r)&0\\0&0&\mu\end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\mu(\delta+\mu+r)} & 0 & 0 \\ \frac{\gamma}{\mu(\gamma+\mu)(\delta+\mu+r)} & \frac{1}{\mu(\gamma+\mu)} & 0 \\ 0 & 0 & -\frac{1}{(\gamma+\mu)(\delta+\mu+r)} \end{bmatrix}$$

$$FV^{-1} = \begin{bmatrix} \frac{\beta\Lambda\gamma}{(\mu+a\Lambda)\left[\mu(\gamma+\mu)(\delta+\mu+r)\right]} & \frac{\beta\Lambda}{(\mu+a\Lambda)\left[\mu(\gamma+\mu)\right]} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The spectral radius R_0 of the matrix $K = FV^{-1}$, is the basic reproduction number of the model $R_0 = \rho\left(FV^{-1}\right)$, so

$$R_0 = \frac{\beta\Lambda\gamma}{(\mu+a\Lambda)\left[\mu(\gamma+\mu)(\delta+\mu+r)\right]}$$

3. Local Stability

Theorem 3.1. *At P_0, the disease-free equilibrium of the system (1) is locally asymptotically stable when $R_0 < 1$.*

Proof.

$$P_0 = \begin{vmatrix} -(\mu) & 0 & -\frac{\beta\Lambda}{\mu+a\Lambda} \\ 0 & -(\gamma+\mu) & \frac{\beta\Lambda}{\mu+a\Lambda} \\ 0 & \gamma & -(\delta+\mu+r) \end{vmatrix}$$

$$-(\mu+\lambda)\left((\gamma+\mu+\lambda)(\delta+\mu+r-\lambda) - \frac{\beta\Lambda\gamma}{\mu+a\Lambda}\right) = 0$$

$$(\mu+\lambda)\left(\lambda^2 + (2\mu+\gamma+\delta+r)\lambda + (\mu+\gamma)(\delta+\mu+r) - \frac{\beta A\gamma}{\mu}\right) = 0$$

$$(\mu+\lambda)\left(\lambda^2 + (2\mu+\gamma+\delta+r)\lambda + (\mu+\gamma)(\delta+\mu+r) - (1-R_0)\right) = 0$$

$$\lambda^2 + p\lambda + q = 0 \qquad (3.1)$$

Where

$$p=\left(2\mu+\gamma+\delta+r\right)>0,$$
$$q=\left(\mu+\gamma\right)\left(\delta+\mu+r\right)-\left(1-R_0\right)$$

all the eigen values of the jacobian matrix are negative, If $R_0<1$, then $q>0$, so both roots of equation (3.1) are negative.

Theorem 3.2. *The system (1) is locally asymptotically stable at* P_* *if* $R_0>1$, *otherwise unstable.*

Proof. At the endemic equilibrium P_*, the jacobian matrix of the system (1) is given by

$$P_*=\begin{pmatrix} -\mu-\dfrac{\beta I^*\left(1+bI^*\right)}{\left(1+aS^*+bI^*\right)^2} & 0 & -\dfrac{\beta S^*\left(1+aS^*\right)}{\left(1+aS^*+bI^*\right)^2} \\ \dfrac{\beta I^*\left(1+bI^*\right)}{\left(1+aS^*+bI^*\right)^2} & -\left(\gamma+\mu\right) & \dfrac{\beta S^*\left(1+aS^*\right)}{\left(1+aS^*+bI^*\right)^2} \\ 0 & \gamma & -\left(\delta+\mu+r\right) \end{pmatrix}$$

The characteristic equation of the Jacobian matrix P_* is given by

$$t^3-Z_1+\left(\gamma+\mu\right)+\left(\delta+\mu+r\right)t^2-\left[Z_4-Z_2\left(\delta+\mu+r\right)-\gamma Z_2\right]t$$
$$-\left[Z_1Z_4+Z_3\left(\gamma Z_2\right)\right]=0$$

$$Z_1=-\mu-\frac{\beta I^*\left(1+bI^*\right)}{\left(1+aS^*+bI^*\right)^2}$$

$$Z_2=\frac{\beta I^*\left(1+bI^*\right)}{\left(1+aS^*+bI^*\right)^2}$$

$$Z_3=\frac{\beta S^*\left(1+aS^*\right)}{\left(1+aS^*+bI^*\right)^2}$$

$$Z_4=\left(\gamma+\mu\right)\left(\delta+\mu+r\right)-\gamma Z_3$$

all the roots are in the left-half plane. Therefore, the endemic equilibrium is stable.

4. Global Stability

Theorem 4.1. *The disease-free equilibrium of the model (1) is globally asymptotically stable if*

$$L=\gamma E+\left(\gamma+\mu\right)I$$
$$L^1=\gamma E^1+\left(\gamma+\mu\right)I^1$$
$$L^1=\gamma\left[\frac{\beta SI}{1+aS+bI}-\left(\gamma+\mu\right)E\right]+\left(\gamma+\mu\right)\left[\gamma E-\left(\delta+\mu+r\right)I\right]$$

$$L^1 = \gamma\left[\frac{\beta S I}{1+aS+bI}\right] - (\gamma+\mu)(\delta+\mu+r)I$$

$$L^1 = \left[\frac{R_0}{1+bI} - 1\right] I$$

If $I = 0, L^1 = 0$ but if $I \neq 0$ and $R_0 < 1, L^1 < 0$ Therefore, the disease free equilibrium is globally asymptotically stable.

Theorem 4.2. *The endemic equilibrium* P_* *of the system (1) is globally asymptotically stable if* $R_0 > 1.$

Proof[3]. In order to prove the result, we use Dulac plus Poincare Bendixson theorem as follow

$$H(S,E,I,R) = \frac{1}{S.E.I.R} \text{ where } S>0, E>0, I>0, R>0$$

Then

$$\nabla.(HF) = \frac{\partial}{\partial S}(H.F_1) + \frac{\partial}{\partial E}(H.F_2) + \frac{\partial}{\partial I}(H.F_3) + \frac{\partial}{\partial R}(H.F_4)$$

$$= \frac{\partial}{\partial S}\left[\frac{1}{SEIR}\left(A - \frac{\beta S I}{1+aS+b\,I} - \mu S\right)\right] +$$

$$\frac{\partial}{\partial E}\left[\frac{1}{SEIR}\left(\frac{\beta S I}{1+a\,S+bI} - (\gamma+\mu)E\right)\right]$$

$$+\frac{\partial}{\partial I}\left[\frac{1}{SEIR}\left(\gamma E - (\delta+\mu+r)I\right)\right]$$

$$\frac{\partial}{\partial R}\left[\frac{1}{SEIR}(\delta+r)I - \mu R\right]$$

$$= -\left[\frac{A}{S^2EIR} + \frac{\beta\, a}{ER(1+aS+bI)^2}\right] - \left[\frac{\beta}{E^2R(1+aS+bI)}\right]$$

$$-\left[\frac{\gamma}{SI^2R}\right] - \frac{(\delta+r)}{SER^2} < 0$$

Hence, by Dulac's criterion, there is no closed orbit in the first quadrant. Therefore, the endemic equilibrium is globally asymptotically stable.

5. Numerical Simulation

In this section, we shall discuss the stability of equilibriums of the model (1) through MATLAB using Lotka Volterra function. Suppose the parameters are $\Lambda = 7, \gamma = 2, \beta = 8, \mu = 2, \delta = 0.7,$ $r = 0.5, b = 3, a = 2.$ Let the initial value of S, E, I are 2,1,1 respectively Then we obtain $R_0 = 0.93 < 1,\ P_0 = (3.5, 0, 0)$, Therefore by theorem 4.1, P_0 is globally asymptotically stable (see in Fig 1)

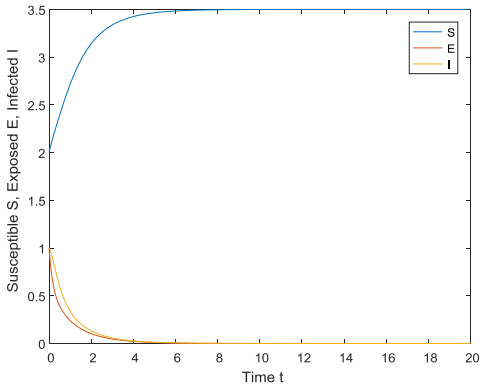

Figure 1. Disease free equilibrium if $R_0 < 1$

Again we take the parameters $\Lambda = 3.1,\ \beta = 9,\ \mu = 0.19,\ \gamma = 1.2,\ \delta = 1.5, r = 0.1,$ $a = 3.1, b = 4.7$ and $(S, E, I) = (2,1,1)$ Then $R_0 = 7.2\ > 1$ Therefore, by theorem 4.2 the endemic equilibrium P_* is globally asymptotically stable (see Fig 2.)

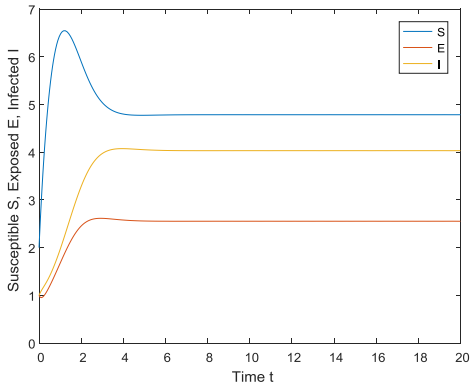

Figure 2. Endemic equilibrium point if $R_0 > 1$

If we change the values of a,b and Keeping other parameter fixed, it has been seen that I_* decreases as a, b are increases and the infected population on sociological and psychological effect rate seems to be similar as shown Fig. 3 and 4

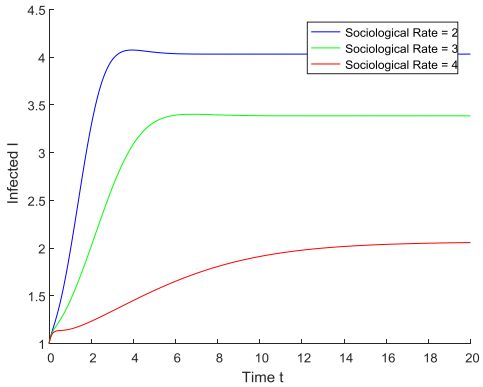

Figure 3. a sociological rate dependence of I^*

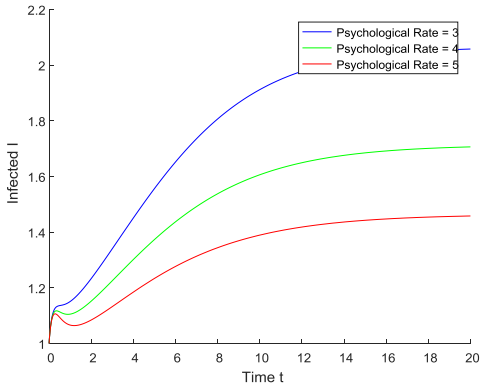

Figure 4. b psychological rate dependence of I^*

6. Conclusion

In this work, we considered a SEIR epidemic model with homogenous transmission function and treatment. We found that basis reproduction number when $R_0 < 1$ then the local and global stability P_0 is disease free equilibrium stable and if $R_0 > 1$ then the local and global stability P_* is endemic equilibrium stable. Our main result shows that the effect of sociological and psychological rate on the infected population.

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