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Brief Summary of Frequently-used Properties of the Floor Function: Updated 2018

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Abstract: *Based on a previous summary on the the frequently-used properties of the floor function, this article collects till 2018 more frequently-used properties of the floor function. This is an update the previous summary and is helpful for scholars of mathematics and computer science and technology.*

Keywords: *Floor function, Inequality, Number theory, Discrete mathematics*

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I. Introduction

The floor function, which is also called the greatest integer function (see in [1]), is a function that takes an integer value. For arbitrary real number *x*, the floor function of *x*, denoted by $\lfloor x \rfloor$, is defined by an inequality of $x-1<\lfloor x \rfloor \le x$. The floor function frequently occurs in many aspects of mathematics and computer science. However, as I stated in article [2], except the Graham's book [3], one can hardly find a general know-of the properties of the floor function though one can find something in the Internet of free wikipedia [4]. Since Graham's book was first published 30 year's ago and its following-up editions made few modification on the part of the floor function, it is necessary to sort out the properties of the function as a reference for researchers.

In 2017, WANG X made a brief summary on the frequently-used properties of the floor function Since new results come into being, this paper updates the previous summaries by adding the new results that could be found in literatures.

II. Definitions and Notations

The floor function of real number x is denoted by symbol $\lfloor x \rfloor$ that satisfies $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$; the fraction part of x is denoted by symbol $\{x\}$ that satisfies $x = \lfloor x \rfloor + \{x\}$; the ceiling function of x is denoted by symbol $\lceil x \rceil$ that fits $x \le \lceil x \rceil < x+1$. In this whole article, $A \Rightarrow B$ means conclusion *B* can be derived from condition *A*; $A \Leftrightarrow B$ means *B* holds if and only if *A* holds. Symbol ***Z*** means the integer set, $x \in \boldsymbol{Z}$ means *x* is an integer and $x \notin Z$ indicates *x* is not an integer.

III. Frequently Used Properties of the Floor Function

The following properties of the floor functions are sorted by basic inequalities, conditional inequalities and basic equalities.

3.1 Basic Inequalities

In the following inequalities, *x* and *y* are real numbers by default.

(P1) [1] $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x+y \rfloor \le \lfloor x \rfloor + \lfloor y \rfloor + 1$

(P2) [5] $\lfloor x \rfloor - \lfloor y \rfloor - 1 \le \lfloor x-y \rfloor \le \lfloor x \rfloor - \lfloor y \rfloor < \lfloor x \rfloor - \lfloor y \rfloor + 1$

(P3)[1][3] $\lfloor 2x \rfloor + \lfloor 2y \rfloor \ge \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor$

(P4)[5] $\lfloor (m+n)x \rfloor + \lfloor (m+n)y \rfloor \ge \lfloor mx \rfloor + \lfloor my \rfloor + \lfloor nx+ny \rfloor$ with *m* and *n* being positive integers

(P5)[5] $\lfloor nx \rfloor + \lfloor ny \rfloor \ge (n-1)\lfloor x+y \rfloor + \lfloor x \rfloor + \lfloor y \rfloor$ with *n* being a positive integer

(P6) [1][5] $\lfloor xy \rfloor \ge \lfloor x \rfloor \lfloor y \rfloor$ with $x, y \ge 0$.

(P7)[6] $\left\lfloor \frac{y}{x} \right\rfloor \le \frac{\lfloor y \rfloor}{\lfloor x \rfloor}$ with $x \ge 1$ and $y > 0$.

(P8) [3] $n\lfloor x\rfloor \le \lfloor nx\rfloor$; $n\lfloor x\rfloor = \lfloor nx\rfloor \Leftrightarrow n\{x\} < 1$, where *n* is a positive integer.

(P9) [7] $\left\lfloor \frac{q}{p} \right\rfloor \ge \frac{q+1}{p} - 1$ for arbitrary positive integers *p* and *q*;

3.2 Conditional Inequalities

In the following inequalities, *x* and *y* are real numbers, and *n* is an integer.

(P10)[3] $x < n \Leftrightarrow \lfloor x\rfloor < n$, $n \le x \Leftrightarrow n \le \lfloor x\rfloor$

(P11)[3] $x < n \le y \Leftrightarrow \lfloor x\rfloor < n \le \lfloor y\rfloor$

(P12) [2] $\lfloor x\rfloor > \lfloor y\rfloor \Rightarrow x > y$

(P13) [2][5] $x \le y \Rightarrow \lfloor x\rfloor \le \lfloor y\rfloor$

3.3 Basic Equalities

In the following equalities, *x* and *y* are real numbers, *m* and *n* are integers.

(P14) [3][5] $\lfloor n + x\rfloor = n + \lfloor x\rfloor$.

(P15)[5] $\left\lfloor \frac{\lfloor x\rfloor}{m} \right\rfloor = \left\lfloor \frac{x}{m} \right\rfloor$ with $m \ge 1$.

(P16)[5] $\lfloor -x\rfloor = \begin{cases} -\lfloor x\rfloor, x \in \mathbf{Z} \\ -\lfloor x\rfloor - 1, x \notin \mathbf{Z} \end{cases}$

(P17)[3][5] $\lfloor nx\rfloor = \lfloor x\rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \ldots + \left\lfloor x + \frac{n-1}{n} \right\rfloor$ with *n*>0, particularly, $\lfloor x\rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2x\rfloor$ and $\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor = \lfloor x\rfloor$.

(P18)[3] $\lfloor x\rfloor = \left\lfloor \frac{x}{n} \right\rfloor + \left\lfloor \frac{1+x}{n} \right\rfloor + \ldots + \left\lfloor \frac{n-1+x}{n} \right\rfloor$, particularly, $\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor = \lfloor x\rfloor$

(P19)[3] $\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$ with $m \ge 1$.

(P20)[1][3] $\lfloor \sqrt{x}\rfloor = \left\lfloor \sqrt{\lfloor x\rfloor} \right\rfloor$ with $x \ge 0$

(P21)[3] $\lfloor \log_b x\rfloor = \lfloor \log_b \lfloor x\rfloor\rfloor$ with $x > 0$

(P22)[3] $\lfloor \log_b m\rfloor + 1 = \lceil \log_b (m+1)\rceil$ with $m \ge 1$.

(P23)[3] $\left\lfloor \frac{\left\lfloor \frac{a}{b} \right\rfloor}{c} \right\rfloor = \left\lfloor \frac{a}{bc} \right\rfloor$ for an arbitrary integer *a* and positive integers *b* and *c*.

(P24) [1][5] $\left\lfloor \frac{m+1}{n} \right\rfloor = \begin{cases} \left\lfloor \frac{m}{n} \right\rfloor & , n \nmid m+1 \\ \left\lfloor \frac{m}{n} \right\rfloor + 1, & n \mid m+1 \end{cases}$

(P25)[5] $\sum_{1 \le n \le x} 1 = \lfloor x\rfloor$

(P26)[7] $\lfloor \sqrt{n} + \sqrt{n+1}\rfloor = \lfloor \sqrt{4n+1}\rfloor = \lfloor \sqrt{4n+2}\rfloor = \lfloor \sqrt{4n+3}\rfloor$

IV. Some New Results

The following equalities and inequalities are found newly in recent two years.

(P27)[1][3] It needs $\lfloor \log_2 N\rfloor + 1$ binary bits to express decimal integer *N* in its binary expression. A positive integer *n* with base b has $\lfloor \log_b n\rfloor + 1$ digits.

(**P28**) [9] Let *N* be an integer; then $N-\lfloor\sqrt{N}\rfloor^2 \ge 0$.

(**P29**)[5] Let m and *p* be positive integers; then number of *p*'s multiples from 1 to *m* is calculated by $\left\lfloor \frac{m}{p} \right\rfloor$.

(**P30**)[8] Let m,n and *p* be positive integers such that $1< p<m<n$; then number of *p*'s multiples from *m* to *n* is calculated by

$$\nu(m,n,p)=\begin{cases}\left\lfloor \frac{n}{p}\right\rfloor-\left\lfloor \frac{m}{p}\right\rfloor, p\nmid m\\ \left\lfloor \frac{n}{p}\right\rfloor-\left\lfloor \frac{m}{p}\right\rfloor+1, p\mid m\end{cases}$$

(**P31**)[10] An arbitrary positive integer *i* yields

$$i-1\le 2\left\lfloor \frac{i}{2}\right\rfloor \le i$$

an arbitrary positive even integer *e* yields

$$2\left\lfloor \frac{e}{2}\right\rfloor = e$$

and an arbitrary positive old integer *o* yields

$$2\left\lfloor \frac{o}{2}\right\rfloor = o-1$$

(P32) [11] Let α and x be positive real numbers; then it holds

$$\alpha\lfloor x\rfloor-1<\lfloor \alpha x\rfloor<\alpha(\lfloor x\rfloor+1)$$

Particularly, if α is a positive integer, say $\alpha=n$, then it yields

$$n\lfloor x\rfloor\le\lfloor nx\rfloor\le n(\lfloor x\rfloor+1)-1$$

(P33) [11] For arbitrary positive real numbers α,x and y with $x>y$, it holds

$$\lfloor \alpha(x-y)\rfloor+\alpha\lfloor y-x\rfloor\le 0$$

(P34) [11]. For an arbitrary odd integer $n\ge 7$, it holds

$$1+\lfloor \log_2 n\rfloor \le \frac{n-1}{2}$$

(**P34**)[12] For positive integer *k* and real number $x>0$, it holds

$$0\ge 2^k\left\lfloor \frac{x}{2^k}\right\rfloor-\lfloor x\rfloor\ge\begin{cases}1-2^k, 0\le k\le\lfloor \log_2 x\rfloor\\ -\lfloor x\rfloor, k>\lfloor \log_2 x\rfloor\end{cases}$$

(**P35**)[13] For positive integer *k* and real number *x* >1, it holds

$$\left\lfloor \frac{x+1}{2^k}\right\rfloor=\begin{cases}\left\lfloor \frac{x-1}{2^k}\right\rfloor+1, k=1\\ \left\lfloor \frac{x-1}{2^k}\right\rfloor, k>1\end{cases}$$

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