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A Necessary and Sufficient Condition for Local and Global Null – Controllability.

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Abstract: *The problem of null-controllability for linear non-autonomous system of the form* $\dot{x}(t) = A(t)x(t) + B(t)u(t), x(t)\epsilon X$ *and* $u(t)\ \epsilon\, \Omega \subset U \equiv \mathbb{R}^m$ *where X and U are complex Banach spaces are considered. Necessary and sufficient condition is given for global Ω-null controllability of the system. In this case the condition that origin will be an interior point of the control set Ω- does not apply.*

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I. Introduction

We consider the problem of steering the state $x(t)$ of the non – autonomous systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), t\ \epsilon\ [0, \infty) \qquad (1.1)$$

to the origin in a finite time t > 0. We take $A(t)$ *and* $B(t)$ to be an n × n and n × m continuous bounded operator matrices; the state $x(t)\epsilon\mathbb{R}^n$ and the control function $u(t)\ \epsilon\ \mathbb{R}^m$. Usually, in controllability problems, the values of controls at each instant of time are free, but in this case, we insist that each of the control value at each instant of time lies in a prescribed set Ω in $\mathbb{R}^m$, *which is not empty*. Now, let us define a function $\mu: \mathbb{R} \to \Omega$, *from* $\mathbb{R}$ *into* Ω and then assume they are measurable on $[0,\infty)$. Then such control function $u(t) \in \mu(\Omega)$is said to be admissible. In this paper, we assume that X and U are complex *Bannach* spaces. According to *Aniaku* and Jackreece [1], a point $x_0\epsilon\ X$ is said to be null-controllable if there is a finite time $T = T(x_0)$ and an admissible control $u(t)\ \epsilon\ \Omega$ such that the corresponding trajectory $x(t)$ of (1.1) initiated at x_0 satisfies $x(T) = 0$. System (1.1) is said to be locally controllable if the setΨ of all null-controllable points contains some neighborhood of the origin. I.e. $0 \in Int\Psi$.The system is then said to be globally controllable if then Ψ = X.

1.2 Notations and Definitions.

Here, we present some notations and definitions which we hope to encounter in this paper. Firstly we have notations.

1.2.1 Notations:

Let X and Y be complex Bannach spaces.$\|.\|$is the norm in the space under consideration.
X* denotes the duel of X.
If M⊂X, the Int M denotes the interior of M in X.
$\mathbb{M}(\Omega)$ denote the set of the function from $\mathbb{R}$ into Ω That are measurable on $[t_0,\infty)$.
Span {M} denotes the span of M.
For each k∈ℕ, M^k denote the set of all vector of the form $x^k = (x_1, x_2, \dots x_k)$with$x_j\ \epsilon\ M$,
$j = 1,2,3, \dots, k$.X^k stands for a Bannach space endowed with the norm $\| x^k \| = \| x_1 \| + \| x_2 \| + \cdots + \| x_k \|$.
ℂ will denote the set of all complex numbers. Complex conjugates of λ is denoted by $\bar{\lambda}$
D_r denotes the disc $\{\lambda\ \epsilon\ \mathbb{C} : |\lambda| \le r\}$
ℕ denotes the set of counting numbers or natural numbers.
If $A\ \epsilon\ L(X, Y)$, then $s(A)$ denotes the spectrum of A.
$L(X, Y)$ denotes the Bannach space of all bounded linear operators.
We now define the controllability which are prescribed to lie in Ω which is very important in our paper. This we precisely call Ω-null controllability. Nevertheless, we have the following definitions.

1.2.2 Definitions:

Definition 1.1: Ω-null controllability [1].

The linear system (1.1) is Ω-null controllability at $(x_0, 0)$ if given the initial point $x(0) = x_0$ there exists a control function $u(t) \epsilon \mu$ (Ω) Such that the solution $x(t)$ of (1.1) satisfies $x(t) = 0 \; for \; t \; \epsilon \; [0, \infty)$.

Definition 1.2: Globally Ω-null controllability [2].

The linear system (1.1) is the globally Ω-null controllable at 0 if the system (1.1) is Ω-null controllable at $(x_0, 0)$ for all$x_0 \epsilon \; \mathbb{R}^n$

Definition 1.3: Locally Ω-null controllability [2].

The linear system (1.1) is locally Ω-null controllable at 0 if there exist an open set V $\epsilon \; \mathbb{R}^n$ containing the origin 0, such that (1.1) is null controllable at $(x_0, 0)$ for all $x_0 \epsilon \; V$.

Definition 1.4: Baire Functions.

These are functions obtained from continuous functions by transfinite iteration of the operation of forming point wise limit of sequence of functions.

Definition 1.5: Support of Function.

The support of a real-valued function f is the subset of the domain containing those elements which are not mapped to zero.

Definition 1.6: Scalar function.

A scalar function $T : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is denoted as

$$J(x_0, T, \lambda) \simeq x_0^1 \varphi^1(\mathrm{T}, t_0)\lambda + \int_o^T H_\Omega(B^1(\tau) Q^1(T, \tau)\lambda) \; d\tau \qquad (1.2)$$

Definition 1.7: Positive Definite Function. [3]

A $function \; F(x, y)$ is called positive definite in a domain D if $F(0,0) = 0 \; and \; F(x, y) > 0$ for all other points (x, y) in D.

II. Review Of Literature.

We note that different necessary and sufficient conditions for global Ω-Null controllability have been studied by different researchers. In the equation (1.1), if A and B are constants, Kalman [4] showed that if $\Omega = \mathbb{R}^m$ then a necessary and sufficient condition for global $\mathbb{R}^m$–null controllability is that the rank (P) = n, where

$$P = [B \; AB \; A^2B \; ... \; A^{n-1}B]$$

We have to point out that Lee and Markus [5] considered constraint set $\Omega \subset R^m$which has the origin as its interior, and showed that the rank(P) = n is necessary and sufficient condition for system (1.1) to be locally Ω-null controllable. He also showed that if each eigenvalue λ of A satisfies$R_e(\lambda) < 0$, then system (1.1) is globally Ω-null controllable.

Saperstone and Yorke [6] eliminated the assumption that the origin must be an interior point of Ω for global Ω-null controllability using m =1 and $\Omega = [0,1]$. In their study, they found that system (1.1) is locally Ω-null controllable if and only if the rank (P) = n and A has no real eigenvalues.

Brammer [7] in his own study, found a more general constrained set where he showed that if there exists a control function $u \in \Omega$ satisfying $Bu = 0$, and the convex hall of Ω has a non empty interor, then the necessary and sufficient conditions for a local Ω-null controllability are rank(P) = n and the non-existence of a real eigenvectors V of A^1 satisfy $V^1Bu = 0$ for all $u \; \epsilon \; \Omega$. He found that if in addition, no eigenvalues of A has a positive real part, then problem (1.1) becomes one in which global Ω-null controllability applies.

Now, for non-autonomous system, Kalman [4] showed controllability result when $\Omega = \mathbb{R}^m$. He showed that system (1.1) is $\mathbb{R}^m$-null controllable if and only if the gramain$W(t_0, t_1)$is positive definite for some $t_1 \in [t_o, \infty)$ Where

$$W(t_{0,} t_1) = \int_{t_0}^{t_1} \varphi(t, \tau) B(\tau) B^1(\tau) \varphi^1(t, \tau) d\tau \qquad (2.1)$$

and $\varphi(t, \tau)$is the fundamental matrix for the system (1.1). The problem Ω-null controllability was also studied by Dauer [8] and also Chukwu and Gronski [9]. Chukwu and Slliman [10] and Granthanand Vincent [11] considered the problem of steering a nonlinear system to a target point. They formulated a way of determining the difference between the set of state which can be steered to a required target point and those which cannot be

steered to it. We have to state at Pachterand Jacobson [12] developed the condition for Ω-null controllability for the case where $A(t)$ and $B(t)$ are time invariant, and Ω is a closed convex cone which contains the origin.
Now, we want to study the problem of Ω-null controllability when $A(t)$ *and* $B(t)$ are time-varying. Our result for global Ω-null controllability is for a constraint set Ω which are compact and contains the origin, which may not be an interior point. This result for Ω-null controllability at (x_0, t_0) has a wider application since it does not require $0 \epsilon\ \Omega$ It does not even require the existence of a control function $u\ \epsilon\ \Omega$such that $Bu = 0$ as in Brammer [7]. So, we can check controllability of system with, for example $m = 1$ *and* $\Omega = [1,2]$, where we know, many of the presently Ω-null controllability result do not apply. We shall show as example an autonomous system (1.1) which is neither globally Ω-null controllable, nor locally Ω -null controllable at some point (x_0, t_0). Note that this our theorem can be used to divide the state space into two sets, viz: initial state which can be steered to the origin and those which cannot be driven to the origin by any admissible control function.

III. The Main Theorem

Before we state our main result, we have to deal with the following proposition.

Proposition 3.1: let Ω be a compact set and G a subset of $\mathbb{R}^n$ which has the origin as interior point. Then system (1.1) is Ω-null controllable at (x_0, t_0) if and only if

$$\min\{J(x_0, T, \lambda): \lambda \in G\} = 0 \qquad (3.1)$$

For some $T \in [t_0, \infty)$
Proof:
Let $A(x_0)$ be the set of states which can be reached from x_0 at finite time T
i.e.

$$A(x_0) = \left\{\varphi(t, o)x_0 + \int_0^T \varphi(T, \tau)B(\tau)u(\tau)d\tau : u(.) \in\ dom\ (\Omega)\right\} \qquad (3.2)$$

We note that the set $A(x_0)$ is convex and compact. It then follows that x_0 can be steered to the origin in a finite time T if and only if $0\ \epsilon\ A(x_0)$ So.

$$0 \le sup\{\lambda^1 a : a \in A(x_0)\} \qquad (3.3)$$

For all vectors $\lambda \in \mathbb{R}^n$
Then (3.2) and (3.3) give

$$\lambda^1 \varphi(T, 0)x_0 + \sup\{\int_0^T \lambda^1 \varphi(T, \tau)B(\tau)u(\tau)d\tau : u(.) \in M(\Omega) \ge 0\} \qquad (3.4)$$

For all $\lambda \in \mathbb{R}^n$
Note that in (3.4), we can commute the supremum and integral operation to get

$$0 \le \lambda^1 \varphi(T, 0)x_0 + \int_o^T H_\Omega(B^1(\tau)Q^1(T, \tau)\lambda)\ d\tau = J(x_0, T, \lambda). \qquad (3.5)$$

For all $\lambda \in \mathbb{R}^n$since $J(x_0, T, \lambda)$. is positively homogeneous in λ we can restrict λ to be in G. this then completes the proof.

We are now ready to state our main result.
Theorem 3.1:
Suppose Ω is a compact set which contains the origin. Then system (1.1) is globally Ω-null controllable at t_0 if and only if

$$\int_{t_0}^T H_\Omega\left(B^1(\tau)Z(\tau)\right) d\tau = +\infty \qquad (3.6)$$

for all non – zero solution $Z(.)$ of the adjoint system

$$\dot{Z}(t) = -A^1(t)z(t), t \in [t, \infty) \qquad (3.7)$$

Proof:
Necessary condition:
Let us suppose that the system (1.1) is globally Ω-controll able at t_0= 0, without loss of generality, let $Z(.)$ be any non – zero solution of adjoint system (1.1). We then show that

$$\int_o^\infty H_\Omega\left(B^1(\tau)Z(\tau)\right) d\tau = +\infty \qquad (3.8)$$

We show this by contradiction. Suppose that there is a non – zero solution $\bar{Z}(.)$ such that

$$\int_o^{\infty} H_\Omega\big(B^1(\tau)\bar{Z}(\tau)\big)\, d\tau \ < \infty$$

Then there is a positive constant $\alpha < \infty$ such that

$$\int_o^{\infty} H_\Omega\big(B^1(\tau)\bar{Z}(\tau)\big)\, d\tau \ < \alpha$$

If we define

$$x_0^+ = \frac{-2\alpha\bar{Z}(0)}{\bar{Z}^1(0)\bar{Z}(0)}, x_0^+ \ \neq 0$$

We claim that x_0^+ cannot be steered to 0 by any admissible control $u(.) \in M(\Omega)$. We prove this as follows: for each $t \in [0,\infty)$ let us define

$$\lambda_t = \varphi^1(0,t)\bar{Z}(0), \lambda_t \neq 0$$

So, given any $t \in [0,\infty)$

$$J(x_0^+, t, \lambda_t) = \ x_0^{+1}\varphi^1(t,0)\lambda_t + \int_o^t H_\Omega(B^1(\tau)\varphi^1(t,\tau)\lambda_t)\, d\tau$$
$$= x_0^+ + \bar{Z}(0) + \int_o^t H_\Omega\big(B(\tau)\bar{z}(\tau)\big)\, d\tau$$
$$= -2\beta + \beta$$
$$< 0$$

If we take $G = \mathbb{R}^n$ in the proposition 3.1 above, we see that

$$\min\{J(x_0^+, t, \lambda) : \lambda \in G\} \leq J(x_0^+, \lambda_t) < 0$$

For all $t \in [0,\infty)$ then, by proposition 3.1 above, the system (1.1) is not Ω-null controllable at $(x_0^+, 0)$.

Sufficiency:

We now assume that (3.8) holds. Then we also go by contradiction. Suppose that the system (1.1) is not globally Ω-null controllable at t_0= 0. Then, there exists an initial point $x_0^+ \neq 0$ which can not be steered to zero in finite time. If we can find a sequence of times $\{t_k\}_{k=1}^{\infty}$ and a sequence of vectors $\{\lambda_k\}_{k=1}^{\infty}$ having the properties

$$(c_1) \lim_{k\to\infty} t_k = +\infty$$

$(c_2)\ J(x_0^+, t, \lambda_t) < 0\ for\ k = 1,2,3, \dots$

Then, we construct an initial point $\overline{z_0} \neq 0$ for (1.1) which makes the integral in (3.8) finite.

Let

$$z_k = \frac{\varphi^1(t_k, 0)\lambda_k}{\parallel \varphi^1(t_k, 0)\lambda_k \parallel}\ k = 1,2, \dots$$

We see that $z_k \neq 0$since $\lambda_k \neq 0$ and $\varphi^1(t_k, 0)$ is invertible. So $\{z_k\}_{k=1}^{\infty}$is a sequence in $\mathbb{R}^n$ which belongs to the set

$$S = \{z \in \ \mathbb{R}^n : \parallel z \parallel = 1\}$$

Since this set S is compact, we can find a subsequence $\{z_{k_j}\}_{j=1}^{\infty}$ which converges to some vector $\overline{z_0} \in S$, we show that $\overline{z_0}$ is the initial point which does the job.

Let $\bar{z}(.)$ be a trajectory of S^1 which is obtained by $z(0) \ = \ z_0$

Let $\{t_{k_j}\}_{j=1}^{\infty}$ be a subsequence of times corresponding to $\{z_{k_j}\}_{j=1}^{\infty}$

By c_1 we have

$$\lim_{j\to\infty} t_{k_j} = +\infty$$

and by c_2 it follows that

$$x_0^{+1}\varphi^1\left(t_{k_j}, 0\right)\lambda_{k_j} + \int_o^{t_{k_j}} H_\Omega\left(B^1(\tau)\varphi\left(t_{k_j}, \tau\right)\lambda_{k_j}\right) d\tau < 0 \qquad j = 1,2,3, \dots$$

Dividing by$\parallel \varphi^1\left(t_{k_j}, 0\right)\lambda_{k_j} \parallel$ and noting that H_Ω is positively homogeneous, we get

$$\int_o^{t_{k_j}} H_\Omega\left(B^1(\tau)\varphi^1(0,\tau)z_{k_j}\right) d\tau \ \leq \parallel x_0^+ \parallel . \parallel z_{k_j} \parallel, \quad j = 1,2,3, \dots$$
$$\leq \parallel x_0^+ \parallel \qquad j = 1,2,3, \dots$$

$\Rightarrow \int_o^{\infty} H_\Omega\big(B^1(\tau)\bar{z}(\tau)\big)\, d\tau \ \leq \parallel x_0^+ \parallel < \infty.$

This is a contradiction which we are looking for. So, the proof is completed.

3.2 Examples:

Example 1.

Consider the control system described by

$$\dot{x}(t) = x(t) + u(t), t \in [0, \infty) \qquad (3.9)$$

Where $x(t)$ and $u(t)$ are scalars.

We know that this system is R^1-null controllable. Suppose we take $\Omega = [0,1]$.Then, this system is not globally Ω -null controllable at t_0= 0.

This conclusion is obtained from Theorem 3.1, since for $z_0 < 0, H_\Omega\big(B^1(\tau)z(\tau)\big) = 0$ and thus

$$\int_o^\infty H_\Omega\big(B^1(\tau)z(\tau)\big)\, d\tau < +\infty$$

Example 2.

Consider the time-varying two dimensional system defined by

$$\begin{cases} \dot{x}_1(t) = u(t)\sin t \\ \dot{x}_2(t) = \frac{1}{(t+1)^2} x_1(t) + u(t)t\sin t, \quad t \in [0, \infty) \end{cases} \qquad (3.10)$$

Let us take the control constraint set to be $\Omega = [0,1]$. Then state fundamental matrix for the adjoint system is

$$\varphi + (t, t_0) = \begin{pmatrix} 1 & \dfrac{t - t_0}{(t+1)(t_0+1)} \\ 0 & 1 \end{pmatrix}$$

So from theorem 3.1, we see that (3.10) is globally Ω-null controllable at t_0= 0 if and only if

$$\int_0^\infty \sup_{t \in [0,1]} t \; (\sin\tau \quad \tau sin\tau) \begin{pmatrix} 1 & \dfrac{\tau}{(\tau+1)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_{0_1} \\ z_{0_2} \end{pmatrix} d\tau = \infty \qquad (3.11)$$

for all non-zero initial point $z_0 = (z_{0_1} \quad z_{0_2})^T$

Evaluating (3.11) we get

$$\int_0^\infty I(\tau)d\tau = \int_0^\infty \max\{0, z_{0_1}\sin\tau + z_{0_2}\tau\sin\tau(1 + \frac{1}{\tau+1})\}d\tau = +\infty \qquad (3.12)$$

for all $z_0 \neq 0.$ Equation (3.2) is indeed true for both

(a) $z_{0_1} \neq 0, z_{0_2} = 0.$

and

(b) $z_{0_1} = 0, z_{0_2} \neq 0.$

For case (a), we have

$$\int_0^\infty I(\tau)d\tau = \int_0^\infty \max\{0, z_{0_1}\sin\tau\}d\tau$$
$$= \int_{\xi_1}^\infty z_{0_1}\sin\tau \; d\tau$$

where $\xi_1 = \{\tau \geq 0 : z_{0_1}\sin\tau > 0\}$ Since the range set $\xi_1$of integration is the union of infinitely many intervals of length π it follows that

$$\int_0^\infty I(\tau)d\tau = +\infty$$

For case (b), let $T^+ = \frac{|z_{0_1}|+1}{z_{0_2}}$

Then it is enough to show that

$$\int_{\xi_2}^\infty I(\tau)d\tau = +\infty$$

where $\xi_2 = \{\tau \geq T^+ : z_{0_2}\sin\tau > 0\}$. Since the range of integration is once again the union of infinitely many intervals of length πwe conclude that (3.10)is globally Ω-null controllable.

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