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Numerical Solutions of Fractional Order Eigen-Value Problems using Bernstein Operational Matrices

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Abstract: *In this article, we offer an easy and active computational technique for finding the solution of the fractional order Sturm-Liouville problems (FOSLPs) with variable coefficients using operational matrices of (BPs). The fractional order derivatives (FODs) are characterized in the Caputo sense. The proposed technique transform the fractional order differential equations (FDEs) into a linear system of algebraic equations, then the eigenvalues can be computed by finding the roots of the characteristics polynomials. Some tested problems are given in order to illustrate the effectiveness and efficiency of the method.*

Keywords: *Fractional order eigen-value problems, Sturm-Liouville problems, Bernstein polynomials.*

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I. Introduction

(FDEs) are generalized of classical integer order ones which are gained by exchanging integer order derivatives by fractional ones.

Most (FDEs) do not ownanalytic solutions, therefore approximate and numerical techniques must to use [1].

various numerical and approximate technique to solve (FDEs) have been given such asAdomian decomposition method [2],Variationaliteration method [3], Homotopyanalysis method [4],Homotopy perturbation method [5], Collocation method [6], wavelet method [7], finite element method [8] and spectral tau method [9].

The concept of operational matrices (OM) recently were acclimated for solving various types of (FDEs). Using the numerical techniques in combination with (OM) in some orthogonal polynomials, for the solution of (FDEs) on finite and infinite intervals, givesvery accurate solutions for (FDEs) [9].

This article looks for finding the approximate values of the eigenvalues of some classes of the (FOSLPs). Some beforehand works have been published about the (FOSLPs) such as [10]-[19]. The essential feature of this work is to find the eigenvalues of the (FOSLPs) using the(OM) of the (BPs). BPs. play an eminent part in various areas of mathematics, these polynomials have extremely been used in the solution of integral equations, differential equations and approximation theory [20], [21].

The (OM) for (BPs) are introduced in order to solve different types of differential equations among them[20] used the (OM) for (BPs) for solving High Even-Order differentials equations,[21]have been used (OM) of (BPs) for solvingVolterra Integral equations, [22]investigated the solution of the nonlinear Volterra-Fredholm-Hammerstein integral equations using the (OM) of (BPs), while, [23]try to solve the physiology problems by the aid of the (OM) of (BPs). [24]have been used (OM) of (BPs) for solving high order delay differential equations.The (OM) of (BPs) have been used for solving multiterm variable order (FDEs) by [25].
The reminder of the article is ordered as follow. A concise review of the definitions ofthe (FODs) and integrations are offered in part 2. (BPs)have been given in part 3. The proposed method will be presented in part 4. Some numerical results are shown in part 5, at last a conclusion have been towed.

II. Fractional Order Derivatives and Integrals

In this part we shall give some basic definitions and properties of the (FODs) and integrals [26].

Def. (1):

The Riemann –Liouville (R-L) fractional integral of order $\alpha > 0$ is defined as follows:

$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (t-\tau)^{\alpha-1} f(\tau) d\tau$, $x > 0$, $\alpha \in R^+$.

Def. (2):

The Caputo fractional derivative of order $\alpha > 0$ is defined as follows:

$$ {}^{c}D_x^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau , & m-1<\alpha<m ; \\ \frac{d^m}{dx^m (x)} f(x) , & \alpha = m . \end{cases} $$

For $\alpha > 0$, we have[1]:

1- ${}^{c}D_x^{\alpha}(I_x^{\alpha} f(x)) = f(x)$.

2-$I_x^{\alpha}\left({}^{c}D_x^{\alpha} f(x)\right) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}$.

3- ${}^{c}D_x^{\alpha}(c)=0$, $c \in \mathbb{R}$.

4- ${}^{c}D_x^{\alpha}(c_1 f(x) + c_2 g(x)) = c_1\ {}^{c}D_x^{\alpha} f(x) + c_2\ {}^{c}D_x^{\alpha} g(x)$.

Where c_1 and c_2 are constants.

III. Bernstein Polynomials (BPs)

The well known (BPs) of degreen [27]are given on [0, 1] as:

$b_i^n(x)=\binom{n}{i} x^i (1-x)^{n-i},\ i = 0,1,\dots,n \cdots (1)$

Eq. (1) can be also written as:

$b_i^n(x)= \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{i} \binom{n-i}{j-i} x^j$,i= 0, ..., n. $\cdots$ (2)The Bernstein vector

$B(x)=[b_0^n(x), b_1^n(x), \cdots, b_n^n(x)] \cdots (3)$

Can be prescribed in the form

$B(x)=AT_n(x) \cdots (4)$

Where

$$ A=\begin{bmatrix} (-1)^0\binom{n}{0} & (-1)^1\binom{n}{0}\binom{n-0}{1} & \cdots & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0} \\ 0 & (-1)^0\binom{n}{i} & \cdots & (-1)^{n-i}\binom{n}{i}\binom{n-i}{n-i} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & (-1)^0\binom{n}{n} \end{bmatrix}_{(n+1)(n+1)} \cdots (5) $$

and

$$ T_n(x) = \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{bmatrix} \cdots (6) $$

III.I. Function Approximation:

Any $f(x) \in L_2(0,1)$, can be decomposed in terms of the (BPs). Hence if f(x) can be decomposed as:

$f(x)=\sum_{i=0}^{n} c_i b_i^n(x) = c^T B(x) \cdots (7)$

Where$c = [c_0, c_1, \cdots, c_n]^T$, $B(x) = [b_0^n(x), b_1^n(x), \cdots, b_n^n(x)]^T$.

Then

$c_i = \int_0^1 f(x) d_i^n(x) dx$,i= 0,1,… , n.

Where $d_i^n(x)$are called the dual basis function to the Bernstein basis of degree n which has been derived in [27] in explicit representation as:

$d_i^n(x)=\sum_{k=0}^{n} \lambda_{jk} b_k^n(x)$, j=0,1… , n.

Where

$$ \lambda_{jk} = \frac{(-1)^{j+k}}{\binom{n}{j}\binom{n}{k}} \sum_{i=o}^{\min(j,k)} (2i+1) \binom{n+i+1}{n-j} \binom{n-i}{n-j} \binom{n+i+1}{n-k} \binom{n-i}{n-k} \cdots (8) $$

For j,k =0,1,… , n.

III.II.The(OM) of the derivatives:

The derivative of the vector B(x) can be written as:

$\frac{d}{dx} B(x)=D^{(1)} B(x) \qquad \cdots (9)$

Where $D^{(1)}$ is handled by:

$D^{(1)}=AVB^* \quad \cdots (10)$

Where

$$V=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{pmatrix}_{(n+1)(n)} \quad \text{and} B^* = \begin{pmatrix} \mathbf{A}^{-1}_{[1]} \\ \mathbf{A}^{-1}_{[2]} \\ \mathbf{A}^{-1}_{[3]} \\ \vdots \\ \mathbf{A}^{-1}_{[n]} \end{pmatrix}_{(n)(n+1)}$$

Where $A^{-1}_{[k]}$ is the k^{th} row of A^{-1} for $k = 1, 2, \ldots, n$.
Employ Eq. (9), it is obvious that

$$\frac{d^n B(x)}{dx^n} = \left(D^{(1)}\right)^n B(x), \quad n \in \mathbb{N} \qquad \ldots(11)$$

and

$$D^{(n)} = \left(D^{(1)}\right)^n \qquad n = 1,2,\ldots,.\cdots(12)$$

In the following theorem the (OM)of the (FODs) of the (BPs) will be offered.
Theorem (1)[27]: Let B(x) be a Bernstein vector and $\alpha > 0$ then

$${}^cD_x^\alpha B(x) \approx D^{(\alpha)} B(x) \cdots (13)$$

Where

$$D^{(\alpha)} = \begin{pmatrix} \sum_{j=\lceil\alpha\rceil}^{n} \omega_{0,j,0} & \sum_{j=\lceil\alpha\rceil}^{n} \omega_{0,j,1} & \cdots & \sum_{j=\lceil\alpha\rceil}^{n} \omega_{0,j,n} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=\lceil\alpha\rceil}^{n} \omega_{i,j,0} & \sum_{j=\lceil\alpha\rceil}^{n} \omega_{i,j,1} & \cdots & \sum_{j=\lceil\alpha\rceil}^{n} \omega_{i,j,n} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=\lceil\alpha\rceil}^{n} \omega_{n,j,0} & \sum_{j=\lceil\alpha\rceil}^{n} \omega_{n,j,1} & \cdots & \sum_{j=\lceil\alpha\rceil}^{n} \omega_{n,j,n} \end{pmatrix} \cdots(14)$$

Here $\omega_{i,j,l}$ is given by

$$\omega_{i,j,l} = (-1)^{j-i} \binom{n}{i} \binom{n-i}{j-i} \left(\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} \right) \sum_{k=0}^{n} \lambda_{lk} \mu_{kj} \qquad \cdots (15)$$

Where λ_{lk} is given in Eq. (8)and μ_{kj} is represented by:

$$\mu_{kj} = \sum_{s=k}^{n} (-1)^{s-k} \binom{n}{k} \binom{n-k}{s-k} \frac{1}{j-\alpha+s+1} \cdots (16)$$

IV. The Proposed Method

Given the following fractional order Sturm-Liouvilleproblem:
${}^cD_x^\alpha y(x) + q(x) y^{(1)}(x) = \lambda H(x) y(x), a \le x \le b, m-1 < \alpha \le m.$ $\cdots (17)$With the (BCs):

$d_{11} y \ (a) + d_{12} y^{(1)} \ (a) = 0 \ldots(18)$

$d_{21} y \ (b) + d_{22} y^{(1)}(b) = 0 \ldots(19)$

Where d_{11}, d_{12}, d_{21} and d_{22} are constants,and q(x),H(x) aregiven functions.We suppose that the solution y(x) can be approximated by using the (BPs) as follows:

$y(x) \approx \sum_{i=0}^{n} c_i b_i^n(x) = c^T B(x) \ \cdots (20)$

Where $c = [c_0, c_1, \ldots, c_n]^T$ and $B(x) = [b_0^n(x), b_1^n(x), \ldots, b_n^n(x)]^T$

The basic thought is to basically get a homogenous system of equations in theunknowns c_i, $0 \le i \le n$, the roots of whose characteristic equationcomprise the eigenvalues of the problem.Using Eq.(13) and Eq.(20) we have:

$${}^cD_x^\alpha \ y(x) \approx c^T \ D^{(\alpha)} B(x) \cdots (21)$$

To begin we apply the (BCs) Eq.(18) and Eq.(19) using Eqs. (20) and (9), so we obtain the following twoequations for the unknown coefficients c_i:

$$\begin{cases} d_{11} c^T B(a) + d_{12} c^T D^{(1)} B(a) = 0 \\ d_{21} c^T B(b) + d_{22} c^T D^{(1)} B(b) = 0 \end{cases} \qquad \cdots (22)$$

Or

$$\begin{cases} c^T \Gamma_a \\ c^T \Gamma_b \end{cases} \qquad \cdots (23)$$

Where

$\Gamma_a = d_{11} B(a) + d_{12} D^{(1)} B(a)$

and

$\Gamma_b = d_{21} B(b) + d_{22} D^{(1)} B(b)$

Next, substituting Eqs.(9),(20)and(21)into Eq.(17) and by using the linearity property, we obtain:

$c^T D^{(\alpha)} B(x) + c^T q(x) D^{(1)} B(x) = \lambda H(x) c^T B(x) \qquad \cdots(24)$

Recall that we have n+1 unknown coefficients c_i, i=0,1,…, n, and only two equations from the (BCs) so we require an additional n−1 equations. To get like equations we collocate Eq.(24) at n−1 points and for appropriate collocation points we use

$x_i = \left(\frac{1}{2}\right)\left(\cos\left(\frac{i\pi}{n}\right) + 1\right)$ i=1,…, n − 1. ⋯(25)

So we have :

$c^T D^{(\alpha)} B(x_i) + c^T q(x_i) D^{(1)} B(x_i) = \lambda H(x_i) c^T B(x_i)$, i=1, …, n − 1.⋯(26)

Or

$c^T \Gamma_{i,\alpha} + c^T q(x_i)\Gamma_{i,1} - \lambda H(x_i) c^T \Gamma_{i,0} = 0$ ⋯(27)

Where

$\Gamma_{i,\alpha} = D^{(\alpha)} B(x_i)$ and $\Gamma_{i,j} = D^{(j)} B(x_i)$, j=0,1.

Combining Eqs.(23)and(27), we get a complete system of n+1 equations which can be written after some simplifications as

(A- λK)c= 0, ⋯ (28)

Where

$$A = \begin{bmatrix} (\Gamma_{i,\alpha} + q(x_i)\Gamma_{i,1})^T \\ {\Gamma_a}^T \\ {\Gamma_b}^T \end{bmatrix}_{(n+1)\times(n+1)} \quad \text{and } K = \begin{bmatrix} (H(x_i)\Gamma_{i,0})^T \\ 0 \end{bmatrix}_{(n+1)\times(n+1)}$$

At last,for getting a nontrivial solution Eq.(28) should possess a nonzero solution which means that

det (A− λK)=0 ⋯(29)

Where det(A− λK) is a polynomial of degree n−2in λ,the eigenvalues of problem(17)−(19) should be those that satisfy Eq.(29).

V. Numerical Results

In this section some Sturm-Liouville problems have been considered in order to find the approximate results of its eigenvalues using the proposed approach given in section 4.

Example1: Consider the following fractional order eigenvalue problem.

$${}^{c}D_x^{\alpha}[y(x)] + y^{(1)}(x) + \lambda y(x) = 0, \quad x \in (0,1) \cdots (30)$$

S.t.:$y^{(1)}(0) = 0$, y(1)=0, where1< α≤ 2. ⋯ (31)

Following tables 1 and 2 represent the approximate values of the 1^{st} three eigenvalues of problem(30)−(31) compared with the results obtained by [17] when α=2.

Table1: The approximate values of the1^{st} three eigenvalues of theproblem (30)−(31)compared with the results obtained by [17] for distinct values of α.

λ_k	α=1.85n=8	α=1.85 n=20[17]	α=1.95 n=8	α=1.95 n=20[17]	α=2 n=8	α=2 n=20 [17]
λ_1	3.688684	3.681348	3.631415	3.628756	3.623089	3.623089
λ_2	20.476917	20.429199	22.236499	22.217377	23.441949	23.442337
λ_3	49.231315	49.087410	57.690948	57.581008	62.996286	62.929723

Table 2: The approximate values of the1^{st} three eigenvalues of theproblem (30)-(31) for distinct values of n with α=1.75./

λ_k	α=1.75					
	n=6	n=8	n=10	n=12	n=14	n=16
λ_1	3.8153558	3.8039381	3.800443	3.797901	3.797529	3.7977257
λ_2	19.721111	19.588887	19.563639	19.553688	19.550035	19.629812
λ_3	41.760745	42.966502	42.974090	42.906804	42.736823	42.698857

Example 2: Consider the singular fractional eigenvalue problem.

${}^{c}D_x^{\alpha}[y(x)] + (\frac{1}{x}+\lambda)\, y(x) = 0, \quad x \in (0,1) \cdots (32)$

S.t.:y(0) =0, $y^{(1)}(1) = 0$.⋯ (33)

Following table3 represent the approximate values of the 1^{st} three eigenvaluesof problem (32)−(33)compared with the results obtained by[11] when α=2.

Table 3: The approximate values of the1^{st} three eigenvalues of theproblem(32)−(33)for distinct values of α.

λ_k	n=8					n=20 ,[11]
	α=1.75	α=1.85	α=1.9	α=1.95	α=2	
λ_1	1.00229	0.86995	0.82720	0.80003	0.78043	0.7805164
λ_2	13.6593	15.41115	16.54381	17.85569	19.35869	19.3607236
λ_3	35.19086	42.74263	47.29187	52.46612	58.35406	58.3363502

Example 3: Consider the following fractional order eigenvalue problem.

$$ {}^{c}D_{x}^{\alpha}[y\quad(x)] + \lambda y(x) = 0.\cdots(34) $$

Subject to: $y^{(1)}(0) = 0,\ y(1) = 0.\cdots(35)$

Following table4 represent the approximate values of the 1st three eigenvalues of problem (34)−(35)for distinct values of α.

Table 4: The approximate valuesof the1^{st} three eigenvaluesof the problem(34)−(35) for distinct values of α.

λ_k	n=8					
	α=1.5	α=1.75	α=1.85	α=1.9	α=1.95	α=2
λ_1	2.12365	2.19259	2.27997	2.335089	2.39747	2.46740
λ_2	13.83558	15.96978	17.97726	19.21322	20.61774	22.20532
λ_3	24.28174	38.45971	46.04442	50.61589	55.81253	61.4169

VI. Conclusions

This article presents powerful technique for calculating the eigenvalues of the fractional order Sturm – Liouvilleproblems. The proposed technique is easy in that it is based on using the operational matrices ofthe (BPs) in order to convert the (FDEs) into a linear system of algebraic equations, then the eigenvalue may be computed by finding the roots of the characteristics polynomials.The proposed method is simple to execute, efficient and yields precise outcomes when its compared with the existing methods.

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