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On Scalar Quasi weak (m,n) - power Commutative Algebras

G.Gopalakrishnamoorthy[1], S.Geetha[2] and S.Anitha[3]
[1]*Advisor, Sri krishnasamy Arts and Science College, Sattur – 626203, Tamilnadu.*
[2]*Dept. of Mathematics, St.Michael College of Engineering and Technology, Kalaiyarkovil, Sivagangai - 630551.*
[3]*Lecturer, Raja Doraisingam Government Arts College, Sivagangai – 630 561, Tamil Nadu.*
Corresponding Author: G.Gopalakrishnamoorthy

Abstract: *A right near-ring N is called Quasi-weak commutative if xyz = yxz[3].A right near-ring N is called quasi weak m- power commutative if xm y z = ymxz for all x,y,z ∈ N,where m ≥1 is a fixed integer [5].An algebra A over a commutative ring R is called scalar quasi-weak commutative if for every x,y,z ∈ A there exists α = α (x,y,z) ∈ R depending on x,y,z such that xyz = α yxz. An algebra A over a commutative ring R is called scalar quasi-weak m - power commutativity if for every x,y,z ∈ A,there exists a scalar α ∈ R depending on x,y,z such that xm y z = α ymxz [8]. In this paper,the concept of scalar quasi-weak m-power commutativity is generalized as scalar quasi- weak commutative (m,n) power commutativity and prove many results.*

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I. Introduction:

Let A be an algebra (not necessarily associative) over a commutative ring R.A is called scalar commutative if for each x,y ∈ A, there exists α ∈ R depending on x,y such that xy = αyx.Rich[11] proved that if A is scalar commutative over a field F,then A is either commutative or anti-commutative.KOH,LUH and PUTCHA [9] proved that if A is scalar commutative with 1 and if R is a principal ideal domain ,then A is commutative. A near-ring N is said to be weak-commutative if xyz = xzy for all x,y,z ∈ N (Definition 9.4, p.289, Pliz[10]). An algebra A over a commutative ring R is called scalar quasi weak commutative if for every x,y,z ∈ A, there exists $\alpha = \alpha$ (x,y,z)∈ R depending on x,y,z such that xyz = αyxz and is called scalar weak m – power commutative if x^m y z = α y^mxz.In this paper we define scalar - quasi weak (m,n) - power commutativity and obtain many results.

II. Preliminaries:

In this section we give some basic definitions and well known results which we use in the sequel.

2.1 Definition [10]:
Let N be a near-ring.N is said to be weak commutative if xyz = xzy for all x,y,z ∈N.

2.2 Definition:
Let N be a near-ring.N is said to be anti-weak commutative if xyz = - xzy for all x,y,z∈N.

2.3 Definition [2]:
Let A be an algebra (not necessarily associative) over a commutative ring R.
A is called scalar commutative if for each x,y ∈ A,there exists $\alpha = \alpha$(x,y) ∈ R depending on x,y such that xy = αyx.A is called scalar anti- commutative if xy = -αyx.

2.4 Lemma[5]:
Let N be a distributive near-ring.If xyz = $\pm$ xzy for all x,y,z ∈ N,then N is either weak commutative or weak anti-commutative.

III. Main Results:

3.1 Definition:
Let A be an algebra (not necessarily associative) over a commutative ring R. A is called scalar quasi- weak (m,n) - power commutative if for every x,y,z ∈ A, there exists scalar α ∈ R depending on x,y,z such that $x^m y^n z = \alpha\, y^m x^n z$.

3.2 Definition:
Let A be an algebra (not necessarily associative) over a commutative ring R. A is said to be scalar quasi- weak (m,n) - power anti-commutative if there exists scalar $\alpha = \alpha(x, y, z)$ ∈ R depending on x,y,z such that $x^m y^n z = -\alpha\, y^m x^n z$..

3.3 Theorem:

Let A be an algebra(not necessarily associative) over a field F.Let m,n $\in z^{+}$.
Let $(x+y)^k = x^k + y^k$ for k = m,n holds for all x,y $\in$ A.Assume $\alpha^k = \alpha\ \forall\ \alpha \in R$, k = m,n.
If for each x,y,z $\in$ A there exists a scalar $\alpha \in F$ depending on x,y,z such that $x^m y^n z = \alpha\, y^m x^n z$.
Then A is either quasi weak (m,n) power commutative or quasi weak (m,n) power anti-commutative.

Proof:
Suppose $x^m y^n z = y^m x^n z$ for all x,y,z $\in$ A,there is nothing to prove.
Suppose not,we shall prove that $x^m y^n z = - y^m x^n z$ for all x,y,z $\in$ A.
First we shall prove that, if $x^m y^n z \neq y^m x^n z$,then $x^{m+n} z = y^{m+n} z = 0$.
So assume $x^m y^n z \neq y^m x^n z$.
Since A is scalar quasi weak (m,n) power commutative,there exists $\alpha = \alpha(x,y,z) \in R$ such that

$x^m y^n z = \alpha\, y^m x^n z$ → (1)

Also there exists a scalar $\gamma = \gamma(x,x+y,z) \in F$ such that $x^m (x+y)^n z = \gamma\, (x+y)^m x^n z$

i.e., $x^m (x^n+y^n) z = \gamma\, (x^m+y^m) x^n z$ → (2)

(1) – (2) gives

$x^m y^n z - x^{m+n} z - x^m y^n z = \alpha\, y^m x^n z - \gamma x^{m+n} z - \gamma\, y^m x^n z.$

$(1-\gamma)\, x^{m+n} z = (\gamma - \alpha)\, y^m x^n z$ → (3)

Now

$y^m x^n z \neq 0$ for if $y^m x^n z = 0$,then from (1) $x^m y^n z = 0$ and so $x^m y^n z = y^m x^n z$,
contradicting our assumption $x^m y^n z \neq y^m x^n z$.
Also $\gamma \neq 1$ for if $\gamma = 1$,then from (3),we get

$$\gamma = \alpha = 1.$$

Then from (1) we get
$x^m y^n z = y^m x^n z$, again a contradiction.

Now

From (3) $x^{m+n} z = \frac{\gamma-\alpha}{1-\gamma}\ y^m x^n z$

i.e., $x^{m+n} z = \beta\, y^m x^n z$ for some $\beta \in F$ → (4)

Similarly $y^{m+n} z = \delta\, y^m x^n z$ for some $\delta \in F$ → (5)

Also correcting to each choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in F,there is an $\eta \in F$ such that

$(\alpha_1 x + \alpha_2 y)^m (\alpha_3 x + \alpha_4 y)^n z = \eta\, (\alpha_3 x + \alpha_4 y)^m (\alpha_1 x + \alpha_2 y)^n z$

$(\alpha_1^m x^m + \alpha_2^m y^m)(\alpha_3^n x^n + \alpha_4^n y^n) z = \eta\, (\alpha_3^m x^m + \alpha_4^m y^m)(\alpha_1^n x^n + \alpha_2^n y^n) z$

$\alpha_1^m \alpha_3^n x^{m+n} z + \alpha_1^m \alpha_4^n x^m y^n z + \alpha_2^m \alpha_3^n y^m x^n z + \alpha_2^m \alpha_4^n y^{m+n} z$

$= \eta\, (\alpha_3^m \alpha_1^n x^{m+n} z + \alpha_3^m \alpha_2^n x^m y^n z + \alpha_4^m \alpha_1^n y^m x^n z + \alpha_4^m \alpha_2^n y^{m+n} z)$

$\alpha_1 \alpha_3 x^{m+n} z + \alpha_1 \alpha_4 x^m y^n z + \alpha_2 \alpha_3 y^m x^n z + \alpha_2 \alpha_4 y^{m+n} z$

$= \eta\, (\alpha_3 \alpha_1 x^{m+n} z + \alpha_3 \alpha_2 x^m y^n z + \alpha_4 \alpha_1 y^m x^n z + \alpha_4 \alpha_2 y^{m+n} z)$ → (6)

$\alpha_1 \alpha_3 \beta\, y^m x^n z + \alpha_1 \alpha_4 x^m y^n z + \alpha_2 \alpha_3 y^m x^n z + \alpha_2 \alpha_4 \delta\, y^m x^n z$

$= \eta\, (\alpha_1 \alpha_3 \beta\, y^m x^n z + \alpha_2 \alpha_3 x^m y^n z + \alpha_1 \alpha_4 y^m x^n z + \alpha_2 \alpha_4 \delta\, y^m x^n z)$

$\alpha_1 \alpha_3 \beta\, \alpha^{-1} x^m y^n z + \alpha_1 \alpha_4 x^m y^n z + \alpha_2 \alpha_3 \alpha^{-1} x^m y^n z + \alpha_2 \alpha_4 \delta\, \alpha^{-1} x^m y^n z$

$= \eta\, (\alpha_1 \alpha_3 \beta\, y^m x^n z + \alpha_2 \alpha_3 y^m x^n z + \alpha_1 \alpha_4 y^m x^n z + \alpha_2 \alpha_4 \delta\, y^m x^n z)$

$(\alpha_1 \alpha_3 \beta\, \alpha^{-1} + \alpha_1 \alpha_4 + \alpha_2 \alpha_3 \alpha^{-1} + \alpha_2 \alpha_4 \delta\, \alpha^{-1})\, x^m y^n z$

$= \eta\, (\alpha_1 \alpha_3 \beta + \alpha_2 \alpha_3 \alpha + \alpha_1 \alpha_4 + \alpha_2 \alpha_4 \delta)\, y^m x^n z$ → (7)

In (7) we choose $\alpha_2 = 0$, $\alpha_1 = \alpha_3 = 1$, $\alpha_4 = -\beta$ the R.H.S of (7) is zero where as the L.H.S of (7) is

$(\beta\, \alpha^{-1} - \beta)\, x^m y^n z = 0$

$\beta(\alpha^{-1} - 1)\, x^m y^n z = 0$

Since $x^m y^n z \neq 0$ and $\alpha \neq 1$, we get $\beta = 0$.
Hence from (4) we get $x^{m+n} z = 0$.
Also if in (7) we choose $\alpha_3 = 0$, $\alpha_2 = \alpha_4 = 1$ and $\alpha = -\delta$,the R.H.S of (7) is zero where as the L.H.S of (7) is

$(-\delta + \delta\alpha^{-1})\, x^m y^n z = 0$

$\delta(\alpha^{-1} - 1)\, x^m y^n z = 0$

Since $x^m y^n z \neq 0$ and $\alpha \neq 1$, we get $\delta = 0$.
Hence from (5) we get $y^{m+n} z = 0$.
Then (6) becomes

$\alpha_1 \alpha_4 x^m y^n z + \alpha_2 \alpha_3 y^m x^n z = \eta\ (\alpha_2 \alpha_3 x^m y^n z + \alpha_1 \alpha_4 y^m x^n z)$

Using (1) we get

$(\alpha_1 \alpha_4 + \alpha_2 \alpha_3 \alpha^{-1})\, x^m y^n z = \eta\, (\alpha_2 \alpha_3 + \alpha_1 \alpha_4\, \alpha^{-1})\, x^m y^n z.$

This is true for any choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in F.

Choose $\alpha_1 = \alpha_3 = \alpha_4 = 1$ and $\alpha_2 = -\alpha^{-1}$, we get

$(1 - (\alpha^{-1})^2)\, x^m y^n z = 0$.

Since $x^m y^n z \neq 0$, $1 - (\alpha^{-1})^2 = 0$, Hence $\alpha = \pm 1$.

Since $\alpha \neq 1$, we get $\alpha = -1$.

Thus $x^m y^n z = - y^m x^n z$.

i.e., A is either quasi weak weak (m,n) power commutative.

3.4 Note:

Taking n = 1,we get Theorem 3.1[8].

3.5 Lemma:

Let A be a algebra (not necessarily associative) over a commutative ring R.Let $m \in z^+$. Suppose A is scalar quasi weak (m,n) power commutative.Then for all x,y,z ∈ A, $\alpha \in R$, $\alpha x^m y^n z = 0$ iff $\alpha y^m x^n z = 0$.Also $x^m y^n z = 0$ iff $y^m x^n z = 0$.

Proof:

Let x,y,z ∈ A and $\alpha \in R$ such that $\alpha x^m y^n z = 0$.Since A is scalar quasi weak (m,n) power commutative,there exists $\beta = \beta(y,x,\alpha z) \in R$ such that $y^m x^n (\alpha z) = \beta x^m y^n (\alpha z)$

i.e., $\alpha y^m x^n z = \alpha \beta x^m y^n z = 0$

Conversely assume $\alpha y^m x^n z = 0$.Since A is scalar quasi weak(m,n) power commutative, there exists $\gamma = \gamma(x,y,\alpha z) \in R$ such that $x^m y^n (\alpha z) = \gamma y^m x^n (\alpha z)$

i.e., $\alpha x^m y^n z = \gamma \alpha y^m x^n z = 0$

Thus $\alpha x^m y^n z = 0$ iff $\alpha y^m x^n z = 0$

Now assume $x^m y^n z = 0$.Since A is scalar quasi weak (m,n) power commutative,there exists a scalar $\delta(y,x,z) \in R$ such that $y^m x^n z = \delta x^m y^n z = 0$.

Conversely assume $y^m x^n z = 0$.Then there exists scalars $\eta = \eta(x,y,z) \in R$ such that $x^m y^n z = \eta y^m x^n z = 0$.

Thus $x^m y^n z = 0$ iff $y^m x^n z = 0$.

3.6 Note:

Taking n = 1,we get Lemma 3.5 [8].

3.7 Lemma:

Let A be an algebra (not necessarily associative) over a commutative ring R.Let m,n $\in Z^+$. Suppose that $(x + y)^k = x^k + y^k$, k = m,n for all x,y ∈ A.Assume that $\alpha^k = \alpha$, k = m,n for all $\alpha \in R$.Let x,y,z,u ∈ A, $\alpha,\beta \in R$ such that $x^m u^n = u^m x^n$, $y^m x^n z = \alpha x^m y^n z$ and $(y+u)^m x^n z = \beta x^m (y+u)^n z$ then $(x^m u^n - \alpha x^m u^n) - \beta(x^m u^n + \alpha\beta x^m u^n) z = 0$.

Proof:

Given $(y+u)^m x^n z = \beta x^m (y+u)^n z \quad \to (1)$

$y^m x^n z = \alpha x^m y^n z \quad \to (2)$

$x^m u = u^m x \quad \to (3)$

From (1) we get

$(y^m + u^m) x^n z = \beta x^m (y^n + u^n) z$

$y^m x^n z + u^m x^n z = \beta x^m y^n z + \beta x^m u^n z \quad \to (4)$

$\alpha x^m y^n z + u^m x^n z = \beta x^m y^n z + \beta x^m u^n z$ (using (2))

$\alpha x^m y^n z + x^m u^n z = \beta x^m y^n z + \beta x^m u^n z$ (using (3))

$x^m(\alpha y^n + u^n - \beta y^n - \beta u^n) z = 0$

$x^m(\alpha y + u - \beta y - \beta u)^n z = 0$

By Lemma (3.5) we get

$(\alpha y + u - \beta y - \beta u)^m x^n z = 0$.

$((\alpha y)^m + u^m - (\beta y)^m - (\beta u)^m) x^n z = 0$.

$(\alpha y^m + u^m - \beta y^m - \beta u^m) x^n z = 0$.

$\alpha y^m x^n z + u^m x^n z - \beta y^m x^n z - \beta u^m x^n z = 0$

$\alpha y^m x^n z + u^m x^n z - \alpha \beta x^m y^n z - \beta u^m x^n z = 0$ (using(2)) $\to (5)$

From (4) we get

$y^m x^n z - \beta x^m y^n z - \beta x^m u^n z + u^m x^n z = 0$

Multiplying by α

$\alpha y^m x^n z - \alpha \beta x^m y^n z - \alpha \beta x^m u^n z + \alpha u^m y^n z = 0 \quad \to (6)$

(5) – (6) gives

$u^m x^n z - \beta u^m x^n z + \alpha \beta x^m u^n z - \alpha u^m y^n z = 0$

$\alpha \beta x^m u^n - \alpha u^m y^n z - \beta u^m x^n z + u^m y^n z = 0$

$(u^m x^n - \alpha u^m x^n - \beta u^m x^n + \alpha \beta x^m u^n) z = 0$

$(x^m u^n - \alpha x^m u^n - \beta x^m u^n + \alpha \beta x^m u^n) z = 0$ (using (3))

3.8 Corollary:

Taking u = x,we get

$(x^{m+n} - \alpha x^{m+n} - \beta x^{m+n} + \alpha\beta x^{m+n}) z = 0$

$(x^{m} - \alpha x^{m}) (x^{n} - \beta x^{n}) z = 0.$

3.9 Note:

Taking n = 1 ,we get Lemma 3.7 [8] and corollary 3.8 [8].

3.10 Theorem:

Let A be an algebra (not necessarily associative) over a commutative ring R. Let m,n $\in Z^{+}$.Assume $(x+y)^{k} = x^{k} + y^{k}$, k = m,n for all x,y ∈ A and that A has no zero divisors. Assume $\alpha^{k} = \alpha$, k = m,n for all $\alpha \in$ R.If A is scalar quasi weak (m,n) power commutative,then A is quasi weak (m,n) power commutative**.**

Proof:

Let x,y,z ∈ A.Since A is scalar quasi weak (m,n) power commutative,there exists scalars $\alpha = \alpha$ (y,x,z) ∈ R and $\beta = \beta$ (y+ x , x,z) ∈ R such that

$(y+x)^{m+n} x^{n} z = \beta x^{m} (y+x)^{n} z \quad \rightarrow (1)$

$y^{m} x^{n} z = \alpha x^{m} y^{n} z \quad \rightarrow (2)$

From (1) we get

$(y^{m} + x^{m}) x^{n} z = \beta x^{m} (y^{n} + x^{n}) z$

i.e., $y^{m} x^{n} z + x^{m+n} z = \beta x^{m} y^{n} z + \beta x^{m+n} z \quad \rightarrow (3)$

$\alpha x^{m} y^{n} z + x^{m+n} z - \beta x^{m} y^{n} z - \beta x^{m+n} z = 0$

$x^{m} (\alpha y^{n} + x^{n} - \beta y^{n} - \beta x^{n}) z = 0$

$x^{m} (\alpha y + x - \beta y - \beta x)^{n} z = 0$

By Lemma 3.5 we get

$(\alpha y + x - \beta y - \beta x)^{m} x^{n} z = 0$

$(\alpha y^{m} + x^{m} - \beta y^{m} - \beta x^{m}) x^{n} z = 0$

$\alpha y^{m} x^{n} z + x^{m+n} z - \beta y^{m} x^{n} z - \beta x^{m+n} z = 0$

$\alpha y^{m} x^{n} z + x^{m+n} z - \alpha \beta x^{m} y^{n} z - \beta x^{m+n} z = 0 \quad$ (using (2)) $\rightarrow (4)$

Multiply (3) by α ,

$\alpha y^{m} x^{n} z + \alpha x^{m+n} z - \alpha \beta x^{m} y^{n} z - \alpha \beta x^{m+n} z = 0 \quad \rightarrow (5)$

(4) – (5) gives ,

$x^{m+n} z - \alpha x^{m+n} z - \beta x^{m+n} z + \alpha \beta x^{m+n} z = 0$

$(x^{m+n} - \alpha x^{m+n} - \beta x^{m+n} + \alpha \beta x^{m+n}) z = 0$

$x^{m+n-2} (x - \alpha x) (x - \beta x) z = 0$

Since A has no zero divisors,

z = 0 or $x^{m} - \alpha x^{m}$, $x^{n} - \beta x^{n} = 0$, x = 0

If x = 0 or z = 0 then $x^{m} y^{n} z = y^{m} x^{n} z \quad \forall$ x,y,z ∈ A.

If x = αx, from (2) we get $y^{m} x^{n} z = x^{m} y^{n} z$.

If x = βx,then from (3) we get

$y^{m} x^{n} z + x^{m+n} z = x^{m} y^{n} z + x^{m+n} z$

$y^{m} x^{n} z = x^{m} y^{n} z$

3.11 Note:

Taking n = 1,we get Lemma 3.10 [8].

3.12 Definition:

Let R be any ring.Let m>1 be a fixed integer.An element a ∈ R is said to be m – potent if $a^{m} = a$.

3.13 Lemma:

Let A be an algebra with unity over a P.I.D R, let m, n $\in Z^{+}$.Assume $(x+y)^{k} = x^{k} + y^{k}$, k = m,n for all x,y ∈ A and that $\alpha^{k} = \alpha$, k = m,n for all $\alpha \in$ R. If A is scalar quasi weak (m,n) power commutative, x ∈ A such that $O(x^{m+n+1}) = 0$,then $x^{m} y^{n} z = y^{m} x^{n} z$ for all y,z ∈ A.

Proof:

Let x ∈ A such that $O(x^{m+n}) = 0$.Let y,z ∈ A.Then there exists scalars $\alpha = \alpha$ (y,x,z) ∈ R and $\beta = \beta$ (y+ x , x,z) ∈ R such that

$(y+x)^{m} x^{n} z = \beta x^{n} (y+x)^{n} z \quad \rightarrow (1)$

and $y^{m} x^{n} z = \alpha x^{m} y^{n} z \quad \rightarrow (2)$

From (2) we get

$(y^{m} + x^{m}) x^{n} z = \beta x^{m} (y^{n} + x^{n}) z$

$y^m x^n z + x^{m+n} z - \beta x^m y^n z - \beta x^{m+n} z = 0$ → (3)

$\alpha x^m y^n z + x^{m+n} z - \beta x^m y^n z - \beta x^{m+n} z = 0$

i.e., $x^m (\alpha y^n + x^n - \beta y^n - \beta x^n) z = 0$

$x^m (\alpha y + x - \beta y - \beta x)^n z = 0$

By Lemma 3.5, we get

$(\alpha y + x - \beta y - \beta x)^m x^n z = 0$

$(\alpha y^m + x^m - \beta y^m - \beta x^m) x^n z = 0$

$\alpha y^m x^n z + x^{m+n} z - \beta y^m x^n z - \beta x^{m+n} z = 0$

$\alpha y^m x^n z + x^{m+n} z - \alpha \beta x^m y^n z - \beta x^{m+n} z = 0$ (using (2)) → (4)

Multiply (3) by α

$\alpha y^m x^n z + \alpha x^{m+n} z - \alpha \beta x^m y^n z - \alpha \beta x^{m+n} z = 0$ → (5)

(4) – (5) gives

$x^{m+n} z - \beta x^{m+n} z - \alpha x^{m+n} z + \alpha\beta x^{m+n} z = 0$

$(1 - \alpha - \beta + \alpha \beta) x^{m+n} z = 0$

$(1 - \alpha)(1 - \beta) x^{m+n} z = 0$ → (6)

Thus for each z ∈ A ,there exists scalars $\gamma \in R$, $\delta \in R$ such that

$\gamma x^{m+n} z = 0$ → (7)

and $\delta x^{m+n} (z + 1) = 0$ → (8)

$\gamma \times (8) - \delta \times (7)$ gives

$\gamma\delta x^{m+n} z + \gamma\delta x^{m+n} - \gamma\delta x^{m+n} z = 0$

$\gamma\delta x^{m+n} = 0$

Since $O(x^{m+n}) = 0$ we get $\gamma = 0$ or $\delta = 0$.

Hence from (6) we get

$(1 - \alpha) = 0$ or $(1 - \beta) = 0.$

If $\alpha = 1$,from (2) we get

$y^m x^n z = x^m y^n z$

If $\beta = 1$ from (3) we get

$y^m x^n z + x^{m+n} z - x^m y^n z - x^{m+n} z = 0$

$y^m x^n z = x^m y^n z$

Hence the theorem.

3.14 Lemma:

Let A be an algebra with identity over a P.I.D R.Let $m \in Z^+$.Suppose $(x+y)^k = x^k + y^k$ for k = m,n and for all x,y ∈ A and that every element of R is m-potent.Suppose that A is scalar quasi weak (m,n) - power commutative.Assume further that there exists a prime p ∈ R such that $p^{mn} A = 0$. Then A is quasi weak.

Proof:

Let x,y ∈ A such that $O(y^m x^n) = p^k$ for some $k \in Z^+$.

We prove by induction on k that $x^m y^n u = y^m x^n u$ for all u ∈ A.

If k = 0,then $O(y^m x^n) = p^0 = 1$ and so $y^m x^n = 0$.

So $y^m x^n u = 0$ for all u ∈ A.

By Lemma 3.5 $x^m y^n u = 0$ for all u ∈ A.

So assume that k > 0 and that the statement is true for all l < k.

If $x^m y^n u - y^m x^n u = 0$, ∀ u ∈ A , there is nothing to prove.

So,let $x^m y^n u - y^m x^n u \neq 0$.

Since A is scalar quasi weak (m,n) power commutative there exists scalars $\alpha = \alpha(x,y,u) \in R$ and $\beta = \beta(x,y+x,u) \in R$ such that

$x^m y^n u = \alpha y^m x^n u$ → (1)

and $x^m (y + x)^n u = \beta (y + x)^m x^n u$ → (2)

From (2) we get

$x^m (y^n + x^n) u = \beta (y^m + x^m) x^n u$

$x^m y^n u + x^{m+n} u = \beta y^m x^n u + \beta x^{m+n} u$ → (3)

$\alpha y^m x^n u + x^{m+n} u = \beta y^m x^n u + \beta x^{m+n} u$ (using(1))

$(\alpha - \beta) y^m x^n u = (\beta - 1) x^{m+n} u$ → (4)

If $(\alpha - \beta) y^m x^n u = 0$,we get $(\beta - 1) x^{m+n} u = 0$.

Since $x^{m+n} u \neq 0$, we get $\beta = 1$.

Hence from (3) we get

$x^m y^n u = y^m x^n u$,contradicting our assumption that $x^m y^n u - y^m x^n u \neq 0$.

So $(\alpha - \beta)\, y^m x^n u \neq 0$.Inparticular $(\alpha - \beta) \neq 0$.
Let $(\alpha - \beta) = p^t \delta$ for some $t \in Z^+$ and $\delta \in R$with $(\delta, p) = 1$.If $t \geq k$,then since $O(y^m x^n) = p^k$ we would get $(\alpha - \beta)\, y^m x^n u = 0$, again a contradiction.
Hence $t < k$.Since $p^k y^m x^n u = 0$ by Lemma 3.5, $p^k x^m y^n u = 0$.
So,from (4) we get $p^{k-t} (\beta - 1)\, x^{m+n} u = p^{k-t} (\alpha - \beta)\, y^m x^n u$

$$= p^{k-t} p^t \delta\, y^m x^n u$$
$$= p^k \delta\, y^m x^n u = 0.$$

Let $O(x^{m+n} u) = p^i$.If $i < k$,then by induction hypothesis $x^m y^n u = y^m x^n u$, a contradiction.so $i \geq k$.
Now

$$p^k \mid p^i \mid p^{(k-t)} (\beta - 1).\text{So } p^t \mid (\beta - 1).$$

Let (- 1) = $p^t \gamma$ for some $\gamma \in R$.
Then from (4) we get

$$p^t \delta\, y^m x^n u = p^t \gamma\, x^{m+n} u.$$
$$p^t (\delta\, y^m - \gamma x^m)\, x^n u = 0$$
$$\text{(i.e.,)} \quad p^t (\delta\, y - \gamma x)^m x^n u = 0$$

Hence by induction hypothesis
$(\delta\, y - \gamma x)^m x^n (uw) = x^m (\delta\, y - \gamma x)^n (uw) = 0$ for all $w \in A$.
Taking u = 1, we get

$$(\delta\, y - \gamma x)^m x^n w = x^m (\delta\, y - \gamma x)^n w = 0$$

i.e., $\delta\, y^m x^n w - \gamma\, x^{m+n} w = \delta\, x^m y^n w - \gamma\, x^{m+n} w$
$\delta\, (y^m x^n w - x^m y^n w) = 0$.
Since $(\delta, p) = 1$,there exists $\mu, \delta \in R$ such that $\mu\, p^m + \gamma\delta = 1$
Therefore $\mu\, p^m (y^m x^n w - x^m y^n w) + \gamma\delta\, (y^m x^n w - x^m y^n w) = y^m x^n w - x^m y^n w$

$$0 + 0 = y^m x^n w - x^m y^n w.$$
$$\text{i.e., } y^m x^n w - x^m y^n w = 0.$$

Hence the Lemma.

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