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Boosters of MV-algebras

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***ABSTRACT**: Similar to the notion of ⊕-stabilizers in MV-algebras, the present paper introduce the notion of boosters. Then some elementary properties of principle boosters and boosters are presented, and a principle booster is proved to an MV-filter. According to the properties of of principle boosters, we prove that the set of all principle boosters is a complete distributive lattice.*

***KEYWORDS**: MV-algebra, booster, principle booster, MV-filters.*

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I. Introduction

As an important branch of non-classical mathematical logic, fuzzy logic is the basis of reasoning mechanism in many fields, such as artificial intelligence and information science. With the development of fuzzy propositional logic system, various kinds of fuzzy logic algebras appear one after another. Various logical algebras have been proposed and researched as the semantical systems of non-classical logical systems. In order to study the basic logic framework of fuzzy set system, Hάjek [1] proposed a new fuzzy logic system— BL-system and corresponding logic algebraic system based on the theoretical framework of continuous triangular modules in residual lattices. MV-algebras were introduced by Chang [2] in order to give an algebraic proof of the completeness theorem of Lukasiewice system of many valued logic.

Ideals and filters are very effective tool for studying various logical algebras and the completeness of the corresponding non-classical logics. In BL-algebras, the focus is deductive systems also called filters, with the lack of a suitable algebraic addition, the notion of ideals is missing in BL-algebras. To fill the gap, Lele and Nganou [3] introduced the notion of ideals in BL-algebras, which generalized in a natural sense the existing notion in MV-algebras and subsequently all the results about ideals in MV-algebras. Follow Lele and Nganou's work, [4,5] investigated the notion of ideals in residual lattices as a natural generalization of the concept of ideals in BL-algebras. In the theory of MV-algebras, ideals theory is at the center, and filters and ideals are dual notions. Turunen [6] gave the notion of co-annihilators of a non-empty set and proved some of their elementary properties on BL-algebras. Meng and Xin [7] introduced the concept of generalized co-annihilators relative to filters as a generalization of co-annihilator in BL-algebras and studied basic properties of generalized co-annihilators. Zou et al. [8] introduced the notion of annihilators and generalized annihilator relative to ideals on BL-algebras, moreover, they further studied generalized annihilator of BL-algebras in [9].

In [8], the annihilator of X is defined as $X^{\perp}=\{a\in L\mid a\wedge x=0,\forall x\in X\}$, where X is a non-void subset of a BL-algebra L. [10] introduced $\oplus$- stabilizer $St_{\oplus}(X)=\{a\in L\mid a\oplus x=x,\forall x\in X\}$ and $\otimes$- stabilizer $St_{\otimes}(X):=\{a\in A\mid x\otimes a^{*}=x,\forall x\in X\}=\{a\in A\mid a\wedge x=0,\forall x\in X\}$ of a non-empty subset X in an MV-algebra A. It is easy to see that the $\otimes$- stabilizer of X is coincide with the annihilator of X in an MV-algebra. While in the paper, we introduce the notion of boosters in MV-algebras, and the booter of a non-void subset X of an MV-algebra A is expressed as $X^{\circ}:=\{a\in A\mid x\oplus a=a,\forall x\in X\}$. We also investigate some related properties of principle boosters and boosters, and show that principle boosters and boosters are MV-filters. We define the set of all principle boosters of A by $B_o(A)$. Then we prove that $B_o(A)$ is a complete distributive lattice.

II. Preliminaries

An algebra structure $(A,\oplus,*,0)$ of type $(2,1,0)$ is called an MV-algebra if it satisfies the following axioms: for any $x,y,z\in A$,

(MV-1) $(x\oplus y)\oplus z=x\oplus(y\oplus z)$, (MV-2) $x \oplus y = y \oplus x$,

(MV-3) $x \oplus 0 = x$,

(MV-4) $x^{**} = x$,

(MV-5) $x \oplus 0^* = 0^*$,

(MV-6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Let $(A, \oplus, *, 0)$ be an MV-algebra, for any $x, y \in A$, we put $1 = 0^*$, $x \otimes y = (x^* \oplus y^*)^*$, $x \to y = x^* \oplus y$, $x \$ y = x \otimes y^*$, $x \vee y = (x^* \oplus y) \oplus y = (x \$ y) \oplus y$, $x \wedge y = (x^* \vee y^*)^* = (x \$ y) \$ y$. With respect to these operations, $(A, \oplus, *, 0)$ is a distributive lattice. In what follows, $(A, \oplus, *, 0)$ is always assumed to be an MV-algebra unless otherwise specified, and will often be referred to by its support set A. The order relation $\leq$ on A is defined as: $x \leq y$ if and only if $x = x \wedge y$ for any $x, y \in A$. The relation $\leq$ can be alternatively characterized as follows.

Lemma 2.1. ([2, 11]) In any MV-algebra A, the following are equivalent: for any $x, y \in A$,

(i) $x \leq y$,

(ii) $x \$ y = 0$,

(iii) $x \otimes y^* = 0$,

(iv) there is $z \in A$ such that $y = x \oplus z$.

Proposition 2.2. ([11]) In any MV-algebra A, the following properties hold: for any $x, y, z \in A$,

(1) $x \oplus 1 = 1$, $x \$ 0 = x$, $x \oplus x^* = 1$,

(2) $x \$ y \leq z$ if and only if $x \leq y \oplus z$,

(3) $x \$ y \leq x \wedge y \leq x \vee y \leq x \oplus y$.

An element $x \in A$ is called complemented if there is an element $y \in A$ such that $x \vee y = 1$ and $x \wedge y = 0$. We denote the set of complemented of A by $B(A)$. If $x \in B(A)$, then the following conditions are equivalent: (i) $x \oplus x = x$, (ii) $x \otimes x = x$, (iii) $x^* \oplus x^* = x^*$, (iv) $x^* \otimes x^* = x^*$, (v) $x \vee x^* = 1$, (vi) $x \wedge x^* = 0$.

Filters, the order dual of ideals, have a variety of applications in logic and topology.

Definition 2.3. ([12]) Let A be an MV-algebra. A nonempty subset F of A is called an MV-filter if it satisfies: for any $x, y \in A$,

(i) $x, y \in F$ implies $x \wedge y \in F$,

(ii) if $x \leq y$ and $x \in F$, then $y \in F$.

It is easy to shown that a nonempty subset F of A is an MV-filter if and only if for any $x, y \in A$, (1) $x, y \in F$ implies $x \otimes y \in F$, (2) if $x \leq y$ and $x \in F$, then $y \in F$. It is shown that a nonempty subset F of A is an MV-filter if and only if for any $x, y \in A$, (1) $1 \in F$; (2) $x \in F$ and $x \to y \in F$ imply $y \in F$.

Definition 2.3. ([1]) Let A, B be two MV-algebras. A function $h : A \to B$ is an MV-homomorphism iff it satisfies the following conditions: for any $x, y \in A$, (1) $h(0) = 0$, (2) $h(x \oplus y) = h(x) \oplus h(y)$, (3) $h(x^*) = h(x)^*$.

III. Booters in MV-algebras

Definition 3.1. Let $\varnothing \neq X \subseteq A$. Then the set

$$X^\circ := \{a \in A \mid x \oplus a = a, \forall x \in X\}$$

is said to be the booster of X. If $X = \{x\}$, we often write $\{x\}^\circ$ simply as x°, which is called the principle booster of x.

Lemma 3.2. ([1]) In an MV-algebra $(A, \oplus, *, 0)$, the following four conditions are equivalent: for any

$x, y \in A$, (i) $y \vee x^* = 1$; (ii) $x \wedge y^* = 0$; (iii) $x \oplus y = y$; (iv) $x \otimes y = x$.

From Definition 3.1, it can be easily observed that $0^\circ = A$ and $1^\circ = \{1\}$. The following lemmas reveals some basic properties of boosters.

Lemma 3.3. For any $x, y \in A$, the following results hold:

(1) if $x \leq y$, then $y^\circ \subseteq x^\circ$;

(2) if $a \in x^\circ$, then $x \leq a$;

(3) if $a \in x^\circ$ and $b \in A$ such that $a \leq b$, then $b \in x^\circ$;

(4) if $a \in x^\circ$ and $b \in x^\circ$, then $a \otimes b \in x^\circ$;

(5) $x^\circ \bigcup y^\circ \subseteq (x \wedge y)^\circ$, $x^\circ \bigcap y^\circ = (x \vee y)^\circ$;

(6) $(x \oplus y)^\circ = x^\circ \bigcap y^\circ$, moreover $(nx)^\circ = x^\circ$, where $0x := 0$, $(n+1)x := nx \oplus x$ for $n \geq 1$;

(7) if $a \in x^\circ$ and $b \in y^\circ$, then $a \otimes b, a \wedge b, a \vee b, a \oplus b \in (x \wedge y)^\circ$.

Proof: (1) For any $a \in y^\circ$, according to Lemma 3.2, we have $a \vee y^* = 1$. Since $x \leq y$, then $y^* \leq x^*$, and so $1 = a \vee y^* \leq a \vee x^*$. Consequently, $a \vee x^* = 1$, which means that $a \in x^\circ$, therefore $y^\circ \subseteq x^\circ$.

(2) Assume that $a \in x^\circ$, then we get that $x \oplus a = a$. From Lemma 2.1, it follows that $x \leq a$.

(3) If $a \in x^\circ$, then $a \vee x^* = 1$. From $a \leq b$, we obtain that $1 = a \vee x^* \leq b \vee x^*$, hence $b \vee x^* = 1$, and so $b \in x^\circ$.

(4) From $a \in x^\circ$ and $b \in x^\circ$, it follows that $x \otimes a = x$ and $x \otimes b = x$ by Lemma 3.2. Hence $x \otimes (a \otimes b) = (x \otimes a) \otimes b = x \otimes b = x$, and so $a \otimes b \in x^\circ$.

(5) For any $a \in x^\circ \bigcup y^\circ$, we have $a \in x^\circ$ or $a \in y^\circ$, so $a^* \wedge x = 0$ or $a^* \wedge y = 0$ by Lemma 3.2. Hence $a^* \wedge (x \wedge y) = 0$, that is, $a \in (x \wedge y)^\circ$, and therefore $x^\circ \bigcup y^\circ \subseteq (x \wedge y)^\circ$.

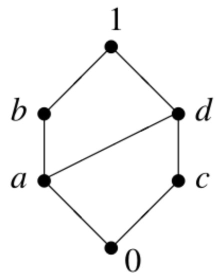
$a \in x^\circ \bigcap y^\circ$ iff $a \in x^\circ$ and $a^* \wedge y = 0$ $a \in y^\circ$ iff $a^* \wedge x = 0$ and iff $a^* \wedge (x \vee y) = 0$ iff $a \in (x \vee y)^\circ$.

(6) Since $x \vee y \leq x \oplus y$, then $(x \oplus y)^\circ \subseteq (x \vee y)^\circ = x^\circ \bigcap y^\circ$ by (1) and (5). To obtain the reverse inclusion, observe that if $a \in x^\circ \bigcap y^\circ$, then $a \in x^\circ$ and $a \in y^\circ$, and so $x \oplus a = a$ and $y \oplus a = a$. Let us calculate $(x \oplus y) \oplus a = x \oplus (y \oplus a) = x \oplus a = a$, thus $a \in (x \oplus y)^\circ$, and so $x^\circ \bigcap y^\circ \subseteq (x \oplus y)^\circ$, which leads to $(x \oplus y)^\circ = x^\circ \bigcap y^\circ$.

(7) If $a \in x^\circ$ and $b \in y^\circ$, using $x^\circ \subseteq (x \wedge y)^\circ$ and $y^\circ \subseteq (x \wedge y)^\circ$, we get that $a, b \in (x \wedge y)^\circ$. From (3) and (4), we see that $a \otimes b, a \wedge b, a \vee b, a \oplus b \in (x \wedge y)^\circ$.

The following example shows that the equation $x^\circ \bigcup y^\circ \subseteq (x \wedge y)^\circ$ for Lemma 3.3 may be not true in general.

Example 3.4. Let $A = \{0, a, b, c, 1\}$ be a set with a Hasse diagram and Cayley tables as follows.



$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	b	b	d	1	1
b	b	b	b	1	1	1
c	c	d	1	c	d	1
d	d	1	1	d	1	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A,\oplus,*,0)$ is an MV-algebra [3]. Routine calculation shows that $b^\circ=\{b,1\}$ and $c^\circ=\{c,d,1\}$, but $b^\circ\bigcup c^\circ=\{b,c,d,1\}\neq(b\wedge c)^\circ=0^\circ=\{0,a,b,c,1\}$.

From the properties (3), (4) and (7) of Lemma 3.3, it is easy to obtain the following result.

Proposition 3.5. Let A be an MV-algebra and $x\in A$. Then x° is a filter of M.

Proposition 3.6. For any two elements $x,y\in A$, we have the following:

(1) $x^\circ=y^\circ$ implies $(x\wedge z)^\circ=(y\wedge z)^\circ$ for any $z\in A$;

(2) $x^\circ=y^\circ$ implies $(x\vee z)^\circ=(y\vee z)^\circ$ for any $z\in A$;

(3) $x^\circ=y^\circ$ implies $(x\oplus z)^\circ=(y\oplus z)^\circ$ for any $z\in A$.

Proof. (1) Suppose $x^\circ=y^\circ$. For any $a\in(x\wedge z)^\circ$, then $1=a\vee(x\wedge z)^*=a\vee x^*\vee z^*$. Consequently, $a\vee z^*\in x^\circ=y^\circ$, and so $1==a\vee z^*\vee y^*=a\vee(y\wedge z)^*$, that is, $a\in(y\wedge z)^\circ$, so $(x\wedge z)^\circ\subseteq(y\wedge z)^\circ$. Similarly, we can get $(y\wedge z)^\circ\subseteq(x\wedge z)^\circ$, hence $(x\wedge z)^\circ=(y\wedge z)^\circ$.

(2) Assume that $x^\circ=y^\circ$. If $a\in(x\vee z)^\circ$, then $1=a\vee(x\vee z)^*=a\vee(x^*\wedge z^*)=(a\vee x^*)\wedge(a\vee z^*)$, it follows that $a\vee x^*=1$ and $a\vee z^*=1$. Hence $a\in x^\circ=y^\circ$ and $a\in z^\circ$, and so $a\in y^\circ\bigcap z^\circ=(y\vee z)^\circ$, thus $(x\vee z)^\circ\subseteq(y\vee z)^\circ$. Similarly, we can prove that $(y\vee z)^\circ\subseteq(x\vee z)^\circ$, and hence $(x\vee z)^\circ=(y\vee z)^\circ$.

(3) Since $(x\oplus z)^\circ=(x\vee z)^\circ$ by Lemma 3.3, then using (2) we get that $(x\oplus z)^\circ=(y\oplus z)^\circ$.

Proposition 3.7. Let $\Phi\neq X,Y\subseteq A$ and F be a MV-filter of A. Hence

(1) $X^\circ=A$ if and only if $X=\{0\}$;

(2) $F^\circ=\{1\}$;

(3) $X^\circ=\bigcap_{x\in X}x^\circ$;

(4) if $X\subseteq Y$, then $Y^\circ\subseteq X^\circ$;

(5) $X^\circ\bigcup Y^\circ\subseteq(X\bigcap Y)^\circ$, $X^\circ\bigcap Y^\circ=(X\bigcup Y)^\circ$;

(6) if $h:A\to A$ is an MV-homomorphism and $x\in A$, then $h(x^\circ)\subseteq(h(x))^\circ$;

(7) $X^\circ\bigcap X\subseteq B(A)\bigcap X$.

Proof. (1) If $X=\{0\}$, then $X^\circ=\{0\}^\circ=\{a\in A\mid 0\oplus a=a\}=A$. Conversely, suppose that $X^\circ=A$, since $0\in A=X^\circ$, then for any $x\in X$ we have $0\vee x^*=x^*=1$, thus $x=0$. Hence $X=\{0\}$.

(2) Let F be a MV-filter of A. For any $a\in F^\circ$, from $1\in F$ we get that $1=1\oplus a=a$, hence $F^\circ=\{1\}$.

(3) $a\in X^\circ$ if and only if $x\oplus a=a$ for any $x\in X$ if and only if $a\in X^\circ$ for any $x\in X$ if and only if $X^\circ=\bigcap_{x\in X}x^\circ$.

(4) Assume that $a \in Y^\circ$, then $y \oplus a = a$ for any $y \in Y$. Since $X \subseteq Y$, then $y \oplus a = a$ for any $y \in X$. Thus $Y^\circ \subseteq X^\circ$.

(5) Since $X \bigcap Y \subseteq X, Y$, then $X^\circ \subseteq (X \bigcap Y)^\circ$ and $Y^\circ \subseteq (X \bigcap Y)^\circ$ by (4), therefore $X^\circ \bigcup Y^\circ \subseteq (X \bigcap Y)^\circ$. Since $X, Y \subseteq X \bigcup Y$, then $(X \bigcup Y)^\circ \subseteq X^\circ$ and $(X \bigcup Y)^\circ \subseteq Y^\circ$, and so $(X \bigcup Y)^\circ \subseteq X^\circ \bigcap Y^\circ$. For any $a \in X^\circ \bigcap Y^\circ$, we have $a \in X^\circ$ and $a \in Y^\circ$, it follows that $a \vee x^* = 1$ for any $x \in X$ and $a \vee y^* = 1$ for any $y \in Y$. For any $z \in X \bigcup Y$, we have $a \vee z^* = 1$, that is, $a \in (X \bigcup Y)^\circ$, thus $X^\circ \bigcap Y^\circ = (X \bigcup Y)^\circ$.

(6) For any $b \in h(x^\circ)$, then there exists $a \in x^\circ$ such that $b = h(a)$. It follows that $x \oplus a = a$, since h is an MV-homomorphism, we get $h(x) \oplus h(a) = h(a)$, so $b = h(a) \in (h(x))^\circ$, hence $h(x^\circ) \subseteq (h(x))^\circ$.

(7) For any $x \in X^\circ \bigcap X$, then $x \in X^\circ$ and $x \in X$. From the definition of X°, we get that $x \oplus x = x$, so $x \in B(A)$. Hence $X^\circ \bigcap X \subseteq B(A) \bigcap X$.

Theorem 3.8. Let A be an MV-algebra. The following hold:
(1) If $\Phi \neq X \subseteq A$, then X° is an MV-filter of A;
(2) $x \in B(A)$ if and only if $x \in x^\circ$.

Proof. (1) If $a \in X^\circ$ and $b \in X^\circ$, then $a \otimes x = x$ and $b \otimes x = x$ for any $x \in X$, therefore $(a \otimes b) \otimes x = a \otimes (b \otimes x) = a \otimes x = x$, and so $a \otimes b \in X^\circ$. If $a \in X^\circ$ and $b \in A$ such that $a \leq b$, then $a \vee x^* = 1$ for any $x \in X$, and so $1 = a \vee x^* \leq b \vee x^*$. It follows that $b \vee x^* = 1$, thus $b \in X^\circ$, consequently, X° is an MV-filter of A.

(2) $x \in B(A)$ if and only if $x \oplus x = x$ if and only if $x \in x^\circ$.

Now, let us denote the set of all principle boosters of A by $B_o(A)$. Then we prove that $B_o(A)$ is a complete distributive lattice.

Theorem 3.9. For an MV-algebra A, the set $B_o(A)$ is a complete distributive lattice, moreover, $(x \vee y)^\circ$ and $(x \wedge y)^\circ$ are the infimum and supremum of both x° and y° in $B_o(A)$, respectively.

Proof. Obviously, $B_o(A)$ is a partially ordered set with respect to the set inclusion. For any $x, y \in A$, we define x° ò $y^\circ := (x \wedge y)^\circ$, x° ó $y^\circ := (x \vee y)^\circ$. Now we show that $(x \wedge y)^\circ$ is the supremum of both x° and y°. Clearly, $(x \wedge y)^\circ$ is an upper bound for both x° and y°. Suppose $x^\circ \subseteq z^\circ$ and $y^\circ \subseteq z^\circ$ for some $z \in A$. For any $a \in (x \wedge y)^\circ$, we have $1 = a \vee (x \wedge y)^* = a \vee x^* \vee y^*$, and so $a \vee x^* \in y^\circ \subseteq z^\circ$. It follows that $a \vee x^* \vee z^* = 1$, hence $a \vee z^* \in x^\circ \subseteq z^\circ$, which implies that $a \vee z^* = 1$, that is, $a \in z^\circ$, thus $(x \wedge y)^\circ \subseteq z^\circ$. Consequently, $(x \wedge y)^\circ$ is the supremum of both both x° and y°. Similarly, we can prove $(x \vee y)^\circ$ is the infimum of both x° and y°. Hence $(B_o(A), $ó$, $ò$, 1^\circ, 0^\circ)$ is a bounded lattice. According to the extension of the property (5) of Lemma 3.3, we can obtain that $(B_o(A), $ó$, $ò$, 1^\circ, 0^\circ)$ is a complete lattice. It can be easily obtained that $(B_o(A), $ó$, $ò$, 1^\circ, 0^\circ)$ is a complete distributive lattice.

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