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Numerical Simulation of One-Dimensional Wave Equation by Non-Polynomial Quintic Spline

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Abstract: *This paper present a numerical algorithm for the linear one-dimensional wave equation. In this method a finite difference approach had been used to describe the time derivative while quintic spline is used as an interpolation function in the space dimension. We discuss the accuracy of the method by expanding the equation based on Taylor series and minimizing the error. The proposed method was eight-order accuracy in space and fourth-order accuracy in time variables. From the computational point of view, the solution obtained by this method is in excellent agreement with those obtained by previous works and also it is efficient to use. Numerical examples are given to show the applicability and efficiency of the method.*

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I. Introduction

One dimensional wave equation arise in many physical and engineering applications such as continuum physics, mixed models of transonic flows, fluid dynamics, etc. Many authors have studied the numerical solutions of linear hyperbolic wave equations by using various techniques. Consider one-dimensional wave equation of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le l, \qquad t \ge 0 \tag{1}$$

with initial condition

$$u(x,0) = f_2(x), \qquad \frac{\partial u(x,0)}{\partial t} = f_3(t), \quad 0 \le x \le l \tag{2}$$

and boundary conditions

$$u(0,t) = p_2(t), \;\; t \ge 0 \quad and \quad u(l,t) = q_2(t), \;\; t \ge 0 \tag{3}$$

where c^2 and l are positive finite real constants and $f_1(x)$, $f_2(x)$, $f_3(x)$, $p_1(t), p_2(t)$, $q_1(t)$ and $q_2(t)$ are real continues functions.

Some phenomena, which arise in many fields of science such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics, can be modeled by partial differential equations. The wave equation is of primary importance in many physical systems such as electro thermal analogy, signal formation, draining film, water transfer in soils, mechanics and physics, elasticity and etc.

There are several numerical schemes that have been developed for the solution of wave equation. In this paper the formation of non-polynomial quintic spline has been developed and the consistency relation obtained is useful to discretize wave equation. We present discretization of the equation by a finite difference approximation to obtain the formulation of proposed method. Truncation error and stability analysis are discussed. In this study, we approximate the functions based on Taylor series to minimize the error term and to obtain the class of methods. Numerical experiments are conducted to demonstrate the viability and the efficiency of the proposed method computationally.

II. Formulation of Tension Quintic Spline Function

In recent years, many scholars have used non-polynomial spline for solving differential equations. The spline function is a piecewise polynomial or non-polynomial of degree n satisfying the continuity of $(n-1)$th derivative. Tension quintic spline is a non-polynomial function that has six parameters to be determined hence it can satisfy the conditions of two end points of the interval and continuity of first, second, third and fourth derivatives. We introduce the set of grid points in the interval $[0, l]$ in space direction.

$$x_i = ih, \;\; h = \frac{l}{n+1}, \;\; i = 0,1,2, \dots \dots \dots, n+1$$

For each segment quintic spline $p_i(x)$ is defined by

$$p_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3 + e_i\left(e^{w(x-x_i)} - e^{-w(x-x_i)}\right) + f_i\left(e^{w(x-x_i)} - e^{-w(x-x_i)}\right), \quad i = 0,1,2,\dots\dots\dots,n \tag{4}$$

Where a_i, b_i, c_i, d_i, e_i and f_i are the unknown coefficients to be determined also ω is free parameter. If $\omega \to 0$ then $p_i(x)$ reduces to quintic spline in the interval $[0,l]$. To derive the unknown coefficients, we define

$$p_i(x_i) = u_i, \quad p_i(x_{i+1}) = u_{i+1},$$
$$p''_i(x) = M_i, \quad p''_{i+1}(x) = M_{i+1}, \quad p_i^{4}(x_i) = S_i, \quad p_i^{4}(x_{i+1}) = S_{i+1} \tag{5}$$

From Eq. (4) and Eq.(5) we can determine the unknown coefficients

$$a_i = u_i - \frac{s_i}{\omega^4},$$
$$b_i = \frac{u_{i+1}}{h} - \frac{u_i}{h} + S_i\left(\frac{h}{\omega^4} + \frac{h}{3\omega^2}\right) + S_{i+1}\left(\frac{h}{6\omega^2} - \frac{1}{h\omega^4}\right) - \frac{h}{3}M_i - \frac{h}{6}M_{i+1}$$
$$c_i = \frac{M_i}{2} - \frac{S_i}{2\omega^2}, \quad d_i = \frac{1}{6i}\left(M_{i+1} - M_i + \frac{S_i}{\omega^2} - \frac{S_{i+1}}{\omega^2}\right)$$
$$e_i = \frac{S_{i+1}}{\omega^4(e^\theta - e^{-\theta})} - \frac{S_i(e^\theta + e^{-\theta})}{2\omega^4(e^\theta - e^{-\theta})}, \qquad f_i = \frac{S_i}{2\omega^4}$$

Where $\theta = \omega h$ and $i = 0, 1, 2 \dots n$

Finally using the continuity of first derivative at the support points for $i = 2, 3, \dots\dots.., n-1$, we have

$$\frac{u_{i+1}}{h} - 2\frac{u_i}{h} + \frac{u_{i-1}}{h} - \frac{h}{6}M_{i-1} - \frac{2h}{3}M_i - \frac{h}{6}M_{i+1} = S_{i-1}\left(h\omega^{-4} - \frac{h}{6\omega^2} - \frac{\left(e^\theta + e^{-\theta}\right)^2}{2\omega^3(e^\theta - e^{-\theta})} + \frac{\left(e^\theta - e^{-\theta}\right)}{2\omega^3}\right) \tag{6}$$

and from continuity of third derivatives, we have

$$\frac{M_{i+1}}{h} - 2\frac{M_i}{h} + \frac{M_{i-1}}{h} = S_{i-1}\left(h\omega^{-2} - \frac{\left(e^\theta + e^{-\theta}\right)^2}{2\omega(e^\theta - e^{-\theta})} + \frac{\left(e^\theta - e^{-\theta}\right)}{2\omega}\right) + S_i\left(-2h\omega^{-2} + 2\frac{e^\theta + e^{-\theta}}{\omega(e^\theta - e^{-\theta})}\right) + S_{i+1}\left(h\omega^{-2} + 2\frac{1}{\omega(e^\theta - e^{-\theta})}\right) \tag{7}$$

From Eq. (6) and Eq. (7), after eliminating S_i we have the following useful relation for $i = 2,3, \dots\dots\dots, n-2$.

$$u_{i+2} + 2u_{i+1} - 6u_i + 2u_{i-1} + u_{i-2} = \frac{h^2}{20}(\alpha M_{i+2} + \beta M_{i+1} + \gamma M_i + \beta M_{i-1} + \alpha M_{i-2}) \tag{8}$$

Where

$$\alpha = \theta^{-4} - \frac{2}{\theta^3(e^\theta - e^{-\theta})} - \frac{1}{3\theta(e^\theta - e^{-\theta})},$$
$$\beta = -4\theta^{-4} + \frac{4 + 2e^\theta + 2e^{-\theta}}{\theta^3(e^\theta - e^{-\theta})} + \frac{e^\theta + e^{-\theta} - 4}{3\theta(e^\theta - e^{-\theta})},$$
$$\gamma = 6\theta^{-4} + \frac{4e^\theta + 4e^{-\theta} - 2}{3\theta(e^\theta - e^{-\theta})} - \frac{4 + 4e^\theta + 4e^{-\theta}}{\theta^3(e^\theta - e^{-\theta})}$$

When $\omega \to 0$so $\theta \to 0$, then $(\alpha, \beta, \gamma) \to (1, 26, 66)$ and the relation defined by Eq.(8) reduce into ordinary quintic spline

$$u_{i+2} + 2u_{i+1} - 6u_i + 2u_{i-1} + u_{i-2} = \frac{h^2}{20}(M_{i+2} + 26 + 66M_i + 26M_{i-1} + M_{i-2}) \tag{9}$$

III. Numerical Technique

By using Eq.(9) for $(j+1)$th, (j)th and $(j-1)$th time level we have

$$u_{i+2}^{j+1} + 2\,u_{i+1}^{j+1} - 6u_i^{j+1} + 2\,u_{i-1}^{j+1} + u_{i-2}^{j+1} = \frac{h^2}{20}\left(\alpha M_{i+2}^{j+1} + \beta M_{i+1}^{j+1} + \gamma M_i^{j+1} + \beta M_{i-1}^{j+1} + \alpha M_{i-2}^{j+1}\right) \tag{10}$$

$$u_{i+2}^{j} + 2\,u_{i+1}^{j} - 6u_i^{j} + 2\,u_{i-1}^{j} + u_{i-2}^{j} = \frac{h^2}{20}\left(\alpha M_{i+2}^{j} + \beta M_{i+1}^{j} + \gamma M_i^{j} + \beta M_{i-1}^{j} + \alpha M_{i-2}^{j}\right) \tag{11}$$

$$u_{i+2}^{j-1} + 2\,u_{i+1}^{j-1} - 6u_i^{j-1} + 2\,u_{i-1}^{j-1} + u_{i-2}^{j-1} = \frac{h^2}{20}\left(\alpha M_{i+2}^{j-1} + \beta M_{i+1}^{j-1} + \gamma M_i^{j-1} + \beta M_{i-1}^{j-1} + \alpha M_{i-2}^{j-1}\right) \tag{12}$$

that we will use these equations to discretize wave equation.

Wave Equation

Finite difference approximation for second order time derivative is

$$u_{tt}^{-j} = \frac{u_i^{j+1} - 2\,u_i^j + u_i^{j-1}}{k^2} = u_{tt}^j + O(k^2) \tag{13}$$

As we consider the space derivative is approximated by non-polynomial tension spline

$$u_{xx}^{-j} = p''\left(x_i, t_j\right) = M_i^j \tag{14}$$

By using Eq.(13) and Eq.(14), we can develop a new approximation for the solution of wave equation, so that the Eq.(1) can be replaced by

$$\eta M_i^{j-1} + (1-2\eta)M_i^j + \eta\, M_i^{j+1} = \frac{u_i^{j+1} - 2\,u_i^j + u_i^{j-1}}{c^2k^2} \tag{15}$$

where $0 \le \eta \le 1$ is a free variable

Again we multiply Eq.(11) by $(1-2\eta)$ and add this to Eq.(10) and Eq.(12) multiplied by η and eliminate M_i^j, then we obtain the following relation for wave Eq(1)

$$\begin{aligned}
&\eta\left(u(i+2,j-1)+u(i-2,j-1)\right) + (1-2\eta)\left(u(i+2,j)+u(i-2,j)\right) \\
&+\eta\left(u(i+2,j+1)+u(i-2,j+1)\right) + 2\eta\left(u(i+1,j-1)+u(i-1,j-1)\right) \\
&+2(1-2\eta)\left(u(i+1,j)+u(i-1,j)\right) - 6\eta\, u(i,j-1) - 6(1-2\eta)\,u(i,j) - 6\eta\, u(i,j+1) \\
&-\frac{h^2}{40c^2k}\alpha\left[u(i+2,j+1)-2u(i+2,j)+u(i+2,j-1)+u(i-2,j+1)-2u(i-2,j)\right. \\
&\left.+u(i-2,j-1)\right] \\
&-\frac{h^2}{40c^2k}\beta\left[u(i+1,j+1)-2u(i+1,j)+u(i+1,j-1)+u(i-1,j+1)-2u(i-1,j)\right. \\
&\left.+u(i-1,j-1)\right] - \frac{h^2}{40c^2k}\gamma\left[u(i,j+1)-2u(i,j)+u(i,j-1)\right] = 0, \qquad j \\
&= 1,2,3\ldots\ldots \quad i = 2,\ldots\ldots N-2
\end{aligned} \tag{16}$$

Error Estimate in Spline Approximation

To estimate the error for wave equation we expand Eq.(16) in Taylor series about $u(x_i,\ t_j)$ and then we find the optimal value for $\alpha,\ \beta$ and γ.

Error Estimate for Wave Equation

For wave equation, we expand Eq.(16) in Taylor series and replace the derivatives involving t by the relation

$$\frac{\partial^{i+j}u}{\partial x^i \partial t^j} = c^j\,\frac{\partial^{i+j}u}{\partial x^{i+j}} \tag{17}$$

then we derive the local truncation error. The principal part of the local truncation error of the proposed method for wave equation is

$$\begin{aligned}
T_i^j = &\left(-\frac{\alpha}{10}-\frac{\gamma}{20}-\frac{\beta}{10}+6\right)h^2\left(D_{x,x}\right)U(0,0) \\
&+\left[\left(-\frac{\alpha}{5}-\frac{\beta}{20}+\frac{3}{2}\right)h^4 + \left(-\frac{\alpha}{120}-\frac{\gamma}{240}-\frac{\beta}{120}+6\eta\right)h^2k^2c\right]\left(D_{x,x,x,x}\right)U(0,0) \\
&+\left[\left(-\frac{\alpha}{15}-\frac{\beta}{240}+\frac{11}{66}\right)h^6 + \left(-\frac{\alpha}{60}-\frac{\beta}{240}+\frac{3}{2}\eta\right)h^4k^2c\right]\left(D_{x,x,x,x,x,x}\right)U(0,0) \\
&+\left(\left(-\frac{\alpha}{3600}-\frac{\gamma}{7200}-\frac{\beta}{3600}+\frac{1}{2}\eta\right)h^2k^4c\right)\left(D_{x,x,x,x,x,x}\right)U(0,0) \\
&+\left(\left(-\frac{2\alpha}{225}-\frac{\beta}{7200}+\frac{43}{3360}\right)h^8\right)\left(D_{x,x,x,x,x,x,x,x}\right)U(0,0)+\cdots
\end{aligned} \tag{18}$$

If we choose $\alpha = \frac{7}{6},\ \beta = \frac{76}{3}$ and $\gamma = 67$ in Eq.(18) we obtain a new scheme of order $O(h^8 + h^4k^4)$, furthermore by choosing $\eta = \frac{1}{12}$ we can optimize our scheme, too.

Stability Analysis

In this section, we discuss stability of the proposed method for the numerical solution of wave equation. We assume that the solution of Eq. (16) at the grid point (li, jk) is

$$u_l^j = \xi^j e^{li\theta} \tag{19}$$

Where $i = \sqrt{-1},\ \theta$ is a real number and ξ is a complex number.

By substituting Eq.(19) in Eq.(16) we obtain a quadratic equation as follow

$$Q\xi^2 + \emptyset\xi + \varphi = 0 \tag{20}$$

For 1D wave equation we have

$$Q = \cos 2\theta \left(2\eta - \frac{h^2\alpha}{10ck^2}\right) + \cos\theta\left(4\eta - \frac{h^2\beta}{10ck^2}\right) - 6\eta - \frac{h^2\gamma}{20ck^2}$$

$$\emptyset = \cos 2\theta \left(2 - 4\eta - \frac{h^2\alpha}{5ck^2}\right) + \cos\theta\left(4 - 8\eta - \frac{h^2\beta}{5ck^2}\right) - 6 + 12\eta + \frac{h^2\gamma}{10ck^2}$$

$$\varphi = \cos 2\theta \left(2\eta - \frac{h^2}{10ck^2}\right) + \cos\theta\left(4\eta - \frac{h^2\beta}{10ck^2}\right) - 6\eta - \frac{h^2\gamma}{20ck^2}$$

Thus we have $(Q - \emptyset + \varphi)\xi^2 + (Q + \emptyset + \varphi) = 0$. In order to $|\xi| < 1$, we must have $\emptyset < 0$ and $Q + \varphi > 0$. Obviously $Q + \varphi > 0$ for each η, if $\eta > \frac{1}{2} + \frac{7h^2}{120ck^2}$ then $\emptyset < 0$, therefore our scheme will be stable for wave equation.

Numerical Example

We applied the presented method to the following wave equation. For this purpose we two examples for wave equation

We applied proposed method with $(\alpha, \beta, \gamma) = (1, 26, 66)$ (method I) which is ordinary quintic spline of with order $O(h^6 + k^4)$ and if we select $(\alpha, \beta, \gamma) = \left(\frac{7}{6}, \frac{76}{3}, 67\right)$ we obtain a new method which is of order $O(h^8 + h4k4$(method II).

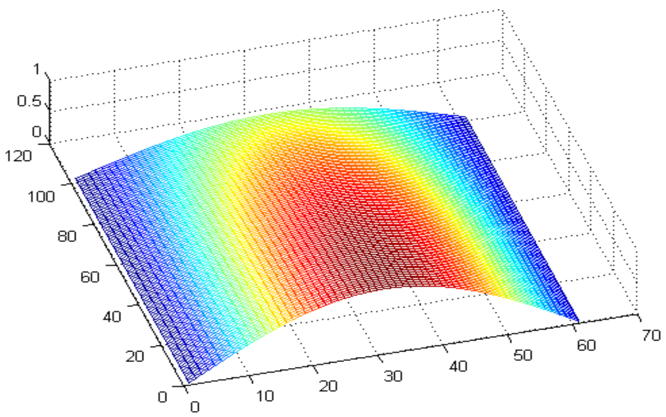
Example 1:

We consider Eq.(1) with $c = 1$, $f_2(x) = 0$, $f_3(x) = \pi\cos x\pi$, $p_2(t) = \sin\pi t$, $q_2(x) = -\sin\pi t$. The exact solution for the problem is

$$u(x, t) = \pi\cos(\pi x)\sin(\pi t)$$

Absolute Error for Example 1.

x_i	t_j	Method I	Method II
0.05	0.03	9×10^{-12}	3×10^{-12}
0.05	0.05	1.6×10^{-11}	7×10^{-12}
0.1	0.03	1.2×10^{-11}	1.7×10^{-11}
0.1	0.05	1.7×10^{-11}	1×10^{-12}
0.2	0.03	7×10^{-12}	1×10^{-12}
0.2	0.05	1×10^{-11}	1.1×10^{-11}



This problem is solved by different values of step size in the x-direction h and the time step size $\Delta t = 0.01$. The solution by proposed method are compared with the exact solution at the grid points and the maximum absolute errors are tabulated in above table. The space-time of the estimated solution is given in the figure. The maximum absolute error of this example by method II is 8.7×10^{-10} and by method I is 1.2×10^{-9}.

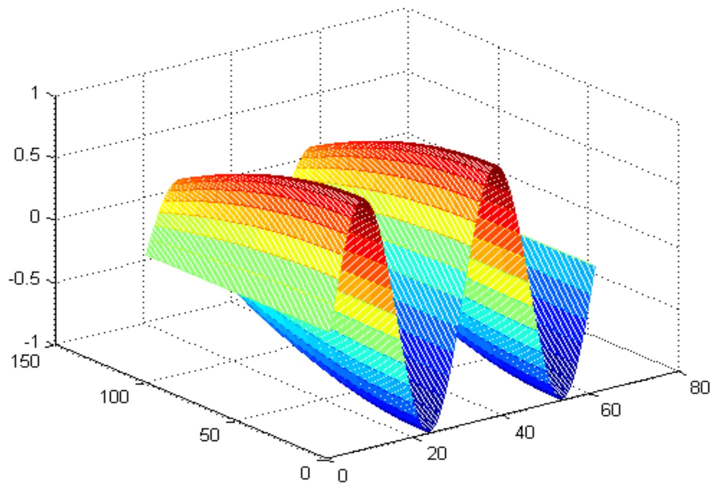
Example 2.

Now we consider Eq.(1) with $c = 1,\ f_2(x) = \cos \pi x\ ,\ f_3(x) = 0,\ p_2(t) = \cos \pi t,\ \ q_2(x) = -\cos \pi t$. The exact solution for the problem is

$$u(x,t) = \frac{1}{2}\cos\big(\pi(x+t)\big) + \frac{1}{2}\cos\big(\pi(x-t)\big)$$

Absolute Error for Example 2.

x_i	t_j	Method I	Method II
0.05	0.0003	5×10^{-10}	1×10^{-9}
0.05	0.0005	1×10^{-10}	1×10^{-9}
0.1	0.0003	2.1×10^{-9}	7×10^{-10}
0.1	0.0005	9.7×10^{-9}	4×10^{-10}
0.2	0.0003	6.2×10^{-9}	1×10^{-10}
0.2	0.0005	1.3×10^{-8}	1×10^{-10}



This problem is solved by different values of step size in the x-direction h and the time step size $\Delta t = 0.01$. The solution by proposed method are compared with the exact solution at the grid points and the maximum absolute errors are tabulated in above table. The space-time of the estimated solution is given in the figure. The maximum absolute error of this example by method II is 6.5×10^{-9} and by method I is 3.84×10^{-8}.

Reference

[1]. Saadatmandi and M. Dehghan, Numerical Solution of the One-Dimensional Wave Equation with An integral Condition, Numerical Methods for Partial Differential Equations, Vol, 23, no.2, pp. 282-292, 2007.

[2]. J. Rashidinia, R. Jalilian, and V. Kazemi, “Spline methods for the solutions of hyperbolic equations,” Applied Mathematics and Computation, vol. 190, no. 1, pp. 882 – 886, 2007.

[3]. R. K. Mohanty and V. Gopal, “High accuracy cubic spline finite difference approximation for the solution of one-space dimensional non- linear wave equations,” Applied Mathematics and Computation, vol. 218, no. 8, pp. 4234 – 4244, 2011.

[4]. Ryo, Fast decay of solutions for linear wave equations with dissipation localized near infinity in an exterior domain, J. Differential Equations, 188 (2) (2003) 390-405.

[5]. Rashidinia J. and Mohsenyzadeha, Numerical Solution of One-Dimensional Heat and Wave Equation by Non-Polynomial Quintic Spline, International Journal of Mathematical Modelling & Computations Vol. 05, No. 04, (2015), 291- 305.

[6]. Won Y. Yang, Wenwu Cao, Tae-Sang Chung, John Morris, Applied Numerical methods using MATLAB, Wiley student Edition, 2013.

[7]. M. Dehghan, On the Solution of An initial-boundary Value Problem tha Combines Neumann and Integral Condition for the Wave Equation, Numerical Methods for Partial Differential Equations, Vol. 21, no. 1, pp:24-40

[8]. P. Karageorgis and Walter A. Strauss, Instability of steady states for nonlinear wave and heat equations , J. Differential Equations, 241 (1) (2007) 184-205.

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