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Research on three different Portfolio Models with singular Covariance Matrix

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Abstract: *In this paper, we will solve three portfolio models with singular covariance matrix. These portfolio models include Mean-Variance Portfolio Model, Value-at-Risk portfolio Model, and Conditional Value-at-Risk portfolio Model. By studying and calculating, we fond: the effective boundary of these three types of portfolio models must be the effective boundary of their maximal linearly independent groups or their maximal linearly independent groups and risk-free assets.*

Key words*: Singular Covariance Matrix; Maximum Linear Unrelated Group; Effective borders*

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I. Introduction

Markowitz first proposed in 1952 that the expected yield of risk assets and the use of variance to quantitatively replace returns and risks, so that abstract risk data can measure and estimate risks. However, both theory and practice show that variance is not an effective risk measure because it treats up and down deviations equally. The risk measure of variance is defined as follows: $P\ (\xi_{\mathrm{p}}) = \sigma^2 = \sum_{i=1}^{N} x_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} x_i x_j \sigma_j$

VaR: Under normal market conditions, the worst expected loss within a holding period of a given confidence interval. It represents the fractional α of the profit and loss distribution of an investment instrument or portfolio. Compared to variance, VaR considers the investor's risk aversion. However, the verification found that the effect of VaR's resulting hand rate distribution and given confidence level only considered the probability of adverse events, and did not consider the degree of loss at the time of adverse events. Therefore, VaR is non-secondary and its non-convex. Therefore, it is not a consistency risk measure. VaR's risk measurement is defined as follows:

$$P(\xi_p) = VaR = \min \{\alpha \in R : \psi(\chi, \alpha) \geq \beta\}$$

CVaR: Proposed by Rockferer and Uryasev，CVaR refers to a loss that exceeds VaR's conditional expectation. CVaR not only retains VaR's point, but also overcomes VaR's limitations. It is not only a consistent risk measure, but also a convex risk measure, and the measurement of risk is more accurate. CVaR's risk measurement is defined as follows:

$$P\ (\xi_p) = CVaR = (1-\beta)^{-1} \int_{f(x,y) \geq \alpha_\beta(x)} f(x, y) p(y) dy$$

Create function by literature:

$$F_\beta(x, \alpha) = (1-\beta)^{-1} \int_{f(x,y) \geq \alpha_\beta(x)} [f(x, y) - \partial]^+ p(y) dy$$

And

$$[t]^+ = \begin{cases} t, t > 0 \\ 0, t \leq 0 \end{cases}$$

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By transforming:

$VaR = \mu(x) + c_1(\beta)\sigma(x)$, and $c_1(\beta) = \sqrt{2}erf^{-1}(2\beta - 1)$

$CVaR = \mu(x) + c_2(\beta)\sigma(x)$, and $c_2(\beta) = \{\sqrt{2}\exp[erf^{-1}(2\beta-1)^2(1-\beta)\}^{-1}$

Then define

$erf(z)$ as $erf(z) = \dfrac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$

$$VaR = -R + c_1(\beta)\sigma(x)$$

$$CVaR = -R + c_2(\beta)\sigma(x)$$

II. The discussion of Mean-Variance Portfolio Model with singular covariance matrix

Theorem 2.1: First set $\xi_1, \xi_2, ..., \xi_r$ （r<n）is a very linearly independent group of N risk assets. When V is a singular Matrix, which means | V | = 0, the optimal selection of the risk asset portfolio in N gets the optimal result, or the best result for the optimal selection of the R risk asset portfolio. Either the optimal result is obtained for the optimal choice of this R type of risk asset and a risk-free asset portfolio.

Lemma 2.1: If there is a real number that is not all zero $k_1, k_2, ..., k_n$, so that $k_1\xi_1 + k_2\xi_2 + ... + k_n\xi_n = a$ (a is a constant), then indicate $\xi_1, \xi_2, ..., \xi_n$ have the relationship of Linear correlation.

Lemma 2.2: If there are real numbers $k_0, k_1, k_2, ..., k_n$, so that $\xi = k_0 + k_1\xi_1 + k_2\xi_2 + ... + k_n\xi_n$, then we can say that ξcan be linearly replaced by $\xi_1, \xi_2, ..., \xi_n$.

Lemma 2.3: If $\xi_1^{\tau_1}, \xi_2^{\tau_2}, ..., \xi_r^{\tau_r}$ is $\xi_1, \xi_2, ..., \xi_n$, whose r maximal linearly independent vectors are rearranged to the first r positions, then $\xi_1^{\tau}, \xi_2^{\tau}, ..., \xi_r^{\tau}$ is the maximal linear independent group of $\xi_1, \xi_2, ..., \xi_n$, and $\xi_{r+1}^{\tau}, \xi_{r+2}^{\tau}, ..., \xi_n^{\tau}$ can be linearly replaced by $\xi_1^{\tau}, \xi_2^{\tau}, ..., \xi_r^{\tau}$.

Proof:

Since V is a singular matrix, then $\xi_1, \xi_2, ..., \xi_n$ must be linearly related. From Lemma 2.1, there are real numbers $k_1, k_2, ..., k_n$ ($k_1, k_2, ..., k_n$ is not all zero), zhen $k_1\xi_1 + k_2\xi_2 + ... + k_n\xi_n = a$ (a is a constant), which means P （$\sum_{i=1}^{n} k_i\xi_i = a) = 1$.

From lemma 3, let $\xi_1^{\tau_1}, \xi_2^{\tau_2}, \dots, \xi_n^{\tau_n}$ is the maximal linearly independent group of $\xi_1, \xi_2, ..., \xi_n$, and $\xi_{r+1}^{\tau_1}, \xi_{r+2}^{\tau_2}, \dots, \xi_n^{\tau_n}$ can be represented by linear representation of $\xi_1^{\tau_1}, \xi_2^{\tau_2}, ..., \xi_r^{\tau_r}$. Also from Lemma 2.2 there are real numbers $k_{t0}, k_{t1}, k_{t2} ..., k_{tr}$, so that $\xi_t^{\tau} = k_{t0} + k_{t1}\xi_1^{\tau} + k_{t2}\xi_2^{\tau} + ... + k_{tr}\xi_r^{\tau}$. Then set $\eta_t = \xi^{\tau}{}_t - \sum_{i=1}^{r} k_{ti}\xi^{\tau}{}_i = k_{t0}$, D（$\eta_t) = 0$, which means η_t is equivalent to ($1 - \sum_{i=1}^{r} k_{ti}$) times risk-free asset investment. From no arbitrage assumption, we have $k_{t0} = E(\eta_t) = (1 - \sum_{i=1}^{r} k_{ti})r_f$, t=r+1, r+2,...n, r_f is risk-free interest rate.

Let the investment ratio vector of n kinds of assets be（$(W_1, \ W_2, ..., W_n) \in R^n$, then the total portfolio return rate $\xi_p = \sum_{i=1}^{n} w_i\xi_i$, $R = (r_1, r_2, .., r_n)^T$ and $V \in R^{n*n}$ are the yield expectation vector and the singular covariance matrix of the asset $\xi_1, \xi_2, ..., \xi_n$, respectively.

Let the investment ratio vector of r assets be $\overline{W} =$（$W^{\tau_1}, W^{\tau_2}, ..., W^{\tau_r}) \in R^r$, and records

$\overline{R}=(r_1^\tau, r_2^\tau,..., r_r^\tau)^T$ and $\overline{V}\in R^{r*r}$ are the yield expectation vector and singular covariance matrix of asset $\xi_1^{\tau_1}, \xi_2^{\tau_2},..., \xi_r^{\tau_r}$, respectively, and $\overline{V}\in R^{r*r}$ is a non-singular matrix.

Case 1: If for all t, there is $1-\sum_{i=1}^{r} k_{ti}=0$, that is, $k_{t0}=E\eta_t=(1-\sum_{i=1}^{r} k_{ti})r_f=0$ (t=r+1, r+2,...n), so $\xi_t^\tau=k_{t0}+k_{t1}\xi_1^\tau+k_{t2}\xi_2^\tau+...+k_{tr}\xi_r^\tau=k_{t1}\xi_1^\tau+k_{t2}\xi_2^\tau+...+k_{tr}\xi_r^\tau$, that is, the tth asset is $\xi_1^{\tau_1}, \xi_2^{\tau_2},\ldots,\xi_n^{\tau_n}$ in proportion to $k_{t0}, k_{t1}, k_{t2}..., k_{tr}$.

$$\xi_p=\sum_{i=1}^{n}w_i\xi_i=\sum_{i=1}^{r}w^\tau{}_i\xi^\tau{}_i+\sum_{t=r+1}^{n}w^\tau{}_i\xi^\tau{}_i=\sum_{i=1}^{r}w^\tau{}_i\xi^\tau{}_i+\sum_{t=r+1}^{n}w^\tau{}_i(k_{t0}+\sum_{i=1}^{r}k_{ti}\xi_i^\tau)=\sum_{i=1}^{r}w^\tau{}_i\xi^\tau{}_i+\sum_{t=r+1}^{n}w_i^\tau(\sum_{i=1}^{r}k_{ti}\xi_i^\tau)$$

$$=\sum_{i=1}^{r}(w^\tau{}_i+\sum_{t=r+1}^{n}w_t{}^\tau k_{ti})\xi_i{}^\tau=\sum_{i=1}^{r}\overline{w}_i^\tau\xi_i^\tau$$

Then $\overline{w}^\tau{}_i=w^\tau{}_i+\sum_{t=r+1}^{n}w_t{}^\tau k_{ti}$. It is easy to prove $\sum_{i=1}^{r}\overline{w}_i^\tau=1=\sum_{i=1}^{n}w_i$

$$u=E\eta_t=\sum_{i=1}^{r}w_i^\tau r_i^\tau=\overline{W}^T\overline{R}$$

$$\sigma^2=Cov(\xi_p,\xi_p)=Cov(\sum_{i=1}^{r}\overline{w}_i^\tau\xi_i^\tau,\sum_{i=1}^{r}\overline{w}_i^\tau\xi_i^\tau)=\overline{W}^T\overline{V}\overline{W}$$

Case 2: If t exists, let $1-\sum_{i=1}^{r} k_{ti}\neq 0$, that is k_{t0}(t = r+1, r+2,..., n) not all be 0.

$$\xi_p=\sum_{i=1}^{n}w_i\xi_i=\sum_{i=1}^{r}w^\tau{}_i\xi^\tau{}_i+\sum_{t=r+1}^{n}w^\tau{}_i\xi^\tau{}_i=\sum_{i=1}^{r}w^\tau{}_i\xi^\tau{}_i+\sum_{t=r+1}^{n}w^\tau{}_i(k_{t0}+\sum_{i=1}^{r}k_{ti}\xi_i^\tau)$$

$$=\sum_{i=1}^{r}(w^\tau{}_i+\sum_{t=r+1}^{n}w_t{}^\tau k_{ti})\xi_i{}^\tau+\sum_{t=r+1}^{n}w_t^\tau k_{t0}=\sum_{i=1}^{r}\overline{w}_i^\tau\xi_i^\tau+\sum_{t=r+1}^{n}w_t^\tau k_{t0}$$

So $u=E\eta_t=\sum_{i=1}^{r}w_i^\tau r_i^\tau+\sum_{t=r+1}^{n}w_t^\tau k_{t0}$

$$\sigma^2=Cov(\xi_p,\xi_p)=Cov(\sum_{i=1}^{r}\overline{w}_i^\tau\xi_i^\tau+\sum_{t=r+1}^{n}w_t^\tau k_{t0},\sum_{i=1}^{r}\overline{w}_i^\tau\xi_i^\tau+\sum_{t=r+1}^{n}w_t^\tau k_{t0})$$

$$=Cov(\sum_{i=1}^{r}\overline{w}_i^\tau\xi_i^\tau,\sum_{i=1}^{r}\overline{w}_i^\tau\xi_i^\tau)=\overline{W}^T\overline{V}\overline{W}$$

$$\sum_{i=1}^{n}w_i=\sum_{i=1}^{r}w_i^\tau+\sum_{t=r+1}^{n}w_i^\tau=\sum_{i=1}^{r}(\overline{w}^\tau{}_i-\sum_{t=r+1}^{n}w_t^\tau k_{ti})+\sum_{t=r+1}^{n}w^\tau{}_i=\sum_{i=1}^{r}\overline{w}^\tau{}_i+\sum_{t=r+1}^{n}(w_t-\sum_{i=1}^{r}w_tk_{ti})$$

Because

$k_{t0}=E\eta_t=(1-\sum_{i=1}^{r} k_{ti})r_f=0$,(t=r+1, r+2,...n), r_f is risk free rate, $r_f>0$, so

$$\sum_{i=1}^{n}w_i=\sum_{i=1}^{r}\overline{w}^\tau{}_i+\frac{1}{r_f}\sum_{t=r+1}^{n}k_{t0}\overline{w}_t\ .$$

III. Optimal solution and its effective frontier of mean - risk model when V is Singular matrix

(1) Mean - variance model

Case 1: If for all t, there is $1-\sum_{i=1}^{r} k_{ti}=0$,it can be converted to

$$\begin{cases} \min \sigma^2 = \overline{W}^T \overline{V}\,\overline{W} \\ s.t. \sum_{i=1}^{n} w_i r_i = \sum_{i=1}^{r} \overline{w}_i^{\tau} r_i^{\tau} = u \\ \sum_{i=1}^{n} w_i = \sum_{i=1}^{r} \overline{w}_i^{\tau} = 1 \end{cases}$$

This is solution of the optimal solution and its effective boundary of mean - variance portfolio without risk-free assets under non-singular matrix.
By reference, we can get:

Optimal solution is $\overline{W}^* = \frac{(uc-b)\overline{V}^{-1}\bar{I}}{\Delta}$

Effective boundary is $\frac{\sigma_p^2}{\frac{1}{c}} - \frac{(u-\frac{b}{c})^2}{\frac{\Delta}{c^2}} = 1$

And let
$\mathrm{a} = \bar{R}^T \overline{VR}$ $\mathrm{b} = \bar{I}^T \overline{V}^{-1} \bar{R}$
$\mathrm{c} = \bar{I}^T \overline{V}^{-1} \bar{I}$ $\Delta = ac - b$ （a>0,c>0）

Case 2: If for all t, there is $1 - \sum_{i=1}^{r} k_{ti} \neq 0$, so that k_{t0} （t=r+1,r+2,...,n）is not all zero.

$$\begin{cases} \min \sigma^2 = \overline{W}^T \overline{V}\,\overline{W} \\ s.t. \sum_{i=1}^{n} w_i r_i = \sum_{i=1}^{r} \overline{w}_i^{\tau} r_i^{\tau} + \sum_{t=r+1}^{n} w_i^{\tau} k_{i0} = u \\ \sum_{i=1}^{n} w_i = \sum_{i=1}^{r} \overline{w}_i^{\tau} + \frac{1}{r_f} \sum_{t=r+1}^{n} k_{io} \overline{w}_t = 1 \end{cases}$$

Optimal solution is $\overline{W}^* = \frac{(u-r_f)}{H} (\overline{V}^{-1}\overline{R} - r_f \overline{V}^{-1}\bar{I})$

Effective boundary is $\sigma_p^2 = \frac{(u-r_f)^2}{H}$

And let
$\mathrm{a} = \bar{R}^T \overline{VR}$ $\mathrm{b} = \bar{I}^T \overline{V}^{-1} \bar{R}$
$\mathrm{c} = \bar{I}^T \overline{V}^{-1} \bar{I}$ $\Delta = ac - b$ （a>0,c>0）

（2）Mean—VaR model

$$\begin{cases} \min VaR = c_1(\beta)\sigma(x) - u \\ s.t. \sum_{i=1}^{n} w_i r_i = u \\ \sum_{i=1}^{n} w_i = 1 \end{cases}$$

From the reference, it is strictly greater than 0 at a given confidence level (α>50%).
In addition, it can be proved that the inequality constraint is tight at the optimal solution, so the mean-VaR model's solution is the same as the standard mean-variance model, with the same boundary combination. So the optimal solution and effective boundary are solved in the following two cases:

Case 1: If for all t, there is $1-\sum_{i=1}^{r} k_{ti} = 0$,it can be converted to

$$\begin{cases} \min VaR = c_1(\beta)\sqrt{\overline{W}^T \overline{V}\overline{W}} - u \\ s.t. \sum_{i=1}^{n} w_i r_i = \sum_{i=1}^{r} \overline{w}_i^{\tau} r_i^{\tau} = u \\ \sum_{i=1}^{n} w_i = \sum_{i=1}^{r} \overline{w}_i^{\tau} = 1 \end{cases}$$

This is solution of the optimal solution and its effective boundary of mean - VaR portfolio without risk-free assets under non-singular matrix.
By reference, we can get:

Optimal solution is $\overline{W}^* = \dfrac{(uc-b)\overline{V}^{-1}\overline{I}}{\Delta}$

Effective boundary is $\dfrac{[(VaR+u)/c_1(\beta)]^2}{\frac{1}{c}} - \dfrac{(u-\frac{b}{c})^2}{\frac{\Delta}{c^2}} = 1$

And let
$a = \overline{R}^T \overline{V}\overline{R}$ $b = \overline{I}^T \overline{V}^{-1}\overline{R}$
$c = \overline{I}^T \overline{V}^{-1}\overline{I}$ $\Delta = ac - b$ （a>0,c>0）

Case 2: If for all t, there is $1-\sum_{i=1}^{r} k_{ti} \neq 0$, so that k_{t0} （t=r+1,r+2,...,n）is not all zero.

$$\begin{cases} \min VaR = c_1(\beta)\sqrt{\overline{W}^T \overline{V}\overline{W}} - u \\ s.t. \sum_{i=1}^{n} w_i r_i = \sum_{i=1}^{r} \overline{w}_i^{\tau} r_i^{\tau} + \sum_{t=r+1}^{n} w_i^{\tau} k_{i0} = u \\ \sum_{i=1}^{n} w_i = \sum_{i=1}^{r} \overline{w}_i^{\tau} + \frac{1}{r_f} \sum_{t=r+1}^{n} k_{io}\overline{w}_t = 1 \end{cases}$$

Optimal solution is $\overline{W}^* = \dfrac{(u-r_f)}{H}(\overline{V}^{-1}\overline{R} - r_f \overline{V}^{-1}\overline{I})$

Effective boundary is $\dfrac{VaR+u}{c_1(\beta)} = \dfrac{u-r_f}{\sqrt{H}}$

And let
$a = \overline{R}^T \overline{V}\overline{R}$ $b = \overline{I}^T \overline{V}^{-1}\overline{R}$
$c = \overline{I}^T \overline{V}^{-1}\overline{I}$ $\Delta = ac - b$ （a>0,c>0）

（3）Mean - CVaR model

$$\begin{cases} \min CVaR = c_2(\beta)\sigma(x) - u \\ s.t.\sum_{i=1}^{n} w_i r_i = u \\ \sum_{i=1}^{n} w_i = 1 \end{cases}$$

The solution of mean-CVaR model is the same as the solution of standard mean-variance, with the same boundary combination. So in both cases the optimal solution and the effective boundary are solved as follows:

Case 1: If for all t, there is $1-\sum_{i=1}^{r} k_{ti} = 0$,it can be converted to

$$\begin{cases} \min CVaR = c_2(\beta)\sqrt{\overline{W}^T \overline{V}\overline{W}} - u \\ s.t.\sum_{i=1}^{n} w_i r_i = \sum_{i=1}^{r} \overline{w}_i^{\tau} r_i^{\tau} = u \\ \sum_{i=1}^{n} w_i = \sum_{i=1}^{r} \overline{w}^{\tau}_i = 1 \end{cases}$$

This is solution of the optimal solution and its effective boundary of mean - CVaR portfolio without risk-free assets under non-singular matrix.

By reference, we can get:

Optimal solution is $\overline{W}^* = \dfrac{(uc-b)\overline{V}^{-1}\overline{I}}{\Delta}$

Effective boundary is $\dfrac{[(CVaR+u)/c_2(\beta)]^2}{\frac{1}{c}} - \dfrac{(u-\frac{b}{c})^2}{\frac{\Delta}{c^2}} = 1$

And let

$a = \overline{R}^T \overline{V}\overline{R}$ $b = \overline{I}^T \overline{V}^{-1}\overline{R}$

$c = \overline{I}^T \overline{V}^{-1}\overline{I}$ $\Delta = ac - b$ （a>0,c>0）

Case 2: If for all t, there is $1-\sum_{i=1}^{r} k_{ti} \neq 0$, so that k_{t0} （t=r+1,r+2,...,n）is not all zero.

$$\begin{cases} \min CVaR = c_2(\beta)\sqrt{\overline{W}^T \overline{V}\overline{W}} - u \\ s.t.\sum_{i=1}^{n} w_i r_i = \sum_{i=1}^{r} \overline{w}_i^{\tau} r_i^{\tau} + \sum_{t=r+1}^{n} w_i^{\tau} k_{i0} = u \\ \sum_{i=1}^{n} w_i = \sum_{i=1}^{r} \overline{w}^{\tau}_i + \frac{1}{r_f}\sum_{t=r+1}^{n} k_{io}\overline{w}_t = 1 \end{cases}$$

Optimal solution is $\overline{W}^* = \dfrac{(u-r_f)}{H}(\overline{V}^{-1}\overline{R} - r_f \overline{V}^{-1}\overline{I})$

Effective boundary is $\dfrac{CVaR+u}{c_2(\beta)} = \dfrac{u-r_f}{\sqrt{H}}$

And let

$a = \overline{R}^T \overline{V} \overline{R}$ $b = \overline{I}^T \overline{V}^{-1} \overline{R}$
$c = \overline{I}^T \overline{V}^{-1} \overline{I}$ $\Delta = ac - b$ （a>0,c>0）

IV. Conclusion

(1)If the V of these three kinds of mean - risk model is a singular covariance matrix, first determine whether there is $1-\sum_{i=1}^{r} k_{ti} = 0$ for all t. If yes, it is case 1, the equivalent is the optimization problem of the risk portfolio composed of the maximally linear irrelevant group; if not all 0, the equivalent is the risk portfolio of the extremely linear irrelevant group and a risk-free asset. Optimization problem.

(2)If A of these three kinds of mean - risk models is a singular covariance matrix, the effective boundary of the n kinds of risk assets is F, and the effective boundary of the risky asset portfolio composed of the extremely linear irrelevant group is the risky asset composed of the extremely linear independent group. The effective boundary of the combination and a risk-free asset is F_2, either $F = F_1$, or $F = F_2$.

References

[1] R.Tyrrell Rockafellar, Stanislav Uryasev. Optimization of conditional value-at-risk.
[2] HaixiangYao, Zhongfei Li, Xun Li, Yan Zeng.Optimal Sharpe ratio in continuous-time markets with and without a risk-free asset.Journal of Industrial and Management Optimization,2017, 13(3):1273-1290.(SSCI , SCI)
[3] Miao Zhang, Ping Chen,Haixiang Yao(Corresponding author). Mean-variance portfolio selection with only risky assets under regime switching.Economic Modelling, 2017, 62: 35–42. (SSCI)
[4] Haixiang Yao,Zhongfei Li, Duan Li.Multi-period mean-variance portfolio selection with stochastic interest rate and uncontrollable liability.European Journal of Operational Research, 2016, 252(3): 837–851.(SSCI , SCI)
[5] Haixiang Yao, Zhongfei Li, Yongzeng Lai. Mean-CVaR portfolio selection: A nonparametric estimation framework. Computers & Operations Research, 2013, 40: 1014-1022. (SCI,SSCI)
[6] Ryan P J. A comment on Mean - Variance portfolio selection with ether a singular or non-singular covariance matrix. The Journal of Finance. 1981
[7] Shumei Ping, Jianxin Yi. A Combination Boundary of Three Kinds of Risky Assets with covariance Matrix Degradation[J]. Journal of Xinyu College, 2004, (05):12-14

Xinyu Ye. " Research on three different Portfolio Models with singular Covariance Matrix" IOSR Journal of Mathematics (IOSR-JM) 14.5 (2018): 33-39.