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Necessary and sufficient conditions for the controllability of fractional systems with control delay

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Abstract: *In this paper, consider the controllability of a class of fractional dynamical systems with control delay. Necessary and sufficient conditions for the controllability of fractional linear systems with control delay are obtained. The results obtained in this paper are important for the study of controllability of nonlinear fractional dynamical systems with control delay. An example is also provided to illustrate the main results.*
Keywords: *Controllability, Delay, Fractional dynamical system, Mittag-Leffler matrix function*

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I. Introduction

In recent years, there has been a growing interest in the fields of analysis of fractional systems. Many scientific phenomena can be modeled by fractional differential equations, for example, electrical circuits, diffusion in porous medium and biological systems. Controllability is one of the most important concepts in mathematical control theory. It means that a controllable system can be steered from every initial system state to every desired final state using a set of admissible control functions. Because the use of fractional derivatives and integral leads to better descriptions of scientific phenomena than those of integer order ones. The controllability of several of types of fractional dynamical systems was investigated by many authors [1]-[4].

A time-delay system means that in which the present rate of change of some unknown function depends upon past values of it. Time-delays may occur in state or in control or in both[1]-[4]. Vijayakumar S. Muni et al. [1] studied the controllability of fractional dynamical systems with a constant times delay in control, and the authors obtained a necessary and sufficient condition for the controllability of the following linear fractional systems with control delays

$$\begin{cases} {}^{C}D^{\alpha}x(t) = Ax(t) + Bu(t-h), \ \ t \in [0,T], \\ x(0) = x_0, \\ u(t) = u_0(t), \ \ t \in [-h,0). \end{cases} \tag{1.1}$$

R. Joice Nirmala et al [2] investigated the controllability of fractional delay dynamical systems with delay in state variables. The solution representations of fractional differential equations with state delay have been given by Laplace transform technique and the Mittag-Leffler function. Necessary and sufficient conditions for the controllability criteria of the following linear fractional delay systems were established

$$\begin{cases} {}^{C}D^{\alpha}x(t) = Ax(t) + Bx(t-h) + Cu(t), \ t \in J, \\ x(t) = \phi(t), \ -h < t \leq 0 \end{cases} \tag{1.2}$$

Binbin He et al. [3] investigated the controllability of the following fractional damped systems with control delay

$$\begin{cases} {}^{C}D^{\alpha}x(t) - A\,{}^{C}D^{\beta}x(t) = Bu(t) + Cu(t-\tau), \ \ t \geq 0, \\ x(0) = x_0, \ \ x'(0) = x_1, \\ u(t) = \varphi(t), \ \ t \in [-\tau,0] \end{cases} \tag{1.3}$$

Sufficient and necessary conditions for the controllability of fractional damped dynamical systems with control delay are established. The controllability of various types of linear and nonlinear fractional dynamical systems was considered by many authors. However, almost all of the previous results on controllability of nonlinear fractional dynamical systems is based on the controllability of the corresponding linear systems. So the controllability of linear fractional dynamical systems is one of the most important issues for the controllability problems. Motivated by the above paper, we consider, in this paper, the controllability of the following fractional dynamical systems with control delay

$$\begin{cases} {}^C D^{\alpha} x(t) = Ax(t) + Bu(t-h) + Cu(t), \quad t \in J := [0,T], \\ x(0) = x_0, \\ u(t) = u_0(t), \ t \in [-h, 0) \end{cases} \tag{1.4}$$

where Caputo fractional derivative of order $0 < \alpha \le 1$, $x \in \mathbf{X}_1$, $u \in \mathbf{X}_2$, A is a $n \times n$ matrix, B and C are $n \times m$ matrices, h is a constant time delay in the control function.

II. Preliminaries for the Fractional Linear Systems with Control Delay

In this section, we shall present some basic Lemmas and Definitions required for the controllability of linear fractional dynamical systems.

In this paper, let $\mathbf{R}^n$ be n-dimensional Euclidean space, ${}^C D^{\alpha}$ represents Caputo fractional derivative, unless otherwise specified, define Banach space

$\mathbf{X}_1 := \{x(\cdot) : [0,T] \to \mathbf{R}^n \mid {}^C D^{\alpha} x(t)$ exists on $[0,T]$ and $x(0) = \lim_{t \downarrow 0} x(t)\}$

endowed with the normal $\| x(\cdot) \|_{\mathbf{X}_1} := \sup_{t \in [0,T]} \| x(t) \|_{\mathbf{R}^n}$

$\mathbf{X}_2 := \{u(\cdot) : [0,T] \in \mathbf{R}^m \mid u(\cdot)$ is continuous with finite number of discontinuous on $[0,T]$ and bounded on $[0,T]\}$

endowed with the norm $\| u(\cdot) \|_{\mathbf{X}_2} := \sup_{t \in [0,T]} \| u(t) \|_{\mathbf{R}^m}$

and

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}$$

represents Mittag-Leffler matrix function, and A^* is the transpose of matrix A.

Definition 2.1 The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha \le n$, is defined as

$${}^C D^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds,$$

where the function $x(t)$ has absolutely continuous derivative up to order n-1.

Definition 2.2 The fractional integral of order $\alpha > 0$ with the lower limit zero for a function $x \in L^1(J, X)$, is defined as

$$I^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Lemma 2.1 ([1]) The systems (1.4) is said to be controllable on $[0,T]$ over $\mathbf{R}^n$, if for every pair of vectors $(x_0, x_T) \in \mathbf{R}^n \times \mathbf{R}^n$ and for every continuous and bounded function $u_0(\cdot) : [-h, 0) \to \mathbf{R}^m$, there exists at least one control function on $[0,T]$, the corresponding solution to the system (1.4) with $x(0) = x_0$, $u(t) = u_0(t)$, $t \in [-h, 0]$, satisfies the condition $x(T) = x_T$.

Lemma 2.2 The solution represent of linear fractional dynamical systems (1.4) at any time $t \in [0,T]$ is given by

$$\mathrm{x}(t) = E_\alpha(At^\alpha)x_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)[Bu(s-h)+Cu(s)]ds$$
$$= E_\alpha(At^\alpha)x_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)Cu(s)]ds$$
$$+\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)Bu(s-h)ds$$
$$= E_\alpha(At^\alpha)x_0 + \int_0^{t\text{-h}} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)Cu(s)]ds$$
$$+\int_0^{t-h} (t-s-h)^{\alpha-1} E_{\alpha,\alpha}(A(t-s-h)^\alpha)Bu(s)ds + \int_{-h}^0 (t-s-h)^{\alpha-1} E_{\alpha,\alpha}(A(t-s-h)^\alpha)Bu(s)ds$$
$$= E_\alpha(At^\alpha)x_0 + a_0(t) + \int_0^{t\text{-h}} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)Cu(s)]ds$$
$$+\int_0^{t-h} (t-s-h)^{\alpha-1} E_{\alpha,\alpha}(A(t-s-h)^\alpha)Bu(s)ds$$
$$= E_\alpha(At^\alpha)x_0 + a_0(t) + \int_0^{t\text{-h}} \left[(t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)C + (t-s-h)^{\alpha-1} E_{\alpha,\alpha}(A(t-s-h)^\alpha)B\right]u(s)ds$$

where

$$a_0(t) = \int_{-h}^0 (t-s-h)^{\alpha-1} E_{\alpha,\alpha}(A(t-s-h)^\alpha)Bu_0(s)ds.$$

Definition 2.3 The $n \times n$ matrix

$$\begin{aligned} W := W(T) = \int_0^{T\text{-h}} &\left[(T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha)C + (T-s-h)^{\alpha-1} E_{\alpha,\alpha}(A(T-s-h)^\alpha)B\right] \\ &\times \left[(T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha)C + (T-s-h)^{\alpha-1} E_{\alpha,\alpha}(A(T-s-h)^\alpha)B\right]^* \mathrm{ds} \end{aligned} \tag{2.1}$$

is called the controllaibility Grammian matrix of the system (1.4)

III. Necessary and Sufficient Conditions of Controllability

Theorem 3.1 The system (1.4) is controllable over $\mathbf{R}^n$ on $[0,T]$, if and only if the controllability Grammian W is non-singular.

Proof: To show the sufficiency, let W be non-singular, therefore it is an invertible matrix. Define a control function for the system (1.4) as following

$$u(t) = \begin{cases} \left[(T-t)^{\alpha-1} E_{\alpha,\alpha}(A(T-t)^\alpha)C + (T-t-h)^{\alpha-1} E_{\alpha,\alpha}(A(T-t-h)^\alpha)B\right]^* W^{-1} \\ \quad \times (x_T - E_\alpha(AT^\alpha)x_0 - a_0(T)) \qquad t \in [0, T-h] \\ u_0(t), \qquad t \in [\text{-}h, 0) \\ u(t) = 0, \qquad t \in (T-h, T] \end{cases} \tag{3.1}$$

It is easy to obtain that

$$x(T) = E_\alpha(AT^\alpha)x_0 + a_0(t)$$
$$+\int_0^{T\text{-h}} \left[(T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha)C + (T-s-h)^{\alpha-1} E_{\alpha,\alpha}(A(T-s-h)^\alpha)B\right]u(s)]ds$$

So we get

$$x(T) = E_\alpha(AT^\alpha)x_0 + a_0(T)$$
$$+\int_0^{T\text{-h}} \left[(T-s)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha)C + (T-s-h)^{\alpha-1} E_{\alpha,\alpha}(A(T-s-h)^\alpha)B\right]$$
$$\times \left[(T-t)^{\alpha-1} E_{\alpha,\alpha}(A(T-s)^\alpha)C + (T-s-h)^{\alpha-1} E_{\alpha,\alpha}(A(T-s-h)^\alpha)B\right]^* W^{-1}$$
$$\times (x_T - E_\alpha(AT^\alpha)x_0 - a_0(T))ds$$

$$= E_{\alpha}(AT^{\alpha})x_0 + a_0(T) + (x_T - E_{\alpha}(AT^{\alpha})x_0 - a_0(T)) = x_T$$

The control function $u(t)$ given in (3.1) steers the state of system (1.4) to from initial state x_0 to finial state x_T, which means that the system (1.4) is controllable on $[0,T]$.

Now we shall prove the necessity by contradiction. First, we assume system (1.4) is controllable on $[0,T]$ and controllability Grammian matrix W is singular. Then there exists non-zero vector Z such that

$$\begin{aligned}
&Z^{*}WZ \\
&= \int_0^{T\text{-h}} Z^{*}\left[(T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^{\alpha})C + (T-s-h)^{\alpha-1}E_{\alpha,\alpha}(A(T-s-h)^{\alpha})B\right] \\
&\quad \times \left[Z^{*}(T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^{\alpha})C + Z^{*}(T-s-h)^{\alpha-1}E_{\alpha,\alpha}(A(T-s-h)^{\alpha})B\right]^{*} \mathrm{ds} \\
&= \int_0^{T\text{-h}} \left[Z^{*}(T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^{\alpha})C + Z^{*}(T-s-h)^{\alpha-1}E_{\alpha,\alpha}(A(T-s-h)^{\alpha})B\right] \\
&\quad \times \left[Z(T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^{\alpha})C + Z(T-s-h)^{\alpha-1}E_{\alpha,\alpha}(A(T-s-h)^{\alpha})B\right]^{*} \mathrm{ds} = 0
\end{aligned}$$

It is easy to get

$$\begin{aligned}
&\left[Z^{*}(T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^{\alpha})C + Z^{*}(T-s-h)^{\alpha-1}E_{\alpha,\alpha}(A(T-s-h)^{\alpha})B\right] \\
&\times \left[Z^{*}(T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^{\alpha})C + Z^{*}(T-s-h)^{\alpha-1}E_{\alpha,\alpha}(A(T-s-h)^{\alpha})B\right]^{*} = 0
\end{aligned}$$

Thus

$$\left[Z^{*}(T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^{\alpha})C + Z^{*}(T-s-h)^{\alpha-1}E_{\alpha,\alpha}(A(T-s-h)^{\alpha})B\right] = 0$$

Noting that system (1.4) is controllable on $[0,T]$, choose initial state $x(0) = E_{\alpha}(A(T)^{\alpha})^{-1}(Z - a_0(T))$ and $u(t) \in \mathbf{X}_2$ such that $x(T) = 0$, then

$$\begin{aligned}
&E_{\alpha}(AT^{\alpha})x_0 + a_0(T) + \int_0^{T\text{-h}} \left[(T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^{\alpha})C + (T-s-h)^{\alpha-1}E_{\alpha,\alpha}(A(T-s-h)^{\alpha})B\right] u(s)]ds \\
&= E_{\alpha}(AT^{\alpha})E_{\alpha}(A(T)^{\alpha})^{-1}(Z - a_0(T)) + a_0(T) \\
&\quad + \int_0^{T\text{-h}} \left[(T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^{\alpha})C + (T-s-h)^{\alpha-1}E_{\alpha,\alpha}(A(T-s-h)^{\alpha})B\right] u(s)]ds = 0
\end{aligned}$$

Therefore

$$\begin{aligned}
Z = &-\int_0^{T\text{-h}} \left[(T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^{\alpha})C + (T-s-h)^{\alpha-1}E_{\alpha,\alpha}(A(T-s-h)^{\alpha})B\right] \\
&\times \left[(T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^{\alpha})C + (T-s-h)^{\alpha-1}E_{\alpha,\alpha}(A(T-s-h)^{\alpha})B\right]^{*} W^{-1} \\
&\times (x_T - E_{\alpha}(AT^{\alpha})x_0 - a_0(T))ds
\end{aligned}$$

Since $\left[Z^{*}(T-s)^{\alpha-1}E_{\alpha,\alpha}(A(T-s)^{\alpha})C + Z^{*}(T-s-h)^{\alpha-1}E_{\alpha,\alpha}(A(T-s-h)^{\alpha})B\right] = 0$

Then $Z^{*}Z = 0$, so $Z = 0$. This is contradiction. Hence W is non-singular.

Remark 3.1 The linear fractional dynamical system (1.4) can be reduced to the first order linear dynamical system when $\alpha = 1$,

$$\begin{cases} x'(t) = Ax(t) + Bu(t\text{-}h) + Cu(t), \quad t \in J := [0,T], \\ x(0) = x_0, \ -h < t \le 0 \\ u(t) = u_0(t), \ t \in [-h, 0) \end{cases} \tag{3.2}$$

The controllability Grammian is given as

$$W(T) = \int_0^{T\text{-h}} \left[e^{A(T-s)}C + e^{A(T-s-h)}B\right] \times \left[e^{A(T-s)}C + e^{A(T-s-h)}B\right]^{*} \mathrm{ds} \tag{3.3}$$

A control function which can steers the state of the first order system form x_0 to x_T is given as

$$u(t)=\begin{cases}\left[e^{A(T-t)}C+e^{A(T-t-h)}B\right]^{*}W^{-1}(x_T-e^{AT}x_0-a_0(T)), & t\in[0,T-h]\\ u_0(t), & t\in[-h,0)\\ 0, & t\in(T-h,T]\end{cases} \tag{3.4}$$

where

$$a_0(t)=\int_{-h}^{0}e^{A(t-s-h)}Bu_0(s)ds.$$

Remark 3.2 Theorem 3.1 and Remark 3.1 are generalized forms of the results obtained in Vijayakumar S. Muni, Venkatesan, Govindaraj and Raju K. George ([1], 2018)

IV. Example

To illustrate the main results, consider the linear fractional systems with control delay of the following type

$$\begin{cases}{}^{C}D^{\frac{1}{2}}x(t)=Ax(t)+Bu(t-0.5)+Cu(t), \quad t\in J:=[0,2],\\ x(0)=x_0,\\ u(t)=u_0(t)=1,\ t\in[-0.5,0]\end{cases} \tag{4.1}$$

with $x_0=\begin{bmatrix}1\\0\end{bmatrix}$, $t=0.5$. Now calculate the controllability Grammian matrix of system (4.1)

$$W:=W(2)=\int_0^{\frac{3}{2}}\left[(2-s)^{-\frac{1}{2}}E_{1/2,1/2}(A\sqrt{2-s})C+(\frac{3}{2}-s)^{-\frac{1}{2}}E_{1/2,1/2}(A\sqrt{\frac{3}{2}-s})B\right]$$
$$\times\left[(2-s)^{-\frac{1}{2}}E_{1/2\ 1/2}(A\sqrt{2-s})C+(\frac{3}{2}-s)^{-\frac{1}{2}}E_{1/2,1/2}(A\sqrt{\frac{3}{2}-s})B\right]^{*}\mathrm{ds}$$

After simple calculation using Matlab based on A, B and C, if $|W|>0$, W is a non-singular matrix, by theorem 3.1, System (4.1) is controllable on $[0, 2]$.

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