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On similarity solutions of the Zabolotskaya--Khokhlov acoustics model

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***Abstract**: The three-dimensional Zabolotskaya--Khokhlov equation (ZK) is a nonlinear second order partial differentialmodel for sound waves propagation. In this work, some ordinary differential reductions of this uation have been givenby two-step using of the Lie point symmetry group method. Then, some new exact similarity solutions of the ZK equationhave been obtained by solvingthe reduced ODEs.The solutions can be used to clarifying the propagation of a boundedtwo-dimensional acoustic beam in nonlinear medias.*

***Keywords:**Zabolotskaya--Khokhlov equation (ZK), Symmetry algebra, Lie point symmetry group, Optimal systemof sub-algebras, Reduction of equation, Similarity solution*

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I. Introduction

1.1 The applications of Lie groups to differential equations

A Lie point symmetry of a differential equation is a Lie point transformation that maps any solution of the equation into its another solution. On the other, similarity solutions of an equation are the solutions which are invariant under the Lie point symmetries. Finding simpler forms of differential equations is called a reduction. Indeed, the order of an ordinary differential equation, and the number of independent variables of a partial differential equation can be reduced by one if they are invariant under one-parameter Lie group of point symmetries. Similarity solutions of an equation are construed by solving its reduced equations.
Marius Sophus Lie was the first to introduce the relationship between Lie point symmetry groups and traditional methods for reducing of differential equations, and finding their similarity solutions. Lie's fundamental idea led to concept of Lie algebra. He declared that Lie groups can be characterized by their infinitesimal generators. Also, satisfying infinitesimal symmetry condition is a distinction for the basic elements of a Lie symmetry algebra. The applications of Lie groups to differential equations were mainly established by Lie, Emmy Noether, and Elie Cartan.

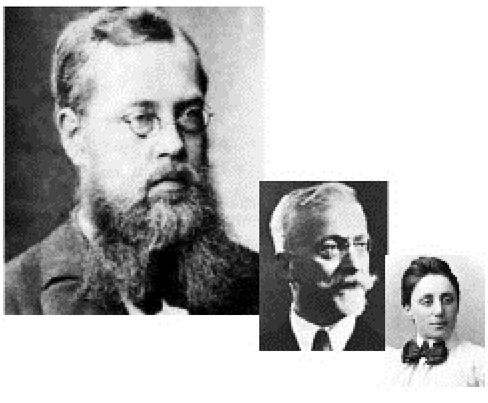

Fig. 1 Marius Sophus Lie (1842-1899), Elie Cartan (1869-1951), Emmy Noether (1882 -1935)

1.2 The Zabolotskaya--Khokhlov equation (ZK)

The Zabolotskaya–Khokhlov equation (ZK) in three dimensions, is a nonlinear partial differential equation in the form

$$\Delta(t,x,y,u) = u_{xt} - (u_x)^2 - uu_{xx} - u_{yy} = 0. \tag{1}$$

where $t, x > 0$ and $-\infty < y < +\infty$. In addition, the value of $u = u(t,x,y)$ is proportional to the deviation of the media density from the balanced density, while the dimensionless variables t, x, y are expressed via the temporal $\mathfrak{t}$ and spatial variables $\mathfrak{x}$, $\mathfrak{y}$ as the following (refer to [2])

$$t = \frac{\mathfrak{t} - \frac{\mathfrak{x}}{c_0}}{\sqrt{\gamma+1}}\sqrt{\rho_0 c_0}, \quad x = \mu\mathfrak{x}, \quad y = \sqrt{\frac{2\mu}{c_0}}\mathfrak{y}, \tag{2}$$

where c_0 is the sound velocity in the media,γ is the isentropic exponent, $\mathfrak{x}$ is the coordinate in the direction of beam propagation, μ is a small parameter, and ρ_0 is the balanced density. According to [1,9,11], this equation is derived from the incompressible Navier–Stokes equation. Furthermore, it is used in order to justify the phenomenon of sound waves propagation in nonlinear medias. In this research, findings about ZK's Lie reductions and its similarity solutions are presented. Since the ZK equation involves three independent variables and one dependent variable, it can be accounted as the total space $E \simeq \mathbb{R}^{3+1}$ with coordinates t, x, y, u. Therefore, Lie point transformations of the equation are via:

$$\begin{aligned} \overline{t} &= t + \varepsilon\xi_1(t,x,y,u) + \mathfrak{O}(\varepsilon^2), \\ \overline{x} &= x + \varepsilon\xi_2(t,x,y,u) + \mathfrak{O}(\varepsilon^2), \\ \overline{y} &= y + \varepsilon\xi_3(t,x,y,u) + \mathfrak{O}(\varepsilon^2), \\ \overline{u} &= u + \varepsilon\varphi_1(t,x,y,u) + \mathfrak{O}(\varepsilon^2), \end{aligned} \tag{3}$$

where ξ_1, ξ_2, ξ_3, φ_1 are C^∞ functions.
A Lie point symmetry of the eq.(1) is a Lie point transformation that maps any solution of the equation into its another solution. Hence, it is a transformation as (3), of E into itself. On the other hand, generators of a symmetry algebra related to the ZK equation, can be regarded as infinitesimals of the transformation (3), namely,

$$v = \xi_1\partial_t + \xi_2\partial_x + \xi_3\partial_y + \varphi_1\partial_u.$$
$$(\ \xi_1, \xi_2, \xi_3, \varphi_1 \ are\ C^\infty functions\ of\ t, x, y, u) \tag{4}$$

In 2008, N.J.C. Ndogmo completed the work in [3,4,6,12], in the line of further introduction of the ZK's symmetry properties. Thereupon, he presented the general form of basic elements of the ZK's symmetry algebras as (refer to [7]):

$$\begin{aligned} v_0 &= 2x\partial_x + y\partial_y + 2u\partial_u, \\ v_g &= g\partial_x - g'\partial_u, \\ v_h &= \tfrac{1}{2}yh'\partial_x + h\partial_y - \tfrac{1}{2}yh''\partial_u, \\ v_f &= f\partial_t + \tfrac{1}{6}(2xf' + y^2f'')\partial_x + \tfrac{2}{3}yf'\partial_y + \tfrac{1}{6}(-4uf' - 2xf'' - y^2f''')\partial_u, \end{aligned} \tag{5}$$

where f, g, h are arbitrary C^∞ functions on the time variable t, defined on some open subset of $\mathbb{R}$. In this article, applying a symmetry method discovers a symmetry algebra for the ZK equation. The infinitesimal symmetries are linear combinations $v_g + v_h + v_f$ corresponded to choices of the functions$f$, g, h as:

$$\begin{gathered} g = -\tfrac{t^2}{2}, h = -t, f = 0, \\ g = -t, h = -1, f = 0, \\ g = 1, h = 0, f = 0, \\ g = 0, h = 0, f = 1. \end{gathered}$$

Obtaining the reductions of the equation by the one and two-dimensional optimal systems of subalgebras and repeating the symmetry method for the reduced equations lead us to some ordinary differential equations. Then, solving the ODEs gives several similarity solutions of the ZK equation.

II. Main discussions

2.1 The symmetry algebra

Considering infinitesimal generators of ZK's symmetry algebra as (4), with constraint $\xi_3 + \varphi_1 = 0$, and implementing the Maple commands result the following linearly independent vector fields

$$\begin{aligned} v_1 &= -\tfrac{1}{2}(y+t^2)\partial_x - t\partial_y + t\partial_u, \\ v_2 &= -t\partial_x - \partial_y + \partial_u, \\ v_3 &= \partial_x, \\ v_4 &= \partial_t. \end{aligned} \tag{6}$$

Such that their second prolongations are according to

$$\begin{aligned} pr^{(2)}v_1 &= -\tfrac{1}{2}(y+t^2)\partial_x - t\partial_y + t\partial_u + (1 + tu_x + u_y)\partial_{u_t} + \tfrac{1}{2}u_x\partial_{u_y} + \\ &(u_x + 2tu_{tx} + 2u_{ty})\partial_{u_{tt}} + (tu_{xx} + u_{xy})\partial_{u_{tx}} + (\tfrac{1}{2}u_{tx} + tu_{xy} + \\ &u_{yy})\partial_{u_{ty}} + \tfrac{1}{2}u_{xx}\partial_{u_{xy}} + u_{xy}\partial_{u_{yy}}, \\ pr^{(2)}v_2 &= -t\partial_x - \partial_y + \partial_u + u_x\partial_{u_t} + 2u_{tx}\partial_{u_{tt}} + u_{xx}\partial_{u_{tx}} + u_{xy}\partial_{u_{ty}}, \\ pr^{(2)}v_3 &= \partial_x, \\ pr^{(2)}v_4 &= \partial_t. \end{aligned}$$

Therefore, $pr^{(2)}v_i(\Delta)\mid_{\Delta=0} = 0$ $(i = 1,2,3,4)$. Thus $\mathfrak{g} = \langle v_1, v_2, v_3, v_4 \rangle$ is a symmetry algebra for the ZK equation.

Theorem 1 *A one-dimensional optimal system of the symmetry subalgebras of the ZK equation is compliant with*

$$\begin{aligned} \{\langle v_3\rangle, \quad \langle v_1 + c_3v_3\rangle, \quad \langle v_2 + c_3v_3\rangle, \quad \langle v_4 + c_3v_3\rangle, \langle c_1v_1 + c_2v_2 + v_3\rangle, \\ \langle c_1v_1 + v_3 + c_4v_4\rangle, \langle c_2v_2 + v_3 + c_4v_4\rangle, \langle c_1v_1 + c_2v_2 + v_3 + c_4v_4\rangle\}, \end{aligned} \tag{7}$$

where c_1, c_2, c_3, c_4 *are nonzero constants.*

Proof The center of $\mathfrak{g}$ is $\langle v_3\rangle$. Therefore, a one-dimensional optimal system of subalgebras of the ZK eq.(1) must be as $\frac{\mathfrak{g}}{\langle v_3\rangle}$ (refer to [5]), such in relations (7).

If $\langle U\rangle$ be a member of the one-dimensional optimal system, providing that U with together an appropriate $V = \sum_{k=1}^{4} c_kv_k$ $(c_k \in \mathbb{R})$ create a Lie algebra, results that $\langle U, V\rangle$ is an element of a 2-dimensional optimal system of the symmetry subalgebras. In fact, the next theorem can be declared:

Theorem 2 *A two-dimensional optimal system of the symmetry subalgebras of the ZK equation is compliant with*

$$\{\langle v_3, c_1v_1 + c_2v_2 + c_4v_4\rangle\}, \tag{8}$$

where c_1, c_2, c_4 *are arbitrary constants.*

More over, taking into account the producers of the 2-dimensional optimal system member, namely, $V_1 = v_1 + v_2 + v_4$ and $V_2 = v_3$, providing that these vectors with a suitable $V = \sum_{k=1}^{4} a_kv_k$; $(a_k \in \mathbb{R})$ create a Lie algebra, concludes that $\langle V, V_1, V_2\rangle$ is an element of a 3-dimensional optimal system. As a result, for the optimal systems of subalgebras, the following table is produced:

Table 1 The optimal systems of subalgebras, and related ordinary invariants

dimension	optimal system element	ordinary invariants
1	$\langle v_3\rangle$	t, y, u
1	$\langle v_1+v_3\rangle$	$t, -x-\frac{y}{t}+\frac{1}{4}\frac{y^2}{t}+\frac{1}{2}ty, y+u$
1	$\langle v_2+v_3\rangle$	$t, y-\frac{x}{t-1}, u+\frac{x}{t-1}$
1	$\langle v_3+v_4\rangle$	$x-t, y, u$
1	$\langle v_1+v_2+v_3\rangle$	$t, -x+\frac{1}{2}y(1+t)+\frac{1}{4}\frac{-6y+y^2}{1+t}, y+u$
1	$\langle v_1+v_3+\frac{1}{2}v_4\rangle$	$y+t^2, -2t+t(y+t^2)+x, -t^2+u$
1	$\langle v_2+v_3+v_4\rangle$	$\frac{1}{2}t^2-t+x, t+y, u-t$
1	$\langle v_1+v_2+\frac{1}{4}v_3+\frac{1}{2}v_4\rangle$	$t^2+2t+y, -\frac{1}{2}t+t(t^2+2t+y)+x,$ $-t^2-2t+u$
2	$\langle v_1+v_2+v_4, v_3\rangle$	$\frac{1}{2}t^2+t+y, -\frac{1}{2}t^2-t+u$
3	$\langle v_1+v_4, v_2, v_3\rangle$	$u+y$

2.2 Reductions and similarity solutions of the ZK equation

Using of the table 1 obtains the reduction formulas as follows (refer to [8]):

i) The reduction by $\langle v_3\rangle$ is done by the formula

$$u(t,x,y)=s(w,z),\quad t=w,\quad y=z, \tag{9}$$

and the reduced equation is

$$\Delta_1: s_{zz}=0. \tag{10}$$

Solving the (10) acquires a similarity solution of the ZK eq.(1) as

$$s=F_1(w)z+F_2(w), \tag{11}$$

where F_1, F_2 are arbitrary C^∞ functions.

ii) The reduction by $\langle v_1+v_3\rangle$ is done by the formula

$$u(t,x,y)=s(w,z)-y,\ t=w,\ x=-z-\frac{y}{w}+\frac{1}{4}\frac{y^2}{w}+\frac{1}{2}wy, \tag{12}$$

and the reduced equation is

$$\Delta_2: (4w^2s+(w^2-2)^2)s_{zz}+4w^2s_{wz}+4w^2s_z^2+2ws_z=0. \tag{13}$$

Eq.(13) has a symmetry algebra with generators $\omega_1=-w\partial_w-\frac{1}{4}\frac{(w^4-4w^2+4)}{w}$ $.\partial_z+s\partial_s$, $\omega_2=\partial_z$, $\omega_3=w\partial_w+\frac{1}{6}\frac{(6zw+w^4+12)}{w}\partial_z$, and a one-dimensional optimal system of sub-algebras $\langle\omega_1\rangle$,$\langle\omega_1+\omega_2\rangle$,$\langle\omega_1+\omega_3\rangle$.

– The reduction by $\langle\omega_1\rangle$ is done by the formula

$$s(w,z)=\frac{v(f)}{w},\ z=f+\frac{1}{12}w^3-\frac{1}{w}-w, \tag{14}$$

and the reduced equation is

$$\Delta_{2.1}: 2v_{ff}v-v_f+2v_f^2=0. \tag{15}$$

Solving the (15) acquires a similarity solution of the ZK eq.(1) as

$$v=c_1\left(LambertW\left(\frac{1}{c_1}e^{\frac{1}{2c_1}(f+c_2)}\right)+1\right). \tag{16}$$

– The reduction by $\langle\omega_1+\omega_2\rangle$ is done by the formula

$$s(w,z)=\frac{v}{w}, z=f+\frac{1}{12}w^3-\ln(w)-\frac{1}{w}-w, \tag{17}$$

and the reduced equation is

$$\Delta_{2.2}: 2(v+1)v_{ff}-v_f+2v_f^2=0. \tag{18}$$

Solving the (18) acquires a similarity solution of the ZK eq.(1) as

$$v=c_1\left(LambertW\left(\frac{1}{c_1}e^{\frac{1}{c_1}(\frac{1}{2}(f+c_2)+1)-1}\right)\right)+c_1-1. \tag{19}$$

– The reduction by $\langle\omega_1+\omega_3\rangle$ is done by the formula

$$s(w,z)=v(f)(f^4-12f^2-12zf-12),\ w=f, \tag{20}$$

and the reduced equation is

$$\Delta_{2.3}: 12f^3v_f-144f^4v^2+19f^2v=0. \tag{21}$$

Solving the (21) acquires a similarity solution of the ZK eq.(1) as

$$v=\frac{1}{-24f^2+c_1f^{3/2}}. \tag{22}$$

iii) The reduction by $\langle v_2+v_3\rangle$ is done by the formula

$$u(t,x,y)=s(w,z)-\frac{x}{w-1},\quad t=w,\quad y=z+\frac{x}{w-1}, \tag{23}$$

and the reduced equation is

$$\Delta_3 : (w-1)s_{wz} + (s+(w-1)^2)s_{zz} + s_z^2 + s_z = 0. \tag{24}$$

Eq.(24) has a symmetry algebra with generators $\alpha_1 = \left(\frac{1}{2}w - \frac{1}{2}\right)\partial_w + z\partial_z + s\partial_s$, $\alpha_2 = \left(\frac{1}{2}w - \frac{1}{2}\right)\partial_w + \left(\frac{1}{2}w^2 - w + 1\right)\partial_z$, $\alpha_3 = \partial_z$, and a one-dimensional optimal system of sub-algebras $\langle \alpha_1 - \alpha_2 + \frac{1}{2}\alpha_3\rangle$,$\langle\alpha_2\rangle$,$\langle\alpha_3\rangle$.

– The reduction by $\langle \alpha_1 - \alpha_2 + \frac{1}{2}\alpha_3\rangle$ is done by the formula

$$s(w,z) = v(f)(f^2 - 2f + 1 - 2z),\ w = f, \tag{25}$$

and the reduced equation is

$$\Delta_{3.1} : (1-f)v_f + 2v^2 - v = 0. \tag{26}$$

Solving the (26) acquires a similarity solution of the ZK eq.(1) as

$$v = \frac{1}{c_1(f-1)+2}. \tag{27}$$

– The reduction by $\langle\alpha_2\rangle$ is done by the formula

$$s(w.z) = v(f),\ z = \left(-f + \frac{1}{2}(w-1)^2 + \ln(w-1)\right), \tag{28}$$

and the reduced equation is

$$\Delta_{3.2} : (v-1)v_{ff} + v_f^2 - v_f = 0. \tag{29}$$

Solving the (29) acquires a similarity solution of the ZK eq.(1) as

$$v = c_1\left(LambertW\left(\frac{e^{\frac{1}{c_1}-1}}{c_1 e^{-\frac{1}{c_1}(f+c_2)}}\right)+1\right)+1. \tag{30}$$

– The reduction by $\langle\alpha_3\rangle$ is done by the formula

$$s(w,z) = v(f),\ w = f, \tag{31}$$

and the reduced equation is trivial.

iv) The reduction by $\langle v_3 + v_4\rangle$ is done by the formula

$$u(t,x,y) = s(w,z),\quad t = x - w,\quad y = z, \tag{32}$$

and the reduced equation is

$$\Delta_4 : (s+1)s_{ww} + s_w^2 + s_{zz} = 0. \tag{33}$$

Eq.(33) has a symmetry algebra with generators $\beta_1 = w\partial_w + \frac{1}{2}z\partial_z + (1+s)\partial_s$, $\beta_2 = \partial_z$, $\beta_3 = \partial_w$, and a one-dimensional optimal system of sub-algebras $\langle\beta_1 - \beta_2 + \frac{1}{2}\beta_3\rangle$,$\langle\beta_2\rangle$,$\langle\beta_3\rangle$.

– The reduction by $\langle\beta_1 - \beta_2 + \frac{1}{2}\beta_3\rangle$ is done by the formula

$$s(w,z) = v(f)(1+2w) - 1,\ z = f\sqrt{1+2w} + 2, \tag{34}$$

and the reduced equation is

$$\Delta_{4.1} : (f^2 v + 1)v_{ff} - 5fvv_f + f^2 v_f^2 + 4v^2 = 0. \tag{35}$$

Solving the (35) acquires a very huge abnormal similarity solution of the ZK eq.(1).

– The reduction by $\langle\beta_2\rangle$ is done by the formula

$$s(w,z) = v(f),\ w = f, \tag{36}$$

and the reduced equation is

$$\Delta_{4.2} : (v+1)v_{ff} + v_f^2 = 0. \tag{37}$$

Solving the (37) acquires a similarity solution of the ZK eq.(1) as

$$v = -\sqrt{c_1 f + c_2} - 1. \tag{38}$$

– The reduction by $\langle \beta_3 \rangle$ is done by the formula

$$s(w,z) = v(f),\ z = f, \tag{39}$$

and the reduced equation is

$$\Delta_{4.3} : v_{ff} = 0. \tag{40}$$

Solving the (40) acquires a similarity solution of the ZK eq.(1) as

$$v = c_1 f + c_2. \tag{41}$$

v) the reduction by $\langle v_1 + v_2 + v_3 \rangle$ is done by the formula

$$\begin{gathered} u(t,x,y) = s(w,z) - y,\ t = w, \\ x = -z + \tfrac{1}{2}y(1+w) + \tfrac{1}{4}\tfrac{-6y+y^2}{1+w}, \end{gathered} \tag{42}$$

and the reduced equation is

$$\begin{gathered} \Delta_5 : (2(w+1)^2 s - (w^2+2w-2)^2)s_{zz} + 2(w+1)s_{wz} + \\ 2(w+1)s_z^2 + s_z = 0. \end{gathered} \tag{43}$$

Eq.(43) has a symmetry algebra with generators
$\gamma_1 = \left(z - \frac{1}{4}\left(\frac{1}{4}w^4 + w^3 - \frac{3}{2}w^2 - 5w\right) + 9\ln(w+1)\right)\partial_z + s\partial_s$, $\gamma_2 = \partial_z$, and a one- dimensional optimal system of sub-algebras $\langle \gamma_1 \rangle$, $\langle \gamma_1 \rangle$.

– The reduction by $\langle \gamma_1 \rangle$ is done by the formula

$$\begin{gathered} s(w,z) = v(f)(16z - f^4 - 4f^3 + 6f^2 + \\ 20f + 144\ln(f+1)),\ w = f, \end{gathered} \tag{44}$$

and the reduced equation is

$$\Delta_{5.1} : 16(f+1)v_f + 256(f+1)v^2 + 8v = 0. \tag{45}$$

Solving the (45) acquires a similarity solution of the ZK eq.(1) as

$$v = \frac{1}{32(f+1) + c_1\sqrt{f+1}}. \tag{46}$$

– The reduction by $\langle \gamma_2 \rangle$ is done by the formula

$$s(w,z) = v(f),\ w = f, \tag{47}$$

and the reduced equation is trivial.

vi) The reduction by $\langle v_2 + v_3 + v_4 \rangle$ is done by the formula

$$u(t,x,y) = s(w,z) + t, \quad x = w + t - \frac{1}{2}t^2, \quad y = z - t, \tag{48}$$

and the reduced equation is

$$\Delta_6 : s_{wz} - s_{zz} - (s+1)s_{ww} - s_w^2 = 0. \tag{49}$$

Eq.(49) has a symmetry algebra with generators $\lambda_1 = (z+4w)\partial_w + 2z\partial_z + (4s+3)\partial_s$, $\lambda_2 = \partial_z$, $\lambda_3 = \partial_w$, and a one-dimensional optimal system of sub-algebras $\langle \lambda_1 \rangle$, $\langle \lambda_2 \rangle$, $\langle \lambda_2 + \lambda_3 \rangle$.

– The reduction by $\langle \lambda_1 \rangle$ is done by the formula

$$s(w,z) = -fz^2v(f) - \frac{3}{4},\ w = fz^2 - \frac{1}{2}, \tag{50}$$

and the reduced equation is

$$\Delta_{6.1} : (4f^3 - f^2v)v_{ff} + (6f^2 - 4fv) - f^2v_f^2 - v^2 = 0. \tag{51}$$

Solving the (51) acquires a very huge abnormal similarity solution of the ZK eq.(1).

– The reduction by $\langle \lambda_2 \rangle$ is done by the formula

$$s(w,z) = v(f),\ w = f, \tag{52}$$

and the reduced equation is

$$\Delta_{6.2} : (v+1)v_{ff} + v_f^2 = 0. \tag{53}$$

Solving the (53) acquires a similarity solution of the ZK eq.(1) as

$$v = -\sqrt{c_1 f + c_2} - 1. \tag{54}$$

– The reduction by $\langle \lambda_3 \rangle$ is done by the formula

$$s(w,z) = v(f),\ w = z, \tag{55}$$

and the reduced equation is done by the formula

$$\Delta_{6.3} : v_{ff} = 0. \tag{56}$$

Solving the (56) acquires a similarity solution of the ZK eq.(1) as

$$v = c_1 f + c_2. \tag{57}$$

vii) The reduction by $\langle v_1 + v_3 + \frac{1}{2}v_4 \rangle$ is done by the formula

$$u(t,x,y) = s(w,z) + t^2,\ x = z + 3t - tw,\ y = w - t^2, \tag{58}$$

and the reduced equation is

$$\Delta_7 : (s - w + 3)s_{zz} + s_{ww} + s_z^2 = 0. \tag{59}$$

Eq.(59) has a symmetry algebra with generators $\kappa_1 = \partial_w + \partial_s$, $\kappa_2 = \left(\frac{1}{2}w - 3\right)\partial_w + z\partial_z + \left(s - \frac{1}{2}w\right)\partial_s$, $\kappa_3 = \partial_z$, and a one-dimensional optimal system of sub-algebras $\langle \kappa_1 \rangle$, $\langle \kappa_2 \rangle$, $\langle \kappa_3 \rangle$.

– The reduction by $\langle \kappa_1 \rangle$ is done by the formula

$$s(w,z) = v(f) + w,\ z = f, \tag{60}$$

and the reduced equation is

$$\Delta_{7.1} : (v+3)v_{ff} + v_f^2 = 0. \tag{61}$$

Solving the (61) acquires a similarity solution of the ZK eq.(1) as

$$v = -\sqrt{c_1 f + c_2} - 3. \tag{62}$$

– The reduction by $\langle \kappa_2 \rangle$ is done by the formula

$$s(w,z) = v(f)(w^2 - 12w + 36) + w - 3,\ z = f(w-6)^2, \tag{63}$$

and the reduced equation is

$$\Delta_{7.2} : 4\left(f^2 + \frac{1}{4}v\right) v_{ff} + v_f^2 - 2fv_f + 2v = 0. \tag{64}$$

Solving the (64) acquires a very huge abnormal similarity solution of the ZK eq.(1).

– The reduction by $\langle \kappa_3 \rangle$ is done by the formula

$$w = f, s(w,z) = v(f), \tag{65}$$

and the reduced equation is done by the formula

$$\Delta_{7.3} : v_{ff} = 0. \tag{66}$$

Solving the (66) acquires a similarity solution of the ZK eq.(1) as

$$v = c_1 f + c_2. \tag{67}$$

viii) The reduction by $\langle v_1 + v_2 + \frac{1}{4}v_3 + \frac{1}{2}v_4 \rangle$ is done by the formula

$$u(t,x,y) = s(w,z) + t^2 + 2t,\ x = z + \tfrac{1}{2}t - tw, \quad y = w - t^2 - 2t, \tag{68}$$

and the reduced equation is

$$\Delta_8 : \frac{1}{2}(2s - 2w + 1)s_{zz} - 2s_{wz} + s_{ww} + s_z^2 = 0. \tag{69}$$

Eq.(69) has a symmetry algebra with generators $\zeta_1 = \partial_w + \partial_s$, $\zeta_2 = \frac{1}{2}(w+1)\partial_w + \left(z + \frac{1}{2}w\right)\partial_z + \left(s - \frac{1}{2}w\right)\partial_s$, $\zeta_3 = \partial_z$, and a one-dimensional optimal system of sub-algebras $\langle \zeta_1 \rangle$, $\langle \zeta_2 \rangle$, $\langle \zeta_3 \rangle$.

– The reduction by $\langle \zeta_1 \rangle$ is done by the formula

$$s(w,z) = v(f) + w,\ z = f, \tag{70}$$

and the reduced equation is

$$\Delta_{8.1} : \frac{1}{2}(2v+1)v_{ff} + v_f^2 = 0. \tag{71}$$

Solving the (71) acquires a similarity solution of the ZK eq.(1) as

$$v = -\frac{1}{2}\sqrt{c_1 f + c_2} - \frac{1}{2}. \tag{72}$$

– The reduction by $\langle \zeta_2 \rangle$ is done by the formula

$$s(w,z) = v(f)(w^2 + 2w + 1) + w + \tfrac{1}{2}, \quad z = f(w^2 + 2w + 1) - w - \tfrac{1}{2}, \tag{73}$$

and the reduced equation is

$$\Delta_{8.2} : \left(v + 4f^2\right) v_{ff} - 2fv_f + v_f^2 + 2v = 0. \tag{74}$$

Solving the (74) acquires a very huge abnormal similarity solution of the ZK eq.(1).

– The reduction by $\langle \zeta_3 \rangle$ is done by the formula

$$s(w,z) = v(f),\ w = f, \tag{75}$$

and the reduced equation is

$$\Delta_{8.3} : v_{ff} = 0. \tag{76}$$

Solving the (76) acquires a similarity solution of the ZK eq.(1) as

$$v = c_1 f + c_2. \tag{77}$$

IX) The reduction by $\langle v_1 + v_2 + v_4, v_3 \rangle$ is done by the formula

$$u(t,x,y) = s(w) + \left(\frac{1}{2}t^2 + t\right),\ y = w - \frac{1}{2}t^2 - t, \tag{78}$$

and the reduced equation is

$$\Delta_9 : s_{ww} = 0. \tag{79}$$

Solving the (79) acquires a similarity solution of the ZK eq.(1) as

$$v = c_1 w + c_2. \tag{80}$$

2.3 Symmetry group action

Recall that for every $w = (t,x,y,u)$ belonging to the total Euclidean space $E \simeq \mathbb{R}^{3+1}$; $v_i \in \mathfrak{g}$ and $\varepsilon \in \mathbb{R}$, the local flow $\exp(\varepsilon v_i)(w)$; $(i = 1,2,3,4)$ is defined by

$$\partial_\varepsilon \exp(\varepsilon v_i)(w) = v_i \mid_{\exp(\varepsilon v_i)(w)},\ \exp(\varepsilon v_i)(w) \mid_{\varepsilon=0} = w, \quad (\text{where } \partial_\varepsilon = \frac{d}{d\varepsilon}).$$

Thus, the behaviors of one- parameter transformations $\exp(\varepsilon v_i) : E \to E$, namely,

$$\exp(\varepsilon v_i) : (t,x,y,u) \mapsto (\widetilde{t}, \widetilde{x}, \widetilde{y}, \widetilde{u}) \quad ((i = 1,2,3,4), \quad \varepsilon \in \mathbb{R}), \tag{81}$$

lead to the actions of one-parameter subgroups of G, as the following table:

Table 2 The one-parameter subgroups actions

	$\widetilde{t}$	$\widetilde{x}$	$\widetilde{y}$	$\widetilde{u}$
$\exp(\varepsilon v_1)$	t	$\frac{1}{4}\varepsilon^2 t - \frac{1}{2}\varepsilon t^2 + x - \frac{1}{2}\varepsilon y$	$-\varepsilon t + y$	$\varepsilon t + u$
$\exp(\varepsilon v_2)$	t	$-\varepsilon t + x$	$y - \varepsilon$	$u + \varepsilon$
$\exp(\varepsilon v_3)$	t	$x + \varepsilon$	y	u
$\exp(\varepsilon v_4)$	$t + \varepsilon$	x	y	u

On the other, the action of G on E is produced by combination of the one-parameter subgroups actions. Therefore, the use of the above table deduces:

Theorem 3 *If $u = F(t,x,y)$ is a solution of the ZK equation, then so are*

$$\begin{gathered} u = F(t, \tfrac{1}{4}\varepsilon^2 t - \tfrac{1}{2}\varepsilon t^2 + x - \tfrac{1}{2}\varepsilon y, -\varepsilon t + y) + \varepsilon t, \\ u = F(t, -\varepsilon t + x, y - \varepsilon) + \varepsilon,\ u = F(t, x + \varepsilon, y),\ u = F(t + \varepsilon, x, y), \end{gathered} \tag{82}$$

where ε is an arbitrary parameter.

Hence, calculating $\exp(\varepsilon(\sum_{k=1}^{4} c_k v_k))$; $(c_k \in \mathbb{R})$ results that:

Corollary 1 *If $u = F(t,x,y)$ is a solution of the ZK equation, then so is*

$$\begin{gathered} u = F(t + c_1\varepsilon, x + (c_2 + c_3 t + c_4 t^2 + c_5 y)\varepsilon + (c_6 + c_7 t)\varepsilon^2 + \\ c_8\varepsilon^3, y + (c_7 + c_9 t)\varepsilon + c_{10}\varepsilon^2) + (c_{11} + c_{12} t)\varepsilon - c_7\varepsilon^2), \end{gathered} \tag{83}$$

where ε, c_i; $(i = 1,2,\cdots,12)$ are arbitrary numbers.

Conclusion

In this research, a symmetry algebra for the three-dimensional Zabolotskaya–Khokhlov equation (ZK) has been determined. Then some new attainments about ZK's reductions and its similarity solutions have been obtained as the table 3. The solutions of this famous nonlinear equation may be applied to clarifying the propagation of a bounded two-dimensional acoustic beam in nonlinear medias.

Table 3 Reductions and Similarity solutions by one-dimensional optimal systems

Reductions by...	Reduced equations	Similarity solutions
v_3	Δ_1	$s = F_1(w)z + F_2(w)$
$v_1 + v_3$	$\Delta_2,$	–
	$\Delta_{2.1},$	$v = c_1\left(LambertW\left(\frac{1}{c_1}e^{\frac{1}{2c_1}(f+c_2)}\right)+1\right),$
	$\Delta_{2.2},$	$v = c_1\left(LambertW\left(\frac{1}{c_1}e^{\frac{1}{c_1}(\frac{1}{2}(f+c_2)+1)-1}\right)\right)+c_1-1,$
	$\Delta_{2.3}$	$v = \frac{1}{-24f^2+c_1f^{3/2}}$
$v_2 + v_3$	$\Delta_3,$	–
	$\Delta_{3.1},$	$v = \frac{1}{c_1(f-1)+2},$
	$\Delta_{3.2}$	$v = c_1\left(LambertW\left(\frac{e^{\frac{1}{c_1}-1}}{c_1e^{-\frac{1}{c_1}(f+c_2)}}\right)+1\right)+1$
$v_3 + v_4$	$\Delta_4,$	–
	$\Delta_{4.1},$	–
	$\Delta_{4.2}$	$v = -\sqrt{c_1f+c_2}-1$
$v_1 + v_2 + v_3$	$\Delta_5,$	–
	$\Delta_{5.1}$	$v = \frac{1}{32(f+1)+c_1\sqrt{f+1}}$
$v_2 + v_3 + v_4$	$\Delta_6,$	–
	$\Delta_{6.1},$	–
	$\Delta_{6.2},$	$v = -\sqrt{c_1f+c_2}-1,$
	$\Delta_{6.3}$	$v = c_1f + c_2$
$v_1 + v_3 + \frac{1}{2}v_4$	$\Delta_7,$	–
	$\Delta_{7.1},$	$v = -\sqrt{c_1f+c_2}-3,$
	$\Delta_{7.2},$	–
	$\Delta_{7.3}$	$v = c_1f + c_2$
$v_1 + v_2 + \frac{1}{4}v_3 + \frac{1}{2}v_4$	$\Delta_8,$	–
	$\Delta_{8.1},$	$v = -\frac{1}{2}\sqrt{c_1f+c_2}-\frac{1}{2},$
	$\Delta_{8.2},$	–
	$\Delta_{8.3}$	$v = c_1f + c_2$

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