IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 14, Issue 5 Ver. I (Sep - Oct 2018), PP 05-09
www.iosrjournals.org

Independence and Domination on Generalized Fibonacci Graphs

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Abstract: *In this study, certain independence and domination properties of generalized Fibonacci graphs are considered. Domination, upper domination, total domination, upper total domination, independent domination and connected domination numbers of generalized Fibonacci graphs are calculated. Several illustrative examples are given.*

Date of Submission: 08-09-2018 Date of acceptance: 26-09-2018

I. Introduction

There are several different studies on generalized Fibonacci graphs. In [1] authors investigate the structure of mincuts in an n-vertex generalized Fibonacci graph of degree 3 and calculate exact value of mincuts in this graph. In [2] authors study the relationship between algebraic expressions and graphs. They consider Fibonacci graph which gives a generic example of non-series-parallel graphs and they simplify the expressions of Fibonacci graphs and find their shortest representations. In [3] certain fundamental properties of generalized Fibonacci graphs of degree k such as number of edges, planarity, diameter, radius, center, girth, etc. are studied and by means of chromatic polynomials their chromatic numbers and chromatic indexes are obtained. Additionally, the incidence chromatic number of generalized Fibonacci graphs are also given. In [4] generalized Fibonacci graphs and their Cartesian product graphs are considered. These graph's certain fundamental properties, such as number of edges, planarity, diameter, radius, center, distance center, Wiener index, etc. are studied. Although generalized Fibonacci graphs are mostly used for communication on networks, but there are also a variety of applications in chemistry (see. [5, 6, 7]).
For the sake of completeness, we briefly recall here the certain essential concepts of graph theory and domination in graphs needed along this paper. For further information, see [8, 9, 10].

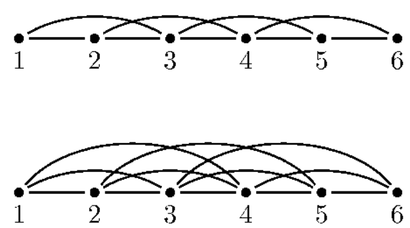

Figure 1:$F_6(2)$ and $F_6(3)$ generalized Fibonacci graphs

A *generalized Fibonacci graph* of degree k has vertices $\{1,2,3,\dots,n\}$ and edges

$$\{\{v,w\} | 1 \le v,w \le n and \quad |w-v| \le k\},$$

and it is usually denoted by $F_n(k)$ (see Figure 1). In particular, if $n = 2$ then it is called a *Fibonacci graph*. The following proposition explains why these graphs are called Fibonacci graphs.

Proposition 1.1 *In a Fibonacci graph F_n the number of different paths, connecting* 1 *and* n *is equal to nth Fibonacci number.*

Proof. For $n = 2$ there is only 1 path $(1 \to 2)$. For $n = 3$ we have 2 choices: $1 \to 2 \to 3$ or $1 \to 3$. For $n = 4$ we have 3 choices: $1 \to 2 \to 3 \to 4$, or $1 \to 3 \to 4$, or $1 \to 2 \to 4$. Let's denote by J_n the number of different paths, connecting 1 and n. We try to figure out what J_{n+1} is, assuming we know the value of J_k for $1 \le k \le n$. If we start with joining 1 and 2, we have J_n ways to connect 2 and n. If we start with joining 1 and 3, we have J_{n-1} different ways to connect 3 and n. These are all the possibilities, and so

$$J_{n+1} = J_n + J_{n-1}.$$

This equation is the same as the equation we have used to compute the Fibonacci numbers. By looking at the beginning values $J_2 = 1$ and $J_3 = 2$ we obtain $J_n = F_n$, which completes the proof.

II. Independence and Domination on Generalized Fibonacci Graphs

A set $S \subset V$ of vertices in a graph $G = (V, E)$ is called *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. We will be interesting in studying minimal dominating sets, where a dominating set S is a *minimal dominating set* if no proper subset $S' \subset S$ is a dominating set. The set of all minimal dominating sets of a graph G is denoted by $MDS(G)$. For instance, the sets {3}, {1,6}, and {2,5} are minimal dominating sets of $F_6(3)$ and

$$MDS(F_6(3)) = \{\{3\}, \{4\}, \{1,5\}, \{1,6\}, \{2,5\}, \{2,6\}\}.$$

The *domination number*$\gamma(G)$ of a graph G equals the minimum cardinality of a set in $MDS(G)$, or equivalently, the minimum cardinality of a dominating set in G.

The following theorem gives the domination number of generalized Fibonacci graph $F_n(k)$.

Theorem 2.1 *If* $k < n/2$ *then*

$$\gamma(F_n(k)) = \left\lceil \frac{n}{2k+1} \right\rceil.$$

Here, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Proof. Let

$$S' = \{(k+1), (k+1) + (2k+1), (k+1) + 2(2k+1), (k+1) + 3(2k+1), \ldots, (k+1) + \left\lfloor \frac{n-k-1}{2k+1} \right\rfloor (2k+1)\},$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Then clearly the set

$$S = \begin{cases} S' & , \quad if \quad (k+1) + \left\lfloor \frac{n-k-1}{2k+1} \right\rfloor (2k+1) + k \geq n \\ S' \cup \{n\} & , \quad otherwise \end{cases}$$

is a minimal dominating set having the minimum cardinality for the generalized Fibonacci graph $F_n(k)$. Since the number of elements of S is $\left\lceil \frac{n}{2k+1} \right\rceil$, which completes the proof.

Note that, if $k = 1$ then $F_n(k) \cong P_n$, where P_n denotes the path graph on n vertices. Then, it is easy to verify that $\gamma(F_n(1)) = \left\lceil \frac{n}{3} \right\rceil$for $n \geq 3$. Additionally, if $k \geq n-1$ then $F_n(k) \cong K_n$ and $\gamma(F_n(k)) = 1$. Here, $K_n$denotes the complete graph with n vertices. Moreover, if $k \geq n/2$ then $S = \{\lceil n/2 \rceil\}$ is a minimal dominating set for $F_n(k)$, and consequently $\gamma(F_n(k)) = 1$.

Table 1: $\gamma(F_n(k))$= $i(F_n(k))$, $\Gamma(F_n(k))$, and $\gamma_t(F_n(k))$ (from left to right respectively)

n \ k	2	3	4	5
2	1	1	1	1
3	1	1	1	1
4	1	1	1	1
5	1	1	1	1
6	2	1	1	1
7	2	1	1	1
8	2	2	1	1
9	2	2	1	1
10	2	2	2	1
11	3	2	2	1
12	3	2	2	2
13	3	2	2	2
14	3	2	2	2
15	3	3	2	2
16	4	3	2	2
17	4	3	2	2
18	4	3	2	2

n \ k	2	3	4	5
2	1	1	1	1
3	1	1	1	1
4	2	1	1	1
5	2	2	1	1
6	2	2	2	1
7	3	2	2	2
8	3	2	2	2
9	3	3	2	2
10	4	3	2	2
11	4	3	3	2
12	4	3	3	2
13	5	4	3	3
14	5	4	3	3
15	5	4	3	3
16	6	4	4	3
17	6	5	4	3
18	6	5	4	3

n \ k	2	3	4	5
2	2	2	2	2
3	2	2	2	2
4	2	2	2	2
5	2	2	2	2
6	2	2	2	2
7	2	2	2	2
8	3	2	2	2
9	3	2	2	2
10	4	2	2	2
11	4	3	2	2
12	4	3	2	2
13	4	3	2	2
14	4	4	3	2
15	5	4	3	2
16	5	4	3	2
17	6	4	3	3
18	6	4	4	3

The *upper domination number*$\Gamma(G)$ equals the maximum cardinality of a set in $MDS(G)$, or equivalently, the maximum cardinality of a minimal dominating set of G.

The following theorem gives the upper domination number of generalized Fibonacci graph $F_n(k)$.

Theorem 2.2

$$\Gamma(F_n(k)) = \left\lceil \frac{n}{k+1} \right\rceil$$

for all n, k positive integers.

Proof. Let

$$S = \left\{1,1 + (k + 1), 1 + 2(k + 1), 1 + 3(k + 1), \ldots ,1 + \left\lceil \frac{n-\mathrm{k}-1}{k+1} \right\rceil (k + 1)\right\}.$$

Then S is clearly minimal dominating set having maximum cardinality for $F_n(k)$. Hence $\Gamma(F_n(k)) = \left\lceil \frac{n}{k+1} \right\rceil$.

An *independent dominating set* of a graph G is a subset S of V such that every vertex not in S is adjacent to at least one vertex of S and no two vertices in S are adjacent. The minimum cardinality of an independent dominating set of G is the *independent domination number*$i(G)$. By the definition of independent dominating set and Theorem 2.1, we obtain the following corollary.

Corollary 2.3*If* $k < n/2$ *then*

$$i(F_n(k)) = \gamma(F_n(k)) = \left\lceil \frac{n}{2k+1} \right\rceil.$$

A set $S \subset V$ is a *total dominating set* of a graph $G = (V, E)$ if for every vertex $v \in V$, there exists a vertex $u \in S$, $u \neq v$, such that u is adjacent to v. If no proper subset of S is a total dominating set of G, then S is a *minimal total dominating set* of G. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. The *upper total domination number* of G, denoted by $\Gamma_t(G)$, is the maximum cardinality of a minimal total dominating set of G.

For instance, the set $S' = \{1,3,5,7\}$ is a total dominating set for $F_8(2)$, while the set $S = \{3,5,7\}$ is a minimal total dominating set. By the definition of generalized Fibonacci graph $F_n(k)$ if $k < \frac{n-1}{4}$ then $S = \{1 + \mathrm{k}, 2 + \mathrm{k}\}$ is a minimal total dominating set and we obtain $\gamma_t(F_n(k)) = 2$. The following theorem gives the total domination number of $F_n(k)$ when $k \geq \frac{n-1}{4}$.

Theorem 2.4 If $k \geq \frac{n-1}{4}$ then

$$\gamma_t(F_n(k)) = \begin{cases} 2\left\lceil\frac{n-2k-1}{2k}\right\rceil & , \quad if\ \frac{n-1}{2k} > \left\lceil\frac{n-2k-1}{2k}\right\rceil \\ 2\left\lceil\frac{n-2k-1}{2k}\right\rceil - 1 & , \quad otherwise. \end{cases}$$

Proof. Let $k \geq \frac{n-1}{4}$. Then the minimal total dominating set$S$ of $F_n(k)$ can be constructed by the following algorithm.

Algorithm:

Set $i = 0$ and $S = \emptyset$.

STEP 1: $i = i + 1$ and $S = S \cup \{1 + ik\}$.

STEP 2: if $\frac{n-1}{2k} > i$ then $S = S \cup \{n - ik\}$.

STEP 3: if $\frac{n-2k-1}{2k} > i$ then go to STEP 1, otherwise stop.

By the number of maximum iteration, the total domination number of $F_n(k)$ can be calculated. Since the number of maximum iteration $\left\lceil\frac{n-2k-1}{2k}\right\rceil$, we complete the proof.

Theorem 2.5

$$\Gamma_t(F_n(k)) = 2\left\lfloor\frac{n+2k-2}{2k}\right\rfloor.$$

Proof. The set $S' = \{1,2,1 + 2k, 2 + 2k, 1 + 4k, 2 + 4k, \ldots, 1 + 2\alpha k, 1 + 2\alpha k + \beta\}$ is a total minimal dominating set of $F_n(k)$ with maximum cardinality. Here,

$$\alpha = \left\lfloor\frac{n-2}{2k}\right\rfloor$$

and β is the smallest positive integer satisfying the following inequalities

$$\beta \geq 1 \quad and \beta \geq n - 1 - 2k\left\lfloor\frac{n+2k-2}{2k}\right\rfloor.$$

Hence, $\Gamma_t(F_n(k)) = 2\left\lfloor\frac{n+2k-2}{2k}\right\rfloor$ which completes the proof.

For instance, the set $S = \{4,7,8,11\}$ is a minimal total dominating set generated by the algorithm with minimum cardinality for the generalized Fibonacci graph $F_{14}(3)$ and $\gamma_t(F_{14}(3)) = 4$. Furthermore, the set $S' = \{1,2,7,8,13,14\}$ is a minimal total dominating set with maximum cardinality for generalized Fibonacci graph $F_{14}(3)$ and $\Gamma_t(F_{14}(3)) = 6$.

If a dominating set S induces a connected subgraph, then it is called a *connected dominating set*. The *connected domination number* of a graph G is the minimum cardinality of a connected dominating set, denoted by $\gamma_c(G)$.

Theorem 2.6

$$\gamma_c(F_n(k)) = \max\left\{1, \left\lfloor\frac{n-2}{k}\right\rfloor\right\}.$$

Proof. Obviously, if $n-k<2$ then $S=\{1\}$ is a connected dominating set with the minimum cardinality. If $n-k\geq 2$ then the set $S=\{1+k, 1+2k, 1+3k, \ldots, 1+\left\lfloor\frac{n-2}{k}\right\rfloor k\}$ is a connected dominating set with the minimum cardinality. Which completes the proof.

For example, the set $S=\{4,7,10\}$ is a connected dominating set with minimum cardinality for the generalized Fibonacci graph $F_{11}(3)$ and $\gamma_c(F_{11}(3))=3$.

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Handan Akyar. " Independence and Domination on Generalized Fibonacci Graphs." IOSR Journal of Mathematics (IOSR-JM) 14.5(2018): 05-09