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A Critical Growth Nonlinear Bi-harmonic Problem in R^4

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Abstract This article concerns with the problem

$$-\Delta^2 u = \mu \frac{u}{|x|^4 In^2 \frac{R}{|x|}} + f(x,u), \quad x \in \Omega;$$
$$u = 0 \qquad\qquad x \in \partial\Omega$$

There f has critical growth at both $+\infty$ and $-\infty$ with the same α_0, through a Hardy Inequality of [4], We prove the existence of a nontrivial solution of above problem by using Mountain Pass Theorem.

Keywords bi-harmonic equation; critical growth; Mountain Pass Theorem

0 Introduction

When $N > p$, the article [1] had discussed the nonlinear harmonic equation involving critical potential. But as $N = p = 2$, the corresponding question hasn't been studied. Then in 1995, in the article [2], D.G.de Figueiredo, Miyagaki and Ruf proved the existence of multiply solutions of nonlinear elliptic problem in $\mathbb{R}^2$, where f has subcritical growth and critical growth. After this article mainly, in 2004, Shen, Yao and Chen [3] have studied the existence of nontrivial solutions for quasi-linear elliptic equation involving critical potential:

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2 \ln^2 \frac{R}{|x|}} = \lambda u, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$, $0 \in \Omega \subset B_R(0)$,$B_R(0)$ is a small ball centering origin with radius R in $\mathbb{R}^2$, and in this article, f has subcritical. Then in 2005, Chen, Shen and Yao[4] have studied the existence of nontrivial solutions for nonlinear biharmonic equation involving critical potential:

$$\begin{cases} \Delta^2 u - \mu \frac{u^2}{|x|^4 \ln^2 \frac{R}{|x|}} + f(x,u), & x \in \Omega \\ u = \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega \end{cases} \tag{0.1}$$

where $\Omega \subset B_R(0) \subset \mathbb{R}^4$ is a bounded domain including the origin, $\mu \in \mathbb{R}$, ν is the unit outer normal vector, and f has subcritical growth(see[2]). According the article [2], we think what will happen if f has critical growth in the problem (0.1). So in this paper, we have discussed the existence of nontrivial solutions for nonlinear bi-harmonic equation involving critical potential (0.1), but in here f has critical growth at $+\infty$ (see[2]), it means if there exists $\alpha_0 > 0$, such that for all $\alpha > \alpha_0$

$$\lim_{t\to+\infty} \frac{|f(x,t)|}{e^{\alpha t^{4/3}}} = 0 \tag{0.2}$$

and for all $\alpha < \alpha_0$

$$\lim_{t\to+\infty} \frac{|f(x,t)|}{e^{\alpha t^2}} = +\infty$$

For easy reference we state new conditions on f that will be assumed bellow:

$$(H_1)\quad f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}\ is\ continuous, f(x,0) = 0$$

$$(H_2)\quad \exists t_0 > 0, \exists M > 0, such\ that$$

$$0 < F(x,t) = \int_0^t f(x,s)ds \le M|f(x,t)|$$

$$(H_3)\quad 0 < F(x,t) \le \frac{1}{2} f(x,t)t, \forall t \in \mathbb{R} - \{0\}, \forall x \in \Omega$$

$$(H_4)\quad \limsup_{t\to 0} \frac{2F(x,t)}{t^2} < \lambda_1, uniformly\ in\ (x,t)$$

Now we state the results which will be proved here. By "solution" in the theorems below we mean weak solution $u \in H_0^2(\Omega)$.

Theorem 0.1 Assume (H_1),(H_2),(H_3),(H^1),$\mu < 1$ and f has critical growth at both $+\infty$ and $-\infty$. Furthermore assume

$$(H_5)\qquad \lim f(x,t)te^{-\alpha_0 t^2} \ge \beta, \quad \beta > \frac{16(4-\mu)}{\alpha_0(1+M)R^4}$$

Then, problem (0.1) has a nontrivial solution.

In this paper, we define $||u||^2 = \int_\Omega |\Delta u|^2, |u|_p = (\int |u|^p)^{1/p}$.

1 The proof of lemmas

We know the functional of equation (0.1) is

$$\Phi(u) = \frac{1}{2}\int_\Omega |\triangle u|^2 dx - \frac{\mu}{2}\int_\Omega \frac{u^2}{|x|^4 \ln^2 R/|x|}dx - \int_\Omega F(x,u)dx$$

We assume (H_1), $(H_2)$and the existence of positive constance$C$

And $\alpha_0 > 0$,when $\alpha > \alpha_0$, $|f(x,t)| \le Ce^{\alpha t}$ $\quad \forall x \in \Omega, t \in R$ (1.1)

It follows easily from (H_1) and (H_2) that

(1) there is a constant $C > 0$, such that

$$F(x,t) \ge Ce^{\frac{1}{M}|t|}, \quad \forall |t| \ge t_0 \tag{1.2}$$

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(2)given >0, there is t_ϵ >0, such that

$$F(x,t) \le \epsilon f(x,t)t, \quad \forall x \in \Omega, \forall |t| \ge t_\epsilon \tag{1.3}$$

Lemma 1.1(see [4]) Assume $u \in H_0^2(\Omega)$,then

$$\int_\Omega \frac{u^2}{|x|^4 \ln^2 R/|x|} dx \le \int_\Omega |\triangle u|^2 dx \tag{1.4}$$

where the constant 1 is optimal.
Lemma 1.2(see [2]) $f(x,u_n) \to f(x,u)$ in $L^1(\Omega)$. where $\{u_n\}$ is a (PS) sequence.
Set
Lemma 1.3 Assume (H_1),(H_2) and (H_3) , if f has critical growth at both $+\infty$ and

$-\infty$ with the same α_0, then Φ satisfies $(PS)_c$for all $c \in (-\infty, 8(1-\mu)\sqrt{\pi/3\alpha_0})$.

Proof: Let $\{u_n\} \subset H_0^1(\Omega)$ be a Palais-Smale sequence, i.e.

$$\frac{1}{2}\int_\Omega |\triangle u_n|^2 dx - \frac{\mu}{2}\int_\Omega \frac{u_n^2}{|x|^4 \ln^2 R/|x|} dx - \int_\Omega F(x,u_n)dx \to c \tag{1.5}$$

$$\int_\Omega \triangle u_n \triangle v dx - \mu \int_\Omega \frac{u_n v}{|x|^4 \ln^2 R/|x|} dx - \int_\Omega f(x,u_n)v dx = o(1)||v|| \tag{1.6}$$

For $\forall v \in H_0^2(\Omega)$.

From (1.3) and (1.5), for any $\epsilon > 0$, we have

$$\begin{aligned} \frac{1}{2}||u_n||^2 - \frac{\mu}{2}\int_\Omega \frac{|u_n|^2}{|x|^4 \ln^2 R/|x|} &\le C + \int_\Omega F(x,u_n)dx \\ &\le C_\epsilon + \epsilon \int_\Omega f(x,u_n)u_n dx \end{aligned} \tag{1.7}$$

We assume $v = u_n$ in (1.6), can obtain

$$||u_n||^2 - \mu \int_\Omega \frac{|u_n|^2}{|x|^4 \ln R/|x|} dx = \int_\Omega f(x,u_n)u_n dx + o(1)||u_n|| \tag{1.8}$$

Substitute (1.8) to (1.7), we have

$$\frac{1}{2}||u_n||^2 - \frac{\mu}{2}\int_\Omega \frac{|u_n|^2}{|x|^4 \ln^2 R/|x|} \le C_\epsilon + \epsilon(||u_n||^2 - \mu\int_\Omega \frac{|u_n|^2}{|x|^4 \ln R/|x|} dx) + \epsilon o(1)||u_n||$$

Set $\epsilon = \frac{1}{4}$, from Lemma 1.1, we know there is a constant C, such that

$$||u_n||^2 \le C$$

Now we take a subsequence of $\{u_n\}$ denoted again by $\{u_n\}$, such that, for some $u \in H_0^2$, we have

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$$u_n \rightharpoonup u \ in \ H_0^2; u_n \to u \ in \ L^q(\Omega), \forall q \geq 1; u_n(x) \to u(x) \ a.e. \ in \ \Omega$$

From Lemma 1.2, when $n \to \infty$, (1.6) become

$$\int_\Omega \triangle u \triangle v dx - \mu \int_\Omega \frac{uv}{|x|^4 \ln^2 R/|x|} dx - \int_\Omega f(x,u) v dx = 0 \quad (1.9)$$

Let $v = u$ in the (1.9), and using (1.3) then

$$2\Phi(u) = \int_\Omega |\triangle u|^2 dx - \mu \int_\Omega \frac{u^2}{|x|^4 \ln^2 R/|x|} dx - 2\int_\Omega F(x,u) dx \geq (\frac{1}{\epsilon} - 2) \int_\Omega F(x,u) dx$$

So, $\Phi(u) \geq 0$. Now we separate the proof into three cases:

Case 1 $c = 0$.

From Lemma 1.2, using Lebesgue dominated convergence theorem, we can get $F(x,u_n) \to F(x,u)$ *in* $L^1(\Omega)$. So, from (1.5) and (1.6), set $v = u_n$, we obtain

$$\lim_{n\to\infty} (\frac{1}{2} \int_\Omega f(x,u_n) u_n dx - \int_\Omega F(x,u_n) dx) = c = 0$$

so

$$\frac{1}{2} \int_\Omega f(x,u_n) u_n dx = \int_\Omega F(x,u_n) dx$$

then, from (1.8), we have

$$0 \leq \Phi(u) \leq \liminf \Phi(u_n) = \frac{1}{2} \int_\Omega f(x,u_n) u_n dx - \int_\Omega F(x,u_n) dx = 0$$

So, $||u_n|| \to ||u||$ and then $u_n \to u \ in \ H_0^2$. The proof is finished in this case.

Case 2 c=0, u=0. In this case, we will show that it cannot happen for a (Ps) sequence. First we claim that, for some q >1, we have

$$\int_\Omega |f(x,u_n)|^q dx \leq const \quad (1.10)$$

From (1.1) ,set a fixed q >1, then

$$\int_\Omega |f(x,u_n)|^q \leq C \int_\Omega e^{\alpha q |u_n|^{4/3}} dx = C \int_\Omega e^{\alpha q ||u_n||^{4/3} (\frac{u_n}{||u_n||})^{4/3}}$$

Using Moser-Trudinger Inequality (N=4)(see[5]): for any $u \in W_0^{1,4}(\Omega)$

$$\sup_{||u||_{W_0^{1,4}} \leq 1} \int_\Omega e^{\gamma |u|^{4/3}} dx \leq C|\Omega|, \quad \forall \gamma \leq 4\omega_3^{1/3}$$

where $||u||_{W_0^{1,4}} = |u|_4 + |Du|_4, \ \omega_3 = 4\pi/3$ is the volume of unit ball when $n = 3$, $|\Omega|$ is the lebesgue measure of Ω.

Then we can deduce the integral (1.10) is bounded independently of n, if

$$\alpha q ||u_n||^{4/3} \leq \gamma \leq 4\omega_3^{1/3}$$

From (1.5) , Lemma 1.1 ,(H_1) and u = 0, we have

$$\lim_{n\to\infty} ||u_n||^2 = \frac{2(c+\epsilon)}{1-\mu}$$

Then it will be indeed the case for $c < 8(1-\mu)\sqrt{\frac{\pi}{3\alpha_0^3}}$, if we choose $q > 1$ sufficiently close to 1 , α sufficiently close to α_0 and sufficiently small.

Let (1.6) subtract (1.9), and assume $v = u_n - u$, then we have

$$\int_\Omega |\triangle(u_n-u)|^2dx - \mu\int_\Omega \frac{|u_n-u|^2}{|x|^4\ln^2 R/|x|}dx - \int_\Omega (f(x,u_n)-f(x,u))(u_n-u)dx = o(1)||u_n-u||$$

We estimate the third integral above using Holder inequality and $|u_n - u|_{L^{q0}} \to 0$, then we have

$$\int_\Omega |(f(x,u_n)-f(x,u))(u_n-u)|dx \to 0$$

So, through Lemma 1.1, we know $||u_n|| \to 0$. But, from (1.5), which implies $\|u_n\|^2 \to \frac{2c}{1-4\mu} \neq 0$. It is contradiction.

Case 3 $\quad c \neq 0, u \neq 0$.

Like case 2, we can proof (1.10). Because $||u_n||^2 \le C$, so it means when $\alpha q C^{2/3} \le 4(\frac{4\pi}{3})^{1/3}$, (1.10) is true. At the same time, we can know $u_n \to u \ in \ H_0^2$. Then the lemma is proved.

In the case 3 of above, we actually can obtain

$\Phi(u) = c \ and \ c < 8(1-\mu)\sqrt{\frac{\pi}{3\alpha_0^3}}$.

Lemma 1.4 Assume (H_1),(H_2),(H_4) and (1.1), then existence $a > 0$,$\rho > 0$, such that $\Phi(u) \ge a$, if $||u|| = \rho$.

Proof: From (H_4), we know there are $\lambda_0 < \lambda_1$,$\delta > 0$, such that

$$F(x,t) \le \frac{1}{2}\lambda_0 t^2, \quad |t| \le \delta$$

In other way, from (1.1), to $q > 2$

$$F(x,t) \le Ce^{\alpha|t|^{4/3}}|t|^q \ |t| > \delta$$

Putting these two estimates together we obtain

$$F(x,t) \le \frac{1}{2}\lambda_0 t^2 + Ce^{\alpha|t|^{4/3}}|t|^q \qquad \forall t \in \mathbb{R} \tag{1.11}$$

From (1.11), and using Holder inequality, for $p > 1$, we have

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_\Omega |\triangle u|^2 - \frac{\mu}{2}\int_\Omega \frac{u^2}{|x|^4\ln^2 R/|x|} - \int_\Omega F(x,u)\\ &\ge \frac{1}{2}(1-\mu)||u||^2 - \frac{1}{2}\lambda_0\int_\Omega u^2 - C\int_\Omega e^{\alpha|u|^{4/3}}|u|^q\\ &\ge \frac{1}{2}(1-\frac{\lambda_0}{\lambda_1})(1-\mu)||u||^2 - C(\int_\Omega e\alpha|u|^{4/3}|u|^q)^{1/p}(\int_\Omega |u|^{qp'})^{1/p'}\\ &\ge \frac{1}{2}(1-\frac{\lambda_0}{\lambda_1})(1-\mu)||u||^2 - C||u||^q\end{aligned}$$

Now choose $\rho > 0$, as the point where the function $g(s) = \frac{1}{2}(1 - \frac{\lambda_0}{\lambda_1})(1-\mu)s^2 - Cs^q$ assumes its maximum. Take $a = g(\rho)$. Then the proof is complete.

Remarks on the conditions above, we easily to prove there is $e \in H_0^2, ||e|| > \rho$, such that $\Phi(e) \leq 0$.

2 The proof of Theorem 0.1

It follows from the assumptions that Φ satisfies $(PS)_c$ for all $c < 8(1-\mu)\sqrt{\frac{\pi}{3\alpha_0^3}}$, see lemma 1.3. At the same time, through lemma 1.4 and (H_4), we can know that Φ has a local minimum at 0. To conclude via the Mountain Pass Theorem it suffices to show that there is a $\omega \in H_0^2, ||\omega||= 1$, such that $max\{\Phi(t\omega) : t \geq 0\} < c$. For that matter we start by introducing the following functions

$$\omega_n(x) = \frac{1}{2\sqrt{2}\pi}\begin{cases}(\ln n)^{1/2}, & 0 \leq |x| \leq \frac{R}{n}\\ \frac{\ln \frac{R}{|x|}}{(\ln n)^{1/2}}, & \frac{R}{n} \leq |x| \leq R\\ 0, & |x| \geq R\end{cases}$$

which indicate that $\omega_n(x) \in H_0^2(B_R(0))$ and $||\omega_n|| = 1$ for all $n = 1, 2, \cdots$.

We claim that there exists n such that

$$max\{\Phi(t\omega_n) : t \geq 0\} < 8(1-\mu)\sqrt{\frac{\pi}{3\alpha_0^3}}$$

Assume by contradiction that this is not the case. So, for all n, this maximum is large or equal to $8(1-\mu)\sqrt{\frac{\pi}{3\alpha_0^3}}$. Set $t_n > 0$, such that

$$max\{\Phi(t\omega_n) : t \geq 0\} = \Phi(t_n\omega_n) \geq 8(1-\mu)\sqrt{\frac{\pi}{3\alpha_0^3}} \tag{2.1}$$

it is to say, from (2.1) and (H_3)

$$\begin{aligned}8(1-\mu)\sqrt{\frac{\pi}{3\alpha_0^3}} &\leq \frac{1}{2}\int_{B_R(0)}|\triangle t_n\omega_n|^2 - \frac{\mu}{2}\int_{B_R(0)}\frac{|t_n\omega_n|^2}{|x|^4\ln^2 R/|x|} - \int_{B_R(0)}F(x, t_n\omega_n)\\ &\leq \frac{1}{2}t_n^2 - \frac{\mu}{2}t_n^2(2\pi^2)\int_{R/n}^{R}\frac{r^3\ln^2 R/r}{8\pi^2 r^4 \ln n \ln^2 R/r}\\ &\leq \frac{1}{2}t_n^2 - \frac{\mu}{8}t_n^2\\ &= \frac{1}{2}(1 - \frac{1}{4}\mu)t_n^2\end{aligned}$$

so it means

$$t_n^2 \geq \frac{16(1-\mu)}{1 - 1/4\mu}\sqrt{\frac{\pi}{3\alpha_0^3}} \tag{2.2}$$

At the same time, we know $\frac{d\Phi(t_n\omega_n)}{dt_n} = 0$, then

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$$t_n^2 - \mu t_n^2 \int_{B_R(0)} \frac{u_n^2}{|x|^4 \ln^2 R/|x|} = \int_{B_R(0)} f(x, t_n\omega_n)t_n\omega_n$$

$$\int_{B_R(0)} f(x, t_n\omega_n)t_n\omega_n \leq (1 - \frac{1}{4}\mu)t_n^2 \tag{2.3}$$

From (H_5), for given $\epsilon > 0$, there existence $s > s_\epsilon$, such that

$$f(x, s)s \geq (\beta - \epsilon)e^{\alpha_0 s^2}, \quad \forall s > s_\epsilon$$

so

$$\begin{aligned}(1 - \frac{1}{4}\mu)t_n^2 &\geq (\beta - \epsilon)\int_{B_R(0)} e^{\alpha_0 t_n^2 \omega_n^2} \\ &\geq (\beta - \epsilon)(2\pi^2)\int_0^{R/n} r^3 e^{\alpha_0 t_n^2 \frac{\ln n}{8\pi^2}} dr \\ &\geq (\beta - \epsilon)\pi^2 \frac{R^4}{2n^4} e^{\alpha_0 t_n^2 \frac{\ln n}{8\pi^2}} \\ &= \frac{1}{2}(\beta - \epsilon)\pi^2 R^4 e^{\ln n(\frac{\alpha_0 t_n^2}{8\pi^2} - 4)}\end{aligned} \tag{2.4}$$

which implies readily that t_n is bounded. And moreover (2.2) together with (2.4), we can deduce that $t_n^2 \to \frac{32\pi^2}{\alpha_0}$.

Then let us estimate (2.3) more precisely.

$$(1 - \frac{1}{4}\mu)t_n^2 \geq (\beta - \epsilon)\int_{B_R(0)} e^{\alpha_0 t_n^2 \omega_n^2} dx$$

Passing to the limit in above and assume $t = \frac{\ln R/r}{\ln n}$, then we can obtain

$$\begin{aligned}\frac{8(4 - \mu)\pi^2}{\alpha_0} &\geq 2\pi^2(\beta - \epsilon)[\int_0^{R/n} e^{32\pi^2 \frac{\ln n}{8\pi^2}} r^3 dr + \int_{R/n}^R e^{32\pi^2 \frac{\ln^2 R/r}{8\pi^2 \ln n}} r^3 dr] \\ &= 2\pi^2(\beta - \epsilon)[\frac{1}{4}R^4 + R^4 \ln n \int_0^1 e^{4t^2 \ln n - 4t \ln n} dt] \\ &= \frac{1}{2}\pi^2(\beta - \epsilon)R^4[1 + 4\ln n \int_0^1 e^{4\ln n(t^2 - t)} dt]\end{aligned} \tag{2.5}$$

which implies $\beta \leq \frac{16(4-\mu)}{\alpha_0(1+M)R^4}$, if we let $M = 4\ln n \int_0^1 e^{4\ln n(t^2-t)} dt[M$ see[2]], then it is contradiction to (H_5).

So, the theorem is proved.

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