IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 14, Issue 4 Ver. IV (Jul - Aug 2018), PP 12-16
www.iosrjournals.org

Structural Point Vortices on Toroidal Surfaces

K. Bitrus[1], U. Sani[2] S. Adamu[3], and C.P. Ezenweke[4]
[1,2,4]*DepartmentOf Mathematical Sciences Federal University Lokoja Kogi State*
[3]*Basic Sciences Department, Federal College OfFreshwater Fisheries Technology, New Bussa Niger State*
Corresponding Author:K. Bitrus

Abstract: *Stream function for the Laplace-Beltrami equation on the surface of a three dimensional ring torus is obtained. The equation is decomposed into two different parts* $\psi_*(\theta)$ *and* $\psi_{**}(\phi)$*, andsolved independently for the explicit representation of the stream function* $\psi(\theta,\phi)$*, on the torus surface. The approach is analytic and the result is first of its kind. The contour plots of the solution was obtained by considering different values of* θ *and* ϕ *as the angles of rotation on* $\tau_{r,R}$ *in the interval* $(0,2\pi]$ *which illustrates the structure of interesting point vortices on the surface considered.*

Keywords: *Ring torus, Vortices, Stream function, Vorticity,*

Date of Submission: 05-08-2018 Date of acceptance: 22-08-2018

I. Introduction

Fluid phenomena are quite intriguing because of its attractive and unsteady composition of fluid patterns produced by interacting vortex structures. Many researchers have attempted to describe the vortex interaction theoretically using mathematical models in order to understand these fluid phenomena. One of these models is the “point vortex model” this is a model of flow in which the vorticity is zero everywhere on the surface except at the point where the vorticity is infinite so that there is nonzero circulation around the point. Vortex is a region in a fluid in which the flow revolves around an axes line, which may be straight or curved [9]. The distributions of velocity, vorticity as well as the concept of circulation are used to characterize vortices. In most vortices, the fluid flow velocity is greatest next to its axis and decreases in inverse proportion to the distance from the axis. This phenomenon is the subject of investigation in this paper.

Geometrically, a torus is a surface of revolution generated by revolving a circle in three-dimensional space about an axis coplanar with circle. A torus is a surface of genus one and it therefore possesses a single hole. The usual 3-D ring torus is known in the old literature as ‘Anchor ring’[8].

New exact solutions are derived for gravitational potential inside and outside a homogeneous torus as rapidly converging series of toroidal harmonics were done in [6].Green’s functions of the Laplacian and biharmonic operators are derived for a three-dimensional toroidal domain in [2]. The evolution equation for N-point vortices from Green’s function associated with the Laplace-Beltrami operator on toroidal surface was derived in [7], it was formulated as a Hamiltonian dynamical system with the help of the symplectic geometry and the uniformizationtheorem. They also investigate the point vortex Equilibria and the motion of two point vortices with the strength of the same magnitude as one of the fundamental vortex interactions, hence found some characteristic interactions between point vortices on the surface of the torus. Inparticular two identical point vortices can be locally repulsive under a certain circumstance. The Green’s function for the Laplace-Beltrami operator on the surface of a three- dimensional ring torus was constructed in [3], in which they use stereographic projection of the torus surface onto the planar annulus and represented the Green’s function in terms of the Schottky-Klein prime function associated with the annulus and the dilogarithm function. They also consider its application to vortex dynamics on the surface of a ring torus. Point vortices on a spherical surface: the Green’s function approach was investigated in [4]. The fundamental Green’s function for sphere was derived and restructured by considering a sour and observation point on the surface. The geometric and topological aspects of the dynamics and energetics of vortex torus knots and un-knots were examined in [5]. The knots were given by small amplitude torus knots solutions in the Local Induction Approximation (LIA) and they studied vortex evolution in the context of the Euler equations by direct numerical integration of the Bio-Savart law and the velocity, helicity and kinetic energy of different vortex knots and un-knots were presented for comparison. Alternative procedure for solving the Laplace’s equation in toroidal coordinates was studied in [1], where the boundary conditions are independent of spherical coordinate θ (rather than the toroidal coordinate η or the azimuthal coordinate ψ) and they consider its application to electrostatics.

The aim of this paper is to obtain a stream function by solving analyticallythe Laplace-Beltrami equation on the torus surface and to present the stream function obtained in an explicit form.

II. Formulation Of The Problem

The surface of the ring torus was denoted by the notation $\tau_{r,R}$, where R is the major radius and r is the minor radius of the Torus with $R > r$ and. $\theta, \phi \in (0, 2\pi]$. The surface $\tau_{r,R}$ is formed by taking a circle of radius r centered a distance $R - r$ from the origin in the $(x, z) - plane$, and rotating it through 2π aboutthe $z - axis$.

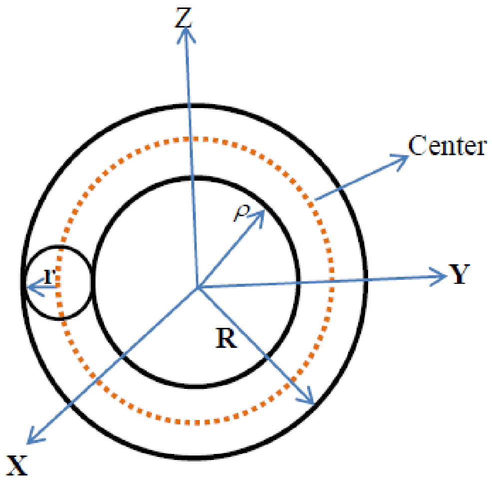

Figure: 1 Geometrical representation of the Ring torus, showing the range of r, R and ρ

Each point on the Torus surface is described by

$(x, y, z) \in R^3$ Where $x = (R - rCos\theta)Cos\phi,\ y = (R - rCos\phi)Sin\phi$ and $z = rSin\theta$

The Laplace-Beltrami operator on the ring torus surface is as follows

$$\nabla^2_{\tau_{r,R}} = \frac{1}{r^2(R - Cos\theta)}\frac{\partial}{\partial\theta}\left[(R - rCos\theta)\frac{\partial}{\partial\theta}\right] + \frac{1}{(R - rCos\theta)^2}\frac{\partial^2}{\partial\phi^2} \quad (1)$$

We seek for a solution, $\psi(\theta, \phi)$ in which the variables θ and ϕ are partially separated for the equation (2) below

$$\nabla^2_{\tau_{r,R}}\psi = 0. \quad (2)$$

We are to solve (2) for the explicit representation of the function ψ .We refer our interested reader to [3] for the concept of the vorticity field and stream function used below

The velocity field of the fluid $\vec{u} = (0, u_\theta, u_\phi)$ where u_θ and u_ϕ are defined as

$$u_\theta = \frac{1}{R - rCos\theta}\frac{\partial\psi}{\phi}, \text{ and } u_\phi = -\frac{1}{r}\frac{\partial\psi}{\theta} \quad (3)$$

is tangential to the surface of the torus, that is in term of the coordinates (r, θ, ϕ), we have for some u_θ, u_ϕ due to the incompressibility condition [3].

$$\nabla_{\tau_{r,R}} \cdot \vec{u} = 0 \quad (4)$$

Introducing the stream function $\psi(\theta, \phi)$ such that

$$\vec{u} = \nabla_{\tau_{r,R}}\psi(\theta, \phi) \times (1, 0, 0) = \left(0, \frac{1}{R - Cos\theta}\frac{\partial\psi}{\partial\phi}, -\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) \quad (5)$$

and introducing the scalar vorticity field $\omega(\theta, \phi)$ defined as

$$\omega(\theta,\phi)=\left(\nabla_{\tau_{r,R}}\times\vec{u}\right)\cdot(1,0,0) \tag{6}$$

On taking the curl of (6), we have

$$\nabla^2_{\tau_{r,R}}\psi(\theta,\phi)=-\omega(\theta,\phi) \tag{7}$$

Because torus surface is a closed compact surface, it follows from Gauss's Divergence theorem that we have an intrinsic topological constrain to enforce on $\vec{u}$; that is

$$\iint_{\tau}\omega(\theta,\phi)\,dA=0 \tag{8}$$

Consider now a point vortex on $\tau_{r,R}$. This provides a δ -function distribution of vorticity. As a consequence of (8), a single point vortex cannot exist on $\tau_{r,R}$, unless an additional source of vorticity is present. One way to resolve this is as in [3] thus the function $\psi(\theta,\phi)$ we are looking for is the stream function of $\nabla^2_{\tau_{r,R}}$.

II. Analytical Solutions

Now set

$$\psi(\theta,\phi)=\psi_*(\theta)+\psi_{**}(\phi) \tag{9}$$

Where $\psi_*(\theta)$ is a function that is independent of ϕ and $\psi_{**}(\phi)$ is a function of θ and ϕ

From (2) we can write

$$\nabla^2_{\tau_{r,R}}\psi_*=\frac{1}{r^2(R-rCos\theta)}\frac{\partial}{\partial\theta}\left[(R-rCos\theta)\frac{\partial\psi_*}{\partial\theta}\right]=\Theta \tag{10}$$

$$\nabla^2_{\tau_{r,R}}\psi_{**}=\frac{1}{(R-rCos\theta)^2}\frac{\partial^2\psi_{**}}{\partial\phi^2}=-\Theta \tag{11}$$

After some algebra it was found that

$$\psi_*(\theta)=\frac{\Theta\left(\frac{R-\rho}{2}\right)^2}{R\theta-\left(\frac{R-\rho}{2}\right)Sin\theta}+\frac{\Theta\left(\frac{R-\rho}{2}\right)^3Cos\theta}{R\theta-\left(\frac{R-\rho}{2}\right)Sin\theta}+\frac{\lambda_1\theta}{R\theta-\left(\frac{R-\rho}{2}\right)Sin\theta}+\lambda_2 \tag{12}$$

At $\theta=\theta_0$, $\psi_*=0$ yield $\lambda_2=0$

Since the torus surface is periodic with period 2π equating (12) to the period and taking $\lambda_2=0$ we have

$$\lambda_1=2\pi R-\left(\frac{R-\rho}{2}\right)Sin(2\pi)-\frac{\Theta}{4}(R-\rho)^2\pi R-\frac{\Theta}{16\pi}(R-\rho)^3Cos(2\pi) \tag{13}$$

After substituting λ_1 into (12) and rearranging we have

$$\psi_*(\theta)=\frac{\beta^2}{8\alpha}\left(\theta^2R+\beta Cos\theta\right)+\frac{\theta}{\alpha}\left(2\pi R-\frac{\beta}{2}Sin(2\pi)-\frac{\pi R\beta^2}{4}-\frac{\beta^3Cos(2\pi)}{16\pi}\right) \tag{14}$$

Similarly

$$\psi_{**}(\phi)=-\Theta\left(R^2-2RrCos\theta+r^2Cos^2\theta\right)\frac{\phi^2}{2}+\kappa_1\phi+\kappa_2 \tag{15}$$

At $\phi=\phi_o$, $\psi_{**}=0$ yield $\kappa_2=0$

$$\kappa_1=\Theta\left(R^2-2RrCos\theta+r^2Cos^2\theta\right)\pi+1 \tag{16}$$

Again by substituting κ_1 into (15) we have

$$\psi_{**}(\phi)=\left(\pi-\frac{\phi}{2}\right)R\beta\phi Cos\theta+\left(\pi+\frac{\phi}{8}\right)\beta^2\phi Cos^2\theta+\left(1+\phi R^2\right)\phi \tag{17}$$

Thus by substituting (14) and (17) in to (9) we have the stream function asfollows

$$\psi(\theta,\phi)=\frac{\beta^2}{8\alpha}\left(R\theta^2+\beta Cos\theta\right)+\frac{\theta}{\alpha}\left(2\pi R-\frac{\beta}{2}Sin(2\pi)-\frac{\beta^2\pi R}{4}-\frac{\beta^3 Cos(2\pi)}{16\pi}\right)$$
$$+\left(\pi-\frac{\phi}{2}\right)R\beta\phi Cos\theta+\left(\pi+\frac{\phi}{8}\right)\beta^2\phi Cos^2\theta+\left(1+\phi R^2\right)\phi \tag{18}$$

where

$\alpha=R\theta-\left(\frac{R-\rho}{2}\right)Sin\theta$ $\beta=R-\rho$ $r=\frac{\beta}{2}$ and $\Theta\in N$

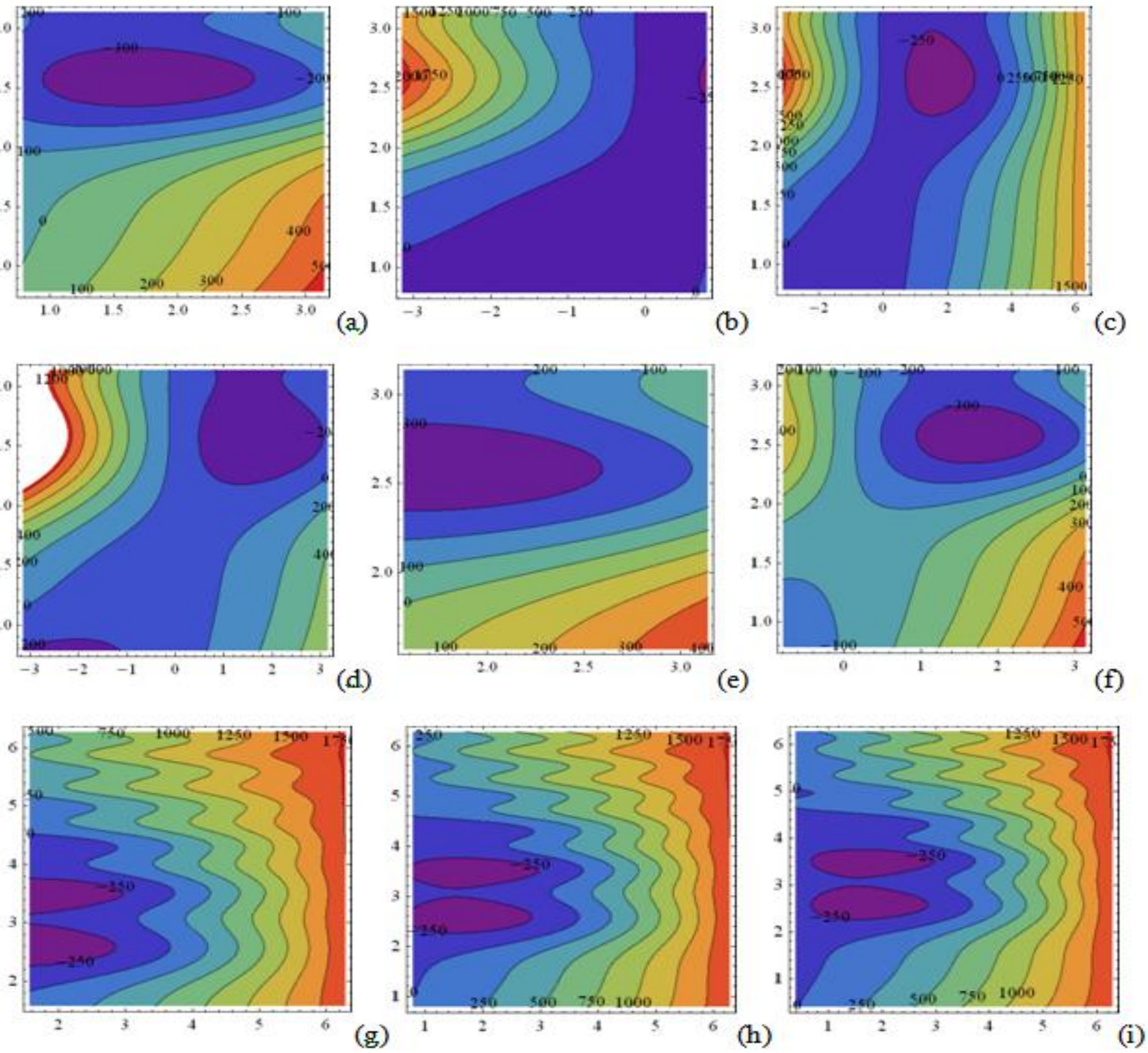

Figure:2

The contour plots of the stream function obtained in (18), illustrating point vortex structure on toroidal surface.
$\theta,\phi\in(0,2\pi]$ $R=10$ $\rho=0.19744$ $r=4.90128$, $\tau_{r,R}\approx\tau_{5,10}$

III. Analysis

The torus surface is a closed compact surface, thus the integral of the scalar vorticity field over the torus surface must be zero according to the Gauss' theorem [3]. This is a global constraint on the vorticity distribution. In order to satisfy this constraint each point vortex on the torus must be counterbalanced by another point vortex on the torus. By observation, we see that for this vortex structure the total vorticity on the torus surface is the sum of the circulations of the point vortex and for each point vortex, there is a point vortex of opposite strength, so that the circulation sum to zero. Below are the display of the angles considered for each contour plots respectively

(a) $\left[\phi,\pi,\frac{\pi}{4}\right]$ and $\left[\theta,\pi,\frac{\pi}{4}\right]$, (b) $\left[\phi,-\pi,\frac{\pi}{4}\right]$ and $\left[\theta,\pi,\frac{\pi}{4}\right]$, (c) $[\phi,-\pi,2\pi]$ and $\left[\theta,\pi,\frac{\pi}{4}\right]$

(d) $[\phi,-\pi,2\pi]$ and $\left[\theta,\pi,\frac{\pi}{4}\right]$, (e) $\left[\phi,\pi,\frac{\pi}{2}\right]$ and $\left[\theta,\pi,\frac{\pi}{2}\right]$, (f) $\left[\phi,-\frac{\pi}{4},\pi\right]$ and $\left[\theta,\pi,\frac{\pi}{4}\right]$

(g) $\left[\phi,\frac{\pi}{2},2\pi\right]$ and $\left[\theta,\frac{\pi}{2},2\pi\right]$,(h) $\left[\phi,\frac{\pi}{4},2\pi\right]$ and $\left[\theta,\frac{\pi}{4},2\pi\right]$,(i) $\left[\phi,\frac{\pi}{8},2\pi\right]$ and $\left[\theta,\frac{\pi}{8},2\pi\right]$

The choice of the angle of rotation in (a)-(f) was deliberately made random so that the structure of the point vortex on the torus surface will be clearly seen distributed all over the surface, but in (g)-(i) a particular trend is chosen so that the two angles decrease at the same rate and it is clearly seen that there exits two point vortices with strong cohesion on the torus surface. It was also found that as the angles decrease, the force of attraction between the two point vortices decreases and finally disappears completely at a particular point slightly after $\left[\phi,\frac{\pi}{32i+1},2\pi\right]$ and $\left[\theta,\frac{\pi}{32i+1},2\pi\right]$ $i=2,3,4,\ldots$

IV. Conclusion

The result obtained in this paper can be applied in solving other problems on the surface of a ring torus provided the guiding equation is the Laplace-Beltrami equation on the torus. The solution has also shed light on the dynamics of point vortices on toroidal surface and flows of superfluid film vortices on toroidal and more generally holey surfaces. The solution obtained is the stream function as shown by the contour plots; therefore the method can be applied to other more complicated surfaces in order to obtain the stream function related to vortex flow on such surfaces.

References

[1]. Andrew, M. (2005). Alternative Separation of Laplace's equation in toroidal coordinates and its application to electrostatics. Journal of Electrostatics, 664-672.

[2]. Bates, J. W. (1997). On toroidal Green's function. AIP, Journal of Mathematical Physics.

[3]. Green, C. C. & Marshall, J. S. (2012). Green's function for the Laplace-Beltrami operator on a toroidal surface.Proceeding of Royal Society. Mathematical, Physical and Engineering Sciences, 1-13

[4]. Kefas, B. M. & Ifidon, E.O, (2016). Point Vortices on Spherical Surface: The Green's function approach. Transaction of the Nigeria Association of Mathematical Physics Volume 2 pp49-58

[5]. Maggioni, F., Alami, S. Z. Barenghi, C. F. & Ricca, R. L. (2013). Vortex Knots dynamics in Eurler fluids.Procedia: elsevier.

[6]. Majc*, M. (2018). Exact gravitational potential of a homogeneous torus in toroidal coordinates and surface integral approach to poission equation.New Zealand: Victoria University of Wellinton.

[7]. Sakajo, T. & Shimizu, Y. (2016). Point Vortex Interactions on a toroidal surface.Sakyo-ku Kyoto, Japan: Kyoto University.

[8]. Tabor, M. (1989). Chaos and Integrability in Nonlinear Dynamics: An Introduction. New York: Wiley.

[9]. "Vortex" Oxford Dictionary. Oxford University Press

K. Bitrus. "Structural Point Vortices on Toroidal Surfaces." IOSR Journal of Mathematics (IOSR-JM) 14.4 (2018): 12-16.