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Numerical Solution of the Heston Stocastic Volatility Model

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Abstract: *This paper has considered the numerical solution of the Heston stochastic volatility model (HSVM) using the Elzaki transform method (ETM). The proposed method seeks the approximate solution of the HSVM by implementing its properties on the HSVM. The ETM proposes the solution as a rapid convergent series that represents the precise interpretation of the HSVM in real life situations. Also, the reckless interest rate is choice as -2.01 in correspondence with [4]. Numerical evidences were obtained with the help of Maple 18 software, and are compared with the homotopy perturbation method (HPM) and variational iteration method (VIM) found in the literature [4]*

Keywords: *Heston partial differential equation, Heston stochastic volatility model, Elzaki transform method, reckless interest rate.*

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I. Introduction

A stock price under a neutral risk scaling is governed by [1]

$$dS(t) = \left[\psi dt + \sqrt{Vt}\, dW_1t\right] St$$

$$dVt = \left[\mathrm{d} - \mathrm{kVt}\, dt + \varphi\sqrt{Vt}\, dW_2t\right] dt$$

where ψ denotes the reckless interest rate, d, k and φ are real and positive constants, W_1t and W_2t are Brownian motion (correlated) under the neutral risk scaling with ρ defined in $-1,1$.

Similarly, the neutral risk scaling price of a stock expires at time $t \leq T$ defined in the HSVM given as

$$CSt, t, Vt = E[e^{-\psi T-t}(St - a)^{+}], \quad t \in [0, T] \qquad (1)$$

Thus, the HSVM for the various values of European option pricing style is the Heston partial differential equation (HPDE) of the form

$$\frac{\partial}{\partial t} cs, v, t + \psi s\frac{\partial}{\partial s} cs, v, t + d - kv\frac{\partial}{\partial v} cs, v, t + \frac{1}{2}s^2 v\frac{\partial^2}{\partial s^2} cs, v, t + \rho\varphi sv\frac{\partial^2}{\partial s\partial v} cs, v, t + \frac{1}{2}\varphi^2 v\frac{\partial^2}{\partial v^2} cs, v, t - \psi cs, v, t = 0 \qquad (2)$$

with initial condition

$$cs, v, 0 = 2s^2 t^2 \qquad (3)$$

where;

- cs, v, t is called the call price,
- v is the velocity
- ψ denotes the interest rate that is reckless,
- s denotes the assert price undergoing Brownian motion geometrically.

If at the time $\psi = T - t$, the variance and its option price are equal to s and v respectively.

Basically, equations of the form (2) is much more relevant in the areas of mathematical finance and statistical inference. At such solving this equation is deemed necessary for the purpose of real-life application. Not much is known for analytic methods for solving the stochastic volatility models (SVMs) due to the effect of the correlated Brownian motions. On this note, numerical methods have become more relevant in seeking these solutions. The numerical methods gives the solution as a rapidly converging series that converges to the analytic solution.

For instance, Tsetimi and Mamadu [2] solved the Fokker-Planck stochastic model using Elzaki transform method [ETM]. Ojobor and Mamadu [3] employed the variational iteration decomposition method for the numerical treatment of Fokker-Planck equation. In like manner, Biazer et. al. [4] employed the variational iteration method (VIM), homotopy perturbation method (HPM) and Adomain decomposition method (ADM) for the numerical solution of the Heston partial differential equation. El-wakil and Abdou [5] solved the diffusion convection reaction equations using the ADM. For more details, see references therein [6-8].

This paper is motivated to seek the numerical solution of the Heston stochastic volatility model using the Elzaki transform method (ETM). This method is necessitated by its less computational rigor, total elimination of round-off, truncation and computational errors as compared with the VIM and HPM as available in literature. Numerical evidences were actualized with the aid of the computational tool Maple 18 software.

II. Elzaki Transform method

An Elzaki transform of the function g(t) in the set

$$\Omega=\left\{g(t): \exists N, \alpha_1 \text{ and } \alpha_2 > 0: |g(t)| < Ne^{|t|/\alpha_j}, \text{ if } t \in (-1)^j \times [0,\infty)\right\} \quad (4)$$

is define as

$$E[g(t)] = r\int_0^\infty g(t)e^{-t/r}dt = T(r), r \in (-\alpha_1, \alpha_2) \quad (5)$$

2.1 Properties of Elzaki Transforms Method

Some relevant Elzaki transforms for this research can be seen in [2], [5], [7] and outline as follows:

i. $E\left[\frac{\partial y(x,t)}{\partial t}\right] = \frac{1}{r}T(x,r) - ry(x,0)$

ii. $E\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right] = \frac{1}{r^2}T(x,r) - r\frac{\partial y(x,0)}{\partial t} - y(x,0)$

iii. $T_m(x,r) = \frac{T(x,r)}{r^m} - \sum_{k=0}^{m-1} r^{2-m+k}\frac{\partial^k y(x,0)}{\partial t^k}$, m is the order of the highest derivative.

iv. $E[t^n] = n!\, r^{n+2}$

v. $E^{-1}\left[r^{n+2}\right] = \frac{t^n}{n!}$

2.2 Construction of Elzaki Tranform Method for Heston PDE

Here, we develop the ETM for resolving the Heston PDE of the form (2).

Applying the Elzaki transform on both sides of equation (2), we have

$$E\left[\frac{\partial}{\partial t}c(s,v,t)\right] = E\left[-\psi s\frac{\partial}{\partial s}c(s,v,t) - d - kv\frac{\partial}{\partial v}c(s,v,t) - \frac{1}{2}s^2v\frac{\partial^2}{\partial s^2}c(s,v,t) - \rho\varphi sv\frac{\partial^2}{\partial s\partial v}c(s,v,t) - \frac{1}{2}\varphi^2v\frac{\partial^2}{\partial v^2}c(s,v,t) + \square\square\square(\square,\square,\square)\right] \quad (6)$$

By property 2.1(iii), we have that

$$E[c(s,v,t)] = \sum_{k=0}^{m-1} r^{2-m+k}\frac{\partial^k c(s,v,0)}{\partial t^k} + rE\left[-\psi s\frac{\partial}{\partial s}c(s,v,t) - d - kv\frac{\partial}{\partial v}c(s,v,t) - \frac{1}{2}s^2v\frac{\partial^2}{\partial s^2}c(s,v,t) - \rho\varphi sv\frac{\partial^2}{\partial s\partial v}c(s,v,t) - 12\square2\square\square2\square\square2\square\square,\square,\square+\square\square\square,\square,\square\right] \quad (7)$$

Taking the inverse on both sides if equation (7), we have

$$c(s,v,t) = E^{-1}\left[\sum_{k=0}^{m-1} r^{2-m+k}\frac{\partial^k c(s,v,0)}{\partial t^k} + rE\left[-\psi s\frac{\partial}{\partial s}c(s,v,t) - d - kv\frac{\partial}{\partial v}c(s,v,t) - \frac{1}{2}s^2v\frac{\partial^2}{\partial s^2}c(s,v,t) - \rho\varphi sv\frac{\partial^2}{\partial s\partial v}c(s,v,t) - 12\square2\square\square2\square\square2\square\square,\square,\square+\square\square\square,\square,\square\right]\right] \quad (8)$$

But, the approximation solution which approximate the analytic solution of the generalized Heston PDE is given by

$$c(s,v,t) = \sum_{n=0}^{\infty} c_n(s,v,t)$$

Thus, equation (8) becomes;

$$\sum_{n=0}^{\infty} c_n(s,v,t)\ E^{-1}\left[\sum_{k=0}^{m-1} r^{2-m+k}\frac{\partial^k c_n(s,v,t)}{\partial t^k} + rE\left[-\psi s\frac{\partial}{\partial s}c_n(s,v,t) - d - kv\frac{\partial}{\partial v}c_n(s,v,t) - \frac{1}{2}s^2v\frac{\partial^2}{\partial s^2}c_n(s,v,t) - \square\square\square\square\square2\square\square\square\square\square\square(\square,\square,\square) - 12\square2\square\square2\square\square2\square\square(\square,\square,\square) + \square\square\square(\square,\square,\square)\right]\right] \quad (9)$$

Comparing both sides of equation (9), we get the recurrence relations

$$c_0(s,v,t) = E^{-1}\left[\sum_{k=0}^{m-1} r^{2-m+k}\frac{\partial^k c_n(s,v,0)}{\partial t^k}\right], \quad (10)$$

$$c_{n+1}s,v,t = E^{-1}\left[rE\left[-\psi s\frac{\partial}{\partial s}c_n(s,v,t) - d - kv\frac{\partial}{\partial v}c_n(s,v,t) - \frac{1}{2}s^2 v\frac{\partial^2}{\partial s^2}c_n(s,v,t) - \rho\varphi sv\frac{\partial^2}{\partial s\partial v}c_n(s,v,t) -\right.\right.$$

12□2□□2□□2□□(□,□,□)+□□□(□,□,□) (11)

Thus, the components $c_1, c_2, c_3, \ldots, c_n$ for $n \geq 1$ are computed using equation (10) and (11) respectively.
Finally, the approximate solution of the Heston PDE becomes

$$c(s,v,t) = \sum_{n=0}^{\infty} c_n(s,v,t).$$

III. Numerical Experiments

In this section we implement the derived ETM for the Heston PDE with the aid of Maple 18 software. Results obtained were compared with the variational iteration method (VIM) and homotopy perturbation method (HPM) [4] for accuracy and convergence.
Now using the ETM scheme (10) and (11) on

$$\frac{\partial}{\partial t}cs,v,t + \psi s\frac{\partial}{\partial s}cs,v,t + d - kv\frac{\partial}{\partial v}cs,v,t + \frac{1}{2}s^2 v\frac{\partial^2}{\partial s^2}cs,v,t + \varphi sv\frac{\partial^2}{\partial s\partial v}cs,v,t + \frac{1}{2}\phi^2 v\frac{\partial^2}{\partial v^2}cs,v,t - \psi cs,v,t = 0\,,$$

with the parameters $d = 0.16$, $k = 0.055$, $\varphi = 0.9$, $\rho = -0.5$ (see [4]), we obtain the following approximations:

$$c_0 = 2s^2t^2$$

$$c_1 := \frac{1}{6}\left(-4\psi s^2 - 4s^2 v\right)t^3$$

$$c_2 := \frac{1}{4}\left(0.6666666667\psi^2 s^2 + 1.333333333\psi s^2 v + 0.1066666667 s^2 - 0.6366666667 s^2 v + 0.6666666667 s^2 v^2\right)t^4$$

$$c_3 := \frac{1}{5}\left(-0.1666666665\psi^3 s^2 - 0.4999999999\psi^2 s^2 v - 0.07999999999\psi s^2 + 0.4774999997\psi s^2 v - 0.4999999997\psi s^2 v^2 + 0.02546666668 s^2 - 0.3670041666 s^2 v + 0.4774999998 s^2 v^2 - 0.1666666666 s^2 v^3\right)t^5$$

$$c_4 := \frac{1}{6}\left(0.03200000000\psi^2 s^2 - 0.02037333332\psi s^2 + 0.03333333330\psi^4 s^2 + 0.1333333333\psi^3 s^2 v - 0.1910000000\psi^2 s^2 v + 0.2000000000\psi^2 s^2 v^2 + 0.2936033333\psi s^2 v - 0.3819999997\psi s^2 v^2 + 0.1333333333\psi s^2 v^3 - 0.1910000000 s^2 v^3 + 0.3528058332 s^2 v^2 + 0.01174413333 s^2 - 0.1831061292 s^2 v + 0.03333333332 s^2 v^4\right)t^6$$

$$\vdots$$

Thus, the required computed solution is

$$c(s,v,t) := 2s^2t^2 + \frac{1}{6}\left(-4\psi s^2 - 4s^2 v\right)t^3 + \frac{1}{4}\left(0.6666666667\psi^2 s^2 + 1.333333333\psi s^2 v + 0.1066666667 s^2 - 0.6366666667 s^2 v + 0.6666666667 s^2 v^2\right)t^4 + \frac{1}{5}\left(-0.1666666665\psi^3 s^2 - 0.4999999999\psi^2 s^2 v - 0.07999999999\psi s^2 + 0.4774999997\psi s^2 v - 0.4999999997\psi s^2 v^2 + 0.02546666668 s^2 - 0.3670041666 s^2 v + 0.4774999998 s^2 v^2 - 0.1666666666 s^2 v^3\right)t^5 + \frac{1}{6}\left(0.03200000000\psi^2 s^2 - 0.02037333332\psi s^2 + 0.03333333330\psi^4 s^2 + 0.1333333333\psi^3 s^2 v - 0.1910000000\psi^2 s^2 v + 0.2000000000\psi^2 s^2 v^2 + 0.2936033333\psi s^2 v - 0.3819999997\psi s^2 v^2 + 0.1333333333\psi s^2 v^3 - 0.1910000000 s^2 v^3 + 0.3528058332 s^2 v^2 + 0.01174413333 s^2 - 0.1831061292 s^2 v + 0.03333333332 s^2 v^4\right)t^6$$

See table 1 for computational results

Table 1: Comparison of results between ETM, VIM and HPM

$c(s, v, t)$	ETM	VIM [4]	HPM [4]
$c(10,0.1,1)$	410.7872814	209.5038332	413.2583333
$c(50,0.2,2)$	83070.86795	82474.33729	81406.34666
$c(70,0.3,4)$	1.761996634E6	1.14781470E6	1.734519680E6
$c(90,0.4,6)$	1.335674406E7	1.950175983E7	1.335877440E7
$c(120,05,8)$	7.613256542E7	7.796505675E7	7.485848702E7
$c(150,06,10)$	2.242732459E8	2.317375113E8	2.937374856E8

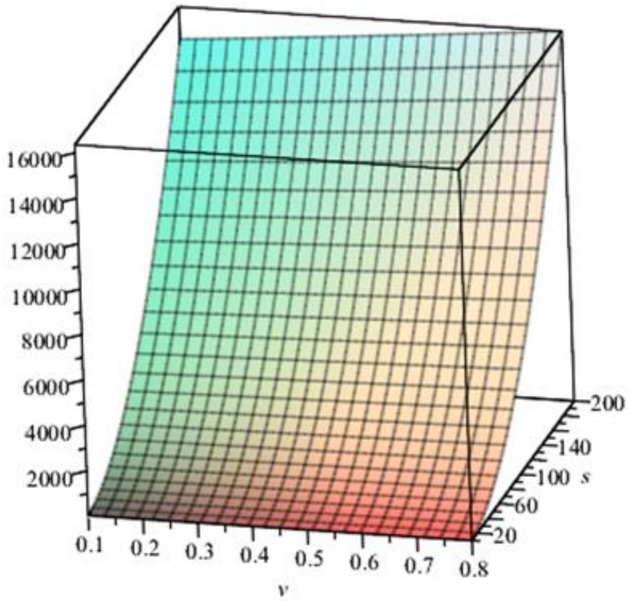

Figure 1. Graphical stimulation of HSVM with the ETM

IV. Conclusion

In this paper, we have considered the ETM for the HSVM. The method executes its properties on the HSVM to arrive at the computed solution. Results presented show that the ETM is a more efficient and reliable numerical scheme for the HSVM than those found in [4] such as VIM and HPM as shown in table 1. Also, the graphical stimulation of the numerical solution of HSVM corresponds with the HPM found in [4].

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