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Investigating the Drum Membrane Shape using The Amplitude Spectra They Produce When Subjected To Symmetrical Disturbances.

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Abstract:*Drums are the world's oldest and most widespread of musical instruments. If basic shapes (like rectangular, circular and triangular), are considered to be the drum membrane; ignoring the air column below, whether it is possible to distinguish them by observing their frequency and amplitude spectra is addressed by solving the wave equation in this research. The wave equation in two dimension was solved in rectangular Cartesian coordinates and cylindrical polar coordinates using the method of separation of variables. Amplitude spectrum for all three cases were determined through numerical implementation of discrete Fourier transforms using Maple software. Results of the study shows that all three drumheads have distinct amplitude spectra when subjected to initial symmetrical disturbances and hence it is possible to distinguish them from the sound they produce. Nowadays the most widely used drumhead is the round shape and rectangular and triangular drumheads are rarely used. The reason behind this is the monotonous hollow sound produced by the latter shapes and the good sound and melody produced by the former. The key aspect of conducting this research is to inspect the reasons as to how this happens.*

Keywords -*Amplitude spectra,Discrete Fourier Transform, frequency spectra, Amplitude spectra*

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I. Introduction

Can one hear the shape of a drum?' This question was first posed by a Polish Mathematician Mark Kac in 1966 and remained unanswered for about quarter century until Gordon, Webb & Wopert (1992) gave an example of two differently shaped drums that sounded exactly the same (i.e. having identical frequency spectra) answering negative to Kac's question.) Nevertheless if basic shapes (like rectangular, circular and triangular), are considered to be the drum membraneit is possible to distinguish them by observing their frequency and amplitude spectra. And also Rowlett & Lu (2015) have uncovered that it is possible to identify the corners of a drumhead. This study is mainly based on observing the frequency and amplitude spectra of rectangular, circular and triangular drumheads and derive conclusions about their shapes. Or simply, 'does the sound made by a drum is effected by its shape?' is investigated by solving 2-D wave equations and comparing amplitude spectra of the drumheads.

II. Methodology

Inthisresearch, weconsideravibratingmembraneinatwodimensionalplane. So the displacement of the membrane can be explained better by using the two dimensional wave equation. 2-D wave equation for the rectangular and triangular membranes was solved by using rectangular coordinate system and for the circular membrane it was solved by cylindrical polar coordinate system, and thereby equations for amplitudes and frequencies of the drum heads were obtained.But here, a few assumptions needed to be considered before modeling the problem.

- The membrane is homogeneous. That is, the density (ρ) of the membrane is constant.
- The membrane is composed of a perfectly flexible material without any resistance to deformation in the z-direction in 3-D plane.
- The membrane is stretched and fixed along a boundary in the xy- plane.
- The Tension per unit length (T) due to stretching is a constant and it is the same everywhere in the membrane.

- Weight of the membrane is negligible.
- The effect of the air column below the membrane is considered negligible and assumed to have no effect on the motion of the drum membrane.

Based on the above assumptions, the first step needed to be taken was generating respective amplitude and frequency equations using 2-D wave equation in order to construct the amplitude spectra for each drum membrane.

A. 2-D wave equation in rectangular coordinates

The 2-D wave equation in rectangular coordinates is of the form,

$$U_{tt} = C^2(U_{xx} + U_{yy})$$

Taking the length of the membrane as 'a' and breadth as 'b' and the edges of the membrane fixed, the boundary conditions can be given as,

$$U(0,y,t) = U(a,y,t) = 0 \ ; 0 \leq y \leq b, t \geq 0$$
$$U(x,0,t) = U(x,b,t) = 0 \ ; 0 \leq x \leq a, t \geq 0$$

And initial conditions are,

$$U(x,y,0) = f(x,y); \ \ (x,y) \in \mathbb{R}$$
$$U_t(x,y,0) = g(x,y); \ \ (x,y) \in \mathbb{R}$$

Then using,

- The separation of variables to produce simple solutions to boundary conditions and
- The principle of superposition to build up a solution that satisfies initial conditions,

equations to find the amplitude and frequency of the vibrating drum membrane when deformed at the center can be generated. Thus the amplitude for this particular problem can be calculated using,

$$Amp_{nm} = a_{nm} = \frac{2}{a}\frac{2}{b}\int_0^a f(x,y)sin(\frac{n\pi x}{a})sin(\frac{m\pi y}{b})dxdy \qquad (1)$$

And to calculate frequency general equation used is,

$$f_{nm} = \frac{\omega_{nm}}{2\pi} \qquad (2)$$

where, $\omega_{nm} = c\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$. Therefore the equation for frequency will be,

$$f_{nm} = \frac{c}{2}\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \qquad (3)$$

Here $f(x,y)$ is the initial displacement function.

B. 2-D wave equation in cylindrical polar coordinates

For the circular drum membrane, wave equation was solved using cylindrical polar coordinates. Therefore initial conditions also needed to be in polar form and when it is expressed in cylindrical polar coordinates it takes the form,

$$U_{tt} = c^2(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2}) \qquad (4)$$

But since we consider the wave equation for a circular disk the above equation (4) reduces as below.

$$U_{tt} = c^2\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right) (5)$$

Since the drum membrane is fixed around the rim, boundary conditions will be,

$$U(a,\theta,t) = 0$$

where 'a' is the radius of the drum membrane. And the initial conditions will be,

$$U(r,\theta,0) = f(r,\theta)$$
$$U_t(r,\theta,0) = g(r,\theta) = 0$$

Using separation of variables method as well as using the special function type 'Bessel functions', solutions to this wave equation can be generated and thereby amplitude and frequency equations can be generated. The formula to calculate amplitude is the same as that of the rectangular membrane. i.e.

$$Amp_{nm} = \sqrt{(a_{nm})^2 + (b_{nm})^2} \qquad (6)$$

And frequency can be calculated using,

$$f_{nm} = \sqrt{\lambda_{nm}^2 - a^2} \qquad (7)$$

where, 'a' is the radius of the drum membrane and $\lambda_{nm} = \left(\frac{\alpha_{nm}}{a}\right)^2$

Here,

$$a_{nm} = \frac{1}{\pi}\frac{\int_0^{2\pi}\int_0^a J_n(\sqrt{\lambda_{nm}}\,r) r f(r,\theta) cos(n\theta) dr d\theta}{\int_0^a r\left(J_n(\sqrt{\lambda_{nm}}\,r)\right)^2 dr} \quad (8)$$

$$b_{nm} = \frac{1}{\pi}\frac{\int_0^{2\pi}\int_0^a J_n(\sqrt{\lambda_{nm}}\,r) r f(r,\theta) sin(n\theta) dr d\theta}{\int_0^a r\left(J_n(\sqrt{\lambda_{nm}}\,r)\right)^2 dr} (9)$$

$$n \geq 1 \geq 1 \, and,$$

$$\int_0^a r\left(J_n(\sqrt{\lambda_{nm}}\,r)\right)^2 dr = \int_0^a J_{n+1}\left(\frac{\alpha_{nm}}{a} r\right)^2$$

C. 2-D wave equation in rectangular coordinates (for the isosceles triangular membrane)

Solving the wave equation using rectangular coordinates, for the triangular membrane is similar to that of the rectangular membrane. Here we consider an isosceles triangle as the drum membrane with length of a side being 'a'. As stated earlier the general equation of the 2-D wave equation for a square, (rectangular with length and breadth $a \times a$) will be,

$$U_{tt} = C^2(U_{xx} + U_{yy})$$

And since the edges of the membrane are fixed here too, the boundary conditions will be,

$$U(0,y,t) = U(a,y,t) = 0; \; 0 \leq y \leq a, t \geq 0$$
$$U(x,0,t) = U(x,a,t) = 0; \; 0 \leq x \leq a, t \geq 0$$

And initial conditions will be,

$$U(x,y,0) = f(x,y); \; (x,y) \in R$$
$$U_t(x,y,0) = g(x,y); \; (x,y) \in R$$

So in solving the wave equation for the square membrane, the only change to that of the rectangular membrane will be length and breadth of the membrane being the same. So using the similar concept of applying variable separable method and superposition principle we can get the solution to the wave equation for a triangular membrane. In order to get that,general solution to the wave equation should be subtracted from the same solution with indices interchanged. Since the points on the main diagonal that are equidistant from the center must have the same solution, this procedure gives a wave function that vanishes along the diagonal and hence by doing so gives the wave equation for the triangular membrane (Isosceles). And thus by using this solution, like in the previous cases amplitude and frequency equations of the membrane can be constructed. The formula to calculate amplitude is,

$$Amp_{nm} = a_{nm} = \frac{2}{a}\frac{2}{a}\int_0^a f(x,y) sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{a}\right) dxdy$$
$$-\frac{2}{a}\frac{2}{a}\int_0^a f(x,y) sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{a}\right) dxdy \quad (10)$$

And to calculate frequency general equation used is,

$$f_{nm} = \frac{\omega_{nm}}{2\pi} \quad (11)$$

where, $\omega_{nm} = c\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$. Therefore the equation for frequency will be,

$$f_{nm} = \frac{c}{2}\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \quad (12)$$

Then using these amplitude and frequency equations, respective amplitudes and frequencies for drum membranes with diff erent sizes can be calculated using Maple software and based on the results, their spectra can be constructed using Minitab software.

III. Results

As stated above using a MAPLE program, amplitude and frequency of both rectangular and circular membranes were calculated when deformed anywhere on the membrane. Here the constant c is taken as 0.01 everywhere. And'c' is the constant in the wave equation where,

$$c = \frac{Tension \; per \; unit \; length}{mass \; density} \quad (13)$$

Minitab software is used to draw the bar charts depicting the amplitude spectrum, which shows the diff erences in the spectra for diff erent shapes diagrammatically.

1) Rectangular membrane

$$a = 2.4\ cm\ and\ b = 1.2\ cm$$

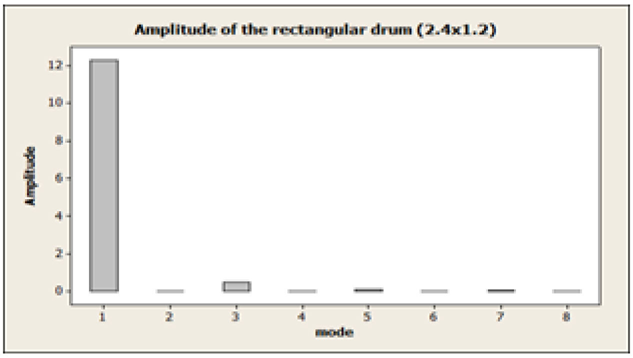


Fig.1: Amplitude of the rectangular membrane of length 2.4 and breadth 1.2

2) Circular membrane

$$Radius = 1.2\ cm$$

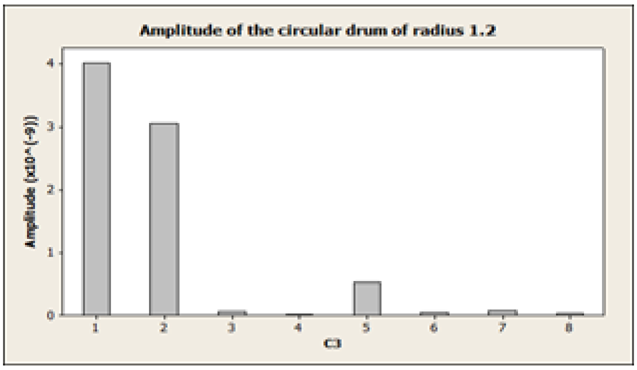
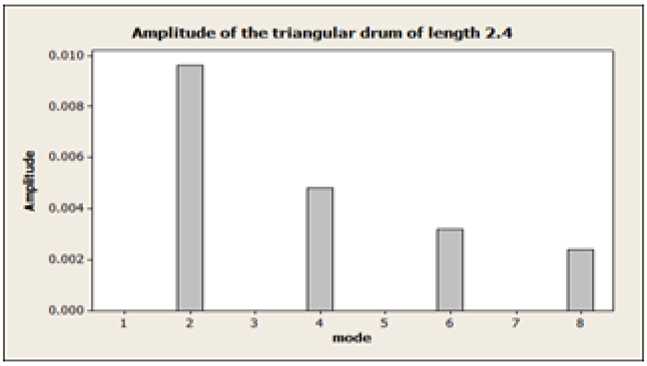


Fig.2: Amplitude of the circular membrane of radius 1.2

3) Triangular membrane (Isosceles)



$$a = 2.4\ cm\ (length\ of\ one\ equal\ side)$$

Fig.3: Amplitude of the triangular membrane of length 2.4

IV. Discussion

A drum when it was struck, produces several amplitudes which determines the sound output. So by a thorough investigation of those amplitudes, shapes of the respective drums can be determined, mainly because those amplitude spectra depends on the shape. A circular drum produces several significant amplitudes while rectangular drum doesn't and the triangular drum produces several amplitudes alternatively. This makes them easier to identify by listening to the loudness. And if one listen carefully, the sound output of each type of membrane they can easily recognize that the rectangular drum gives a hollow, monotonous sound, circular drum has a goodmelody due to the overlapping of the frequencies and triangular has a high pitch and loudness. So the amplitude spectra output shown in this can be considered accurate and can be used to differentiate the membrane shapes. And also it is clear from the results size of the membrane does not have a overall big impact on the amplitude spectra output. And the amplitude spectra doesn't drastically change when the size of the membrane was changed. Even though there were some minute fluctuations in the amplitude values overall effect remains the same.

V. Conclusion

Based on the results for rectangular and circular drums we can derive several conclusions. The major conclusion that we can procure through this study is in fact the confirmation of our research purpose. That is we can distinguish the three shapes (rectangular, circular and triangular) via spectral analysis. It is clearly seen that even though rectangular drum produces several amplitudes only one value is significant, there by affecting the loudness of the drum and as a result of that giving out a hollow monotonous sound output. But in the case of circular drum it produces a series of significant amplitudes and as a result of that it has a good sound and a melody. But in the case of triangular drums they produce significant amplitudes for high frequency nodes alternatively. Therefore they usually have a high pitch and a loudness. And also the results doesn't change even when slight changes are made to the size of the membrane considered.

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