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Study of Limit Cycle for Fitzhug-Nagumo System

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Abstract: *In the present study we have investigated the complete Fitzhug-Nagumosystem with $I \neq 0$: we have shown that one or two limit cycles may bi-furcate at the origin. Bendixons theorem has been used in our study toprove non-existence of limit cycles. We have also proved that the systemhas unique limit cycle through change of the parameters. .*

Keywords: *Limit cycle, Lie0nard equation, Hopf-bifurcation, Fitzhug-Nagumosystem .*

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I. Introduction

In the present paper, we revisit the problem of bifurcation of limit cycles. We give criterion for the study model (Fitzhug-Nagumo system) to have or not tohave limit cycles with $I \neq 0$: We also demonstrate that the model exhibits aHopf-bifurcation. Now we consider the following Li_enard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0.$$

The above equation may be written into two dimensional autonomous dynamical system

$$\dot{x} = y, \dot{y} = -g(x) - f(x)y.:$$

In Lienard plane above equations becomes

$$\begin{aligned} \dot{x} &= y - F(x), \\ \dot{y} &= -g(x) \end{aligned} \qquad (1.1)$$

where $F(x) = \int_0^x f(t)dt$:

The main part of this paper is devoted to explain the existence and uniquenessof limit cycles of Fitzhugh-Nagumosystem which is expressed through
followingdeferential system

$$\begin{aligned} \dot{x} &= y - Ax(x - B)(x - \lambda) + I; \\ \dot{y} &= -\epsilon(x - \delta y): \end{aligned} \qquad (1.2)$$

This system has been extensively studied with particular emphasis on bifurcatelimit cycles as well as in model been of certain phenomenon. Literature review,indicates that, most of the articles studies the system taking some parametersas zeros, for instance see (Mattias 2006, Nikola &Dragana 2003, Enno 2006),(Arnaud 2002, Rabinovitch& Friedman 2009, Romel et al 2001 and Baili 2004)(LuoDingjun,et al 1997)investigated the particular case of taking $(1 + \lambda) = 0$;and proved the niqueness of limit cycle. In (Ringkvist& Zhou 2009) thereis a general analysis of the system for bifurcation of limit cycles from Hopf-bifurcation.In this paper, we study the system (1.2) with all parameters notzeros and prove the uniqueness of limit cycle. The paper is organized as follows.

In section 2, we prosed the main system equations when all parameters arenot zero . The su_cient conditions that the system has at least two limit cyclesare shown by using Hopf-bifurcation methods.

Section 3 is devoted for special cases $\delta = 0$ and prove the uniqueness oflimit cycle, and then the case of quadric system. Finally, case of saddle pointwith limit cycle is presented, theorems and lemmas in section 4 along with theconcluding remakes .

II. Main system equation

In this section, we investigate the Fitzugh-Nagumo system with the parameters $A, B, \delta, \epsilon, \lambda$and I being not zeros. In particular, we study the system under thecase $= B = 1\, and\, \lambda \neq 1; wher\, \delta \in (-1;\, 0);\, I \in R$:

In order to study the existence and non-existence of limit cycles we makechange of variables to get Li_enard type (1.1). Let $x - \propto\, \rightarrow\, x$ and $y + \delta\epsilon x + \frac{\alpha}{\delta} \rightarrow\, y$
whereα is the root of equilibrium equation $\alpha x^3 - \delta(1 + \lambda)x^2 - (\delta\lambda - 1)x - I\delta = 0$:

Then the system becomes,

$$\dot{x} = y - \left[x^3 + \left(3\alpha - (1 + \lambda)\right)x^2 + (3\alpha^2 - 2(1+\lambda)\alpha + \lambda + \delta\epsilon)x\right]$$
$$\dot{y} = -\delta\varepsilon\left[x^3 + \left(3\alpha - (1 + \lambda)\right)x^2 + \left(3\alpha^2 - 2(1+\lambda)\alpha + \lambda + \frac{1}{\delta}\right)x\right] \quad (2.1)$$

We note that $F(0) = 0$; $g(0) = 0$:The other two roots of *F(x) = 0* and *g(x) = 0*respectively are

$$x = \frac{1}{2}\left[-\left(3\alpha - (1+\lambda)\right) \pm \sqrt{A^2 - 4(2\alpha^2 - 2(1+\lambda)\alpha + \lambda + \delta\varepsilon)}\right] \quad (2.2)$$

$$y = \frac{1}{2}\left[-\left(3\alpha - (1+\lambda)\right) \pm \sqrt{A^2 - 4\left(3\alpha^2 - 2(1+\lambda)\alpha + \lambda - \frac{1}{\delta}\right)}\right] \quad (2.3)$$

where$A = \left(3\alpha - (1 + \lambda)\right)$.

The system has unique singular point for$\left(3\alpha - (1 + \lambda)\right)^2 - 4\left(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda - \frac{1}{\delta}\right) < 0$, and for $\left(3\alpha - (1 + \lambda)\right)^2 - 4\left(3\alpha^2 - 2(1 + \lambda)\alpha + \lambda - \frac{1}{\delta}\right) > 0$, wehave three singular points.

2.1 Brief Note On Anti-Saddle Bifurcation Case

Let us consider the system (2.1) in the case O as anti-saddle ie

$$\delta\varepsilon\left(3\alpha^2 - 2(1+\lambda)\alpha + \lambda - \frac{1}{\delta}\right) > 0, \quad (2.4)$$

Since $\delta\varepsilon > 0$ then $3\alpha^2 - 2(1+\lambda)\alpha + \lambda - \frac{1}{\delta}$ must be greater then than zero.Then we have $\lambda - \frac{1}{\delta} > 0$ and $(1 + \lambda)^2 - 3\left(\lambda - \frac{1}{\delta}\right) < 0$.

The first three focal values are [8]:

$$W_1 = \frac{3\alpha^2 - 2(1+\lambda)\alpha + \lambda + \delta\varepsilon}{\sqrt{\mu}}$$
$$W_2 = \frac{\delta\varepsilon\left[2A^2 - 3\left(3\alpha^2 - 2(1+\lambda)\alpha + \lambda - \frac{1}{\delta}\right)\right]}{8\mu^2\sqrt{\mu}} \quad (2.5)$$
$$W_3 = -\frac{15c\delta\varepsilon}{\mu^3\sqrt{\mu}}$$

From Lemma 3.3.2 in [8] O is unstable (stable)strong focus when $W_1 > 0(W_1 < 0)$, unstable (stable) weak focus of order one when $W_2 > 0$ $(W_2 < 0)$and unstable (stable) weak focus of order two when $W_3 > 0$ $(W_3 < 0)$. Thus,from Hopf-bifurcation one stable limit cycle appears in the case $W_1 > 0$and$(W_2 < 0)$. Therefore, *F(x)* has three critical points and the system has only onesingular point *g(0) = 0*. To prove the uniqueness of limit cycle we can apply thefollowing lemma:

Lemma 2.1 [8]

Suppose that system (1 .1) satisfies the following conditions:

1. There exist $c_1 < p_1 < q_1 < 0 < q_2 < p_2 < c_2$ such that $p_1, 0, p_2$ arezero points of *F(x)*,$c_1, 0, c_2$ are zero points of $g(x), xg(x) > 0$ for$x \in (c_1; 0)$ [$(0; c_2)$ and q_1; q_2 are zero points of $f(x)$; $f(x) < 0$ for$x \in (q_1; q_2)$ and *f(x) > 0* otherwise.
2. If the simultaneous equations

$$F(u) = F(v);\ G(u) = G(v);\ c_1 < u < 0 < v < c_2$$

have no solution *(u,v)*, then system (1.1) has no closed orbit in the strip$\{c_1 < x < c_2; -\infty < y < +\infty\}$ or if it has at most one solution andthe function *f(x)g(x)* is monotonically decreasing (increasing) in $x \in (c_1; p_1)$or $x \in (p_2; c_2)$; then (1.1) has at most one limit cycle in the strip $\{x \in (c_1; c_2); y \in (-\infty, \infty)g$ and it is stable (unstable) if it exists,

where$G(x) = \int_0^x g(x)dx$.

Lemma 2.2

For $\left(\lambda - \frac{1}{2}\right)^2 + \frac{3}{4} - 3\delta\varepsilon < 0$ the system (2.1) has no limit cycles.

Proof

Considering the equation

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 3x^2 + 2\left(3\alpha - (1 + \lambda)\right)x + (3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \delta\varepsilon), \quad (2.6)$$

we define

$$N(x) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \ 3x^2 + 2\big(3\alpha - (1 + \lambda)\big)x + (3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \delta\varepsilon), \qquad (2.7)$$

The discriminant of above polynomial root is

$$\Delta = (1 + \lambda)^2 - 3(\lambda + \delta\varepsilon) = \left(\lambda + \frac{1}{2}\right)^2 - \frac{3}{4} - 3\delta\varepsilon.$$

In the lemma we have $\Delta < 0$ and since $(1 + \lambda) \neq 0$ implies that *N(x)* isdefinite in sign and non-zero, then by Bendeixsons theorem [4], we concludethat there are no limit cycles.

Lemma 2.3

If $W_1 = 0$; then W_2not identical zero.

Proof Let $W_1 = 0$; then we have $3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \delta\varepsilon = 0$ substitute thisequation in the the value of W_2; then we get

$$W_2 = 2(1 + \lambda)^2 - 3(\lambda + \delta\varepsilon) - 3\left(\lambda - \frac{1}{\delta}\right) = 2\lambda^2 - 2\lambda + 2 - 3\left(\delta\varepsilon - \frac{1}{\delta}\right),$$

we define

$$H(\lambda) = 2\lambda^2 - 2\lambda + 2 - 3\left(\delta\varepsilon - \frac{1}{\delta}\right).$$

Since $\delta \in (-1; 0)$,so $2 - 3\left(\delta\varepsilon - \frac{1}{\delta}\right) < 0$. All limit cycles would lie in one ofthe regions $H(\lambda) < 0 \; or \; H(\lambda) > 0$.For$H(\lambda) < 0$ we have $W_2 < 0$ and $W_1 > 0$,and for $H(\lambda) > 0$ we have $W_2 > 0$ and $W_1 < 0$. Thus, $H(\lambda)$ cannot be identicalto zero and we have the following lemma.

Lemma 2.4

For $W_1 > 0$ and $W_2 < 0$ the system (3) has at least one limit cycle surroundingO.

Proof First let $W_1 = 0$ when $W_2 < 0$ then O is stable focus of order one,and when W_1 increasing from zero one stable limit cycle appear surrounding O, by Hopf-bifurcation.

From above lemma we have

$$\alpha < \frac{1}{3}\sqrt{(1 + \lambda)^2 - 3(\lambda - \delta\varepsilon)} + 1 + \lambda = \rho_1$$

and

$$\alpha < \frac{1}{3}\sqrt{3(\lambda - \delta\varepsilon) - (1 + \lambda)^2} + 1 + \lambda = \rho_2$$

Let $\rho = \max\{\rho_1, \rho_2\}$, then we have the following theorem.

Theorem 2.5 For $\alpha < \frac{1}{3}\sqrt{\rho} + 1 + \lambda$, the system (2.1) has unique limitcycle.

Proof Now we apply lemma 2.1. From the condition of W2 > 0 we see that**g(x)** has only one zero point $g(0) = 0$, then we can easily find that $xg(x) > 0$.From the condition W_1 we find that *F(x)* has three zeros, therefor *f(x)* has twozeros. And also from the case $3\alpha^2 - 2(1 + \lambda)\alpha + \lambda + \delta\varepsilon < 0$ we deduce that *f(x) < 0*for $x \in (q_1, q_2)$ and *f(x) > 0* otherwise. So condition (1) satisfied. From thelocal position of *f(x)* and the case that *xg(x) > 0* we get $\frac{f(x)}{g(x)}$ increasing for all$x > p_2$. So we just need to prove the condition of the simultaneous equations*F(u) = F(v); G(u) = G(v)*. After simplify and by putting *s* = *u*+*v*and *r* = *uv*we get,

$$h(s) = \frac{1}{4}s^3 - \frac{1}{2}\big(3\alpha\alpha - 2(1 + \lambda)\big)s^2 + \left(\frac{1}{3}\left(3\alpha - (1 + \lambda)^2 + \frac{1}{2}\delta\varepsilon - \frac{1}{2}\frac{1}{\delta}\right)\right)s + \frac{1}{3}\big(3\alpha - (1 + \lambda)\big)\big(3\alpha^2 - 2(1 + \lambda)\big)\alpha + \lambda + \delta\varepsilon.$$

Since *h(s)* with odd degree, then *h(s)* has at most one solution, so for prove theonly one solution let consider discriminant of $h'(s)$,

$$h'(s) = \frac{3}{4}s^2 - \big(3\alpha\alpha - 2(1 + \lambda)\big)s + \left(\frac{1}{3}\left(3\alpha - (1 + \lambda)^2 + \frac{1}{2}\delta\varepsilon - \frac{1}{2}\frac{1}{\delta}\right)\right),$$

it is easy to find $\Delta = -\frac{3}{2}\left(\delta\varepsilon - \frac{1}{\delta}\right) < 0$.

And hence the theorem has been proved.

Remark 6 An inequality $\alpha < \frac{1}{3}\sqrt{p} + 1 + \lambda$equivalent to $H(\lambda) < 0$.

2.2 A saddle bifurcation case

In this case we have

$$\delta\varepsilon\big(3\alpha^2 - 2(1+\lambda)\big)\alpha + \lambda + \frac{1}{\delta}) < 0,$$

we discuss saddle bifurcation in the case

$\delta\varepsilon < 0$and$\big(3\alpha^2 - 2(1+\lambda)\big)\alpha + \lambda + \frac{1}{\delta} > 0$,

From these situations, we can _nd that the discriminant of the roots of *g(x)*

$$\Delta = \big(3\alpha - (1+\lambda)\big)^2 - 4\left(3\alpha^2 - 2(1+\lambda)\alpha + \lambda - \frac{1}{\delta}\right) < 0.$$

Thus, the system (2.1) has unique singular point which is hyperbolic saddleat the origin, and therefore no limit cycle is possible. Thus we get the followingresult.

Theorem 2.7

For $\delta\varepsilon$< 0 and $(3\alpha^2 - 2(1+\lambda)\alpha + \lambda - 1/\delta) > 0$ the system (2.3) has no limit cycles.

2.3 The existence of two limit cycles

Theorem 2.8 For$\delta\epsilon$>0 and $H(\lambda) > 0$ the system (2.1) has at least twolimit cycles.

Proof Since from the system as $W_1 = W_2 = 0, O$ is stable weak focusof order two. Initially keep $W_1 = 0$ and let W_2 increases from zero, then onestable limit cycle L_1 bifurcates. Then, change W_1 to the negative such that L_1does not disappear but O change its stability again and unstable limit cycle L_2bifurcates in the interior of L_1: Hence, the conclusion is obtained.

III. A special case of $\boldsymbol{\delta = 0}$

In this case the system after B = 1 becomes

$$\dot{x} = y - Ax(x-1)(x-\lambda) + I$$
$$\dot{y} = \epsilon x. \qquad (3.1)$$

This system has unique singular point (0; I), so by putting $y = y + I$; thenwe have the following Lifienard type

$$\dot{x} = y - [Ax^3 - (1+\lambda)x^2 + \lambda x]$$
$$\dot{y} = \epsilon x \qquad (3.2)$$

this system has the origin as unique singular point. The Jacobianis given by

$$J(0,0) = \begin{bmatrix} -\lambda & 1 \\ \epsilon & 0 \end{bmatrix}$$

then$det(J) = -\epsilon$; for $\epsilon > 0$ O is saddle and no limit cycle possible. Thus,for existence of limit cycles we must consider $\epsilon < 0$:

Lemma 3.1

A and λ are both rotated parameters of system (3.1)

ProofDenote the right hand sides by P and Q respectively. Then we have

$$P\frac{\partial Q}{\partial A} - \frac{Q\partial P}{\partial A} = -\epsilon x^4 \geq 0 \; and \; P\frac{\partial Q}{\partial \lambda} - Q\frac{\partial P}{\partial \lambda} = -\epsilon x^2 \geq 0.$$

The first three focal values (3.10) are see [8]

$$W_1 = -\frac{\lambda}{\sqrt{-\epsilon}}, W_2 = \frac{3\epsilon A}{8\epsilon^2\sqrt{-\epsilon}}, \; W_3 = 0.$$

For $\lambda \neq 0$ O is strong focus stable(unstable) if $\lambda > 0(\lambda < 0)$ for$\lambda = 0$ thenO weak focus of order one stable(unstable) if *A > 0(A < 0)* thus, we can getthe following results.

Lemma 3.2

To create limit cycles of system (3.2) we have $\lambda A \neq 0$.

Proof Consider the case $\lambda = 0$; then for A = 0 system has O as center,and for $A \neq 0$ no limit cycle from rotated vector field. Similarly the case *A = 0*:

Lemma 3.3

If $\lambda < 0, A > 0(\lambda > 0, A < 0)$ stable (unstable) limit cycle for system (3.2)surrounding O appears via a Hopf-bifurcation.

Remark 3.4 From above lemma we deduced that a limit cycle can appearsjust in the two cases $\lambda A \neq 0$ and $\lambda A < 0$.

Lemma 3.5 [1],[5]

Let *f(x)* and *g(x)* be continuously differentiable functions for $k_1 < x < k_2$ where$k_1 < 0 < k_2$ such that for $k_1 < x < k_2$ the following conditions are satisfied:

1. *g(x) > 0(< 0)* for *x > 0(< 0);*
2. there exist x_0 such that $f(x_0) = 0$ and $f(x) > 0(< 0)$ for $x > 0(< 0)$;
3. $\frac{f(x)}{g(x)}$is an increasing function both for $x < 0$ for $x > x_0$.

Then the Lifienard system has at most one periodic orbit, and if exist itmust be a limit cycle with negative characteristic exponent.

Theorem 3.6 For $\lambda < 0$ and $A > 0$ system (3.2) has unique stable limitcycle.

Proof Now we apply lemma 3.5, its easily to see that *f(x)* and *g(x)* arecontinuously differentiable functions. Since $\epsilon < 0$ then condition (1) holds.

For second condition consider $f(x) = 3Ax^2 - 2(1 + \lambda)x + \lambda\Delta = 4(1 + \lambda)^2 - 12A\lambda > 0$ so f(x) has two singular points $x_1 < 0 < x_2$; and since $\lambda < 0$hen we deduce that condition (2) satisfied. For third one let $\frac{f(x)}{g(x)} = \frac{3Ax^2-2(1+\lambda)x+\lambda}{-\epsilon x}$ then $\left(\frac{f(x)}{g(x)}\right)' = \frac{-6\epsilon Ax^2+\epsilon\lambda}{\epsilon^2 x^2} > 0\ for\ all\ x.$ Thus condition (3) holds and the theorem is proved.

IV. A saddle case with limit cycle

In this section we study the saddle case with the following quadratic system

$$\begin{aligned} \dot{x} &= y + A(\lambda + 1)x^2 - \lambda Ax + I \\ \dot{y} &= \epsilon(x - \delta y), \end{aligned} \quad (4.1)$$

where $\in (0,1)$, $I \in R$, $(\lambda + 1) < 0$.

For studying limit cycles, we may transform the system to the following Lienard system

$$\begin{aligned} \dot{x} &= y - [-(\lambda + 1)x^2 + (-(\lambda + 1)\alpha + \lambda + \delta\epsilon)x] \\ \dot{y} &= -\delta\epsilon\left[-(\lambda + 1)x^2 + \left(-(\lambda + 1)\alpha + \lambda - \frac{1}{\delta}\right)x\right]. \end{aligned} \quad (4.2)$$

The system has two critical points with O as saddle, and C(c; 0) is an antisaddlesuch that

$$c = \frac{-2(\lambda + 1)\alpha + \lambda - \frac{1}{\delta}}{(\lambda + 1)}$$

Since $O(0;\ 0)$ as saddle, and $\delta\epsilon > 0$ then we have $-2(\lambda + 1)\alpha + \lambda - \frac{1}{\delta} < 0$,tostudy the existence of limit cycles we translate C to the origin, then we can find

$$\begin{aligned} \dot{x} &= y - \left[-(\lambda + 1)x^2 + \left(2(\lambda + 1)\alpha - \lambda + \delta\epsilon + \frac{2}{\delta}\right)x\right] \\ \dot{y} &= -\delta\epsilon\left[(\lambda + 1)x^2 - \left(-2(\lambda + 1)\alpha + \lambda - \frac{1}{\delta}\right)x\right]. \end{aligned} \quad (4.3)$$

Change $(x, t)\ to\ (-x, -t)$ then we get

$$\begin{aligned} \dot{x} &= y - \left[-(\lambda + 1)x^2 + \left(2(\lambda + 1)\alpha - \lambda + \delta\epsilon + \frac{2}{\delta}\right)x\right] \\ \dot{y} &= -\delta\epsilon\left[(\lambda + 1)x^2 - \left(-2(\lambda + 1)\alpha + \lambda - \frac{1}{\delta}\right)x\right]. \end{aligned} \quad (4.4)$$

The critical points of *F(x)* and *g(x)* respectively are

$$O(0,0);\ x_1 = \frac{-2(\lambda + 1)\alpha - \lambda + \delta\epsilon + \frac{2}{\delta}}{(\lambda + 1)}$$

$$O(0,0);\ x_2 = \frac{2(\lambda + 1)\alpha + \lambda - \frac{1}{\delta}}{(\lambda + 1)}$$

The _rst three focal values are [8]:

$$W_1 = \frac{2(\lambda + 1)\alpha - \lambda + \delta\epsilon + \frac{2}{\delta}}{(\lambda + 1)}$$
$$W_2 = -\frac{(\lambda + 1)^2\alpha + \lambda - \frac{1}{\delta}}{8\mu^2\sqrt{\mu}} \qquad (4.5)$$
$$W_3 = -0.$$

Since $W_2 < 0$ so for creating limit cycles W_1 must be positive, for instance see[6]. Thus, for $W > 0$ and from Hopf-bifurcation the system has stable limitcycle. Thus, for this situation we deduce that $0 < x_1 < x_2$ and *F(x)* has minimumvalue and *g(x)* has maximum value. For uniqueness of limit cycle we gotthe following theorem;

Theorem 4.1 The system (4.12) has unique limit cycle.

Proof Now we apply Lemma 3.5 for system (4.15), since *f(x)* and *g(x)* arepolynomials function, then these functions are continuously differentiable, andeasily to see that conditions 1 and 2 holds. For the third one and after simplifylet $\frac{f(x)}{g(x)} = \frac{-2(\lambda+1)x}{\delta\epsilon((\lambda+1)x^2+bx)}$ then $\left(\frac{f(x)}{g(x)}\right)' = \frac{2(\lambda+1)^2-2a(\lambda+1)x-ab}{((\lambda+1)x^2+bx)^2} = \frac{N(x)}{((\lambda+1)x^2+bx)^2}$.

Its enough to prove that $N(x) > 0, N(0) = -ab > 0$ and $\Delta = 4a(\lambda + 1)^2(a + 2b) = (-)(+) = -$. Thus _ < 0 and $N(x) > 0\ \forall x$. Hence the theorem is proved.

Concluding Remarks

A complete FitzHugh-Nagumo system with $I \neq 0$ is studied and analyzed in detail by adapting Hopf-bifurcation theory. It was shown that one or twolimit cycles bifurcates from the origin. Bendixons theorem is used to prove nonexistenceof limit cycles. Also we proved that the system has unique limit cycleunder some change of parameters.

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