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The Cauchy - Euler Differential Equation and Its Associated Characteristic Equation

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Abstract: *Many natural phenomena, from physics to biology, through the fields of medicine and engineering, can be described by means of differential equations. This article aims to present the solutions of a homogeneous Cauchy-Euler differential equation from the roots of the characteristic equation associated with this differential equation, in order to facilitate the life of the university student. The great algebraic difficulty that a graduate student encounters in solving the homogeneous Cauchy-Euler differential equations is that his solution depends on a polynomial equation of degree n called the characteristic equation. It is hoped that this work can contribute to minimize the lags in teaching and learning of this important Ordinary Differential Equation.*

Keywords: *Ordinary Differential Equation. Equation of Cauchy-Euler homogeneous. Characteristic equation.*

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I. Introduction

[1] Many of the principles, or laws, that govern the behavior of the physical world are propositions, or relationships, involving the rate at which things happen. Expressed in mathematical language, relations are equations and rates are derived. Equations containing derivatives are Differential Equations.

The ordinary differential equations $F\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}\right) = 0$ encompass a very broad area of mathematics and of fundamental importance to explain various models of everyday life. [1] [2] [3] Geometrically, the general solution of a differential equation represents a family of curves that are called the integral curves. This solution is called primitive or integral of the differential equation.

Solving a differential equation means finding an adequate family of curves. Many natural phenomena, from physics to biology, through the fields of medicine and engineering, can be described by means of these differential equations. The solutions of these equations are used, for example, to design automobiles, construct buildings and bridges, identify population growth, explain electric circuits, among many other applications. [4] [5] A single differential equation can serve as a mathematical model for several different phenomena.

The purpose of this paper is to present the solution of an Ordinary Differential Equation, called the Cauchy-Euler Equation from the roots of the Characteristic Equation associated with this differential equation. We first define the homogeneous Cauchy-Euler equation of order n. Then we will use the particular case, n = 2, to present its solutions, according to the roots of its characteristic equation. In the following section, several Equations Characteristics associated with their Cauchy-Euler Equations will be presented, and finally a conclusion will be reported.

II. The Equation of Cauchy - Euler homogeneous

[6] Leonhard EULER (1708-1783), physicist, mathematician, German-speaking Swiss geometer had as his advisor the mathematician Johann Bernoulli. He developed several works in the area of Calculus and Graph Theory and spent much of his life in St. Petersburg and Berlin. He wrote several works, among them, Complete Treaty of Mechanics, Institutions of Integral Calculus and Introduction to Analysis of Infinitesimals.

[6] Augustin Louis CAUCHY (1789-1857), French mathematician, was the creator of Theory of Analytical Functions and the Method to determine the number of real roots. He created the modern notion of continuity for the functions of real or complex variable.

The Equation of Cauchy - Euler homogeneous of nth order is any Ordinary Differential Equation of the form:

$$A_n x^n \frac{d^n y}{dx^n} + A_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + A_2 x^2 \frac{d^2 y}{dx^2} + A_1 x \frac{dy}{dx} + A_0 y = 0 \ , \qquad (1)$$

Where $A_n, A_{n-1}, \cdots, A_2, A_1$ and A_0 are constant.

In this case the solution will be some function whose first derivative multiplied by x, the second derivative multiplied by x^2, the third derivative multiplied by x^3, and so on, up to the nth derivative multiplied by x^n being linearly dependent on the original function.
The function that has this property is the function:

$$y = x^m \ , \qquad (2)$$

in which m it is a real constant whatever.

III. Study of the Solutions of the Cauchy-Euler Equation of the Type $Ax^2 \frac{d^2y}{dx^2} + Bx\frac{dy}{dx} + Cy = 0$

[4] Over time, mathematicians have crafted ingenious methods to solve some very specialized equations; therefore, there are, not surprisingly, a large number of differential equations that can be solved analytically.

To solve the Cauchy - Euler equation of type $Ax^2 \frac{d^2y}{dx^2} + Bx\frac{dy}{dx} + Cy = 0$ (1), we must look for a solution of the form $y = x^m$ (2). Substituting (2) into (1), we have:

$$Ax^2 m(m-1)x^{m-2} + Bxmx^{m-1} + Cx^m = 0$$

Hence, $Am(m-1) + Bm + C = 0 \ \therefore \ Am^2 + (B-A)m + C = 0$ is called the characteristic equation associated with the Cauchy - Euler equation of the type $Ax^2 \frac{d^2y}{dx^2} + Bx\frac{dy}{dx} + Cy = 0$. The solution of this Ordinary Differential Equation depends on the roots of its associated Equation. The solution of the higher-order equation follows analogously. Note that there are three cases for the solution of the characteristic equation:

Case 1: If the roots are real and distinct.
Suppose that α and β are the roots of the characteristic equation, then the solution of the Cauchy-Euler equation is of the form:

$y = C_1 x^\alpha + C_2 x^\beta$, where C_1 and e C_2 are arbitrary constants.

Case 2: If the roots are real and equal.
Suppose that α = β are the roots of the characteristic equation, then the solution of the Cauchy - Euler equation is of the form:

$y = C_1 x^\alpha + C_2 x^\alpha \ln x$, where C_1 and e C_2 are arbitrary constants.

Case 3: If the roots are complex and conjugate.

Suppose that are the roots of the characteristic equation, then the solution of the Cauchy - Euler equation is of the form:

$y = x^\alpha [C_1 \cos(\ln(x^\beta)) + C_2 sen(\ln(x^\beta))]$, where C_1 and e C_2 are arbitrary constants.

IV. Several Equations Characteristics associated with the Cauchy-Euler Equation and Examples.

In this section, for each homogeneous Equation of Cauchy - Euler of nth order (Table I), we will present, respectively, its characteristic Equation that will be a polynomial equation of degree *n*. Then three examples will be discussed, one for each case.

Table I - The Cauchy - Euler equation homogeneous associated with its characteristic equation (until fifth order)

$Ax\frac{dy}{dx}+By=0 \rightarrow Am+B=0$
$Ax^2\frac{d^2y}{dx^2}+Bx\frac{dy}{dx}+Cy=0 \rightarrow Am^2+(B-A)m+C=0$
$Ax^3\frac{d^3y}{dx^3}+Bx^2\frac{d^2y}{dx^2}+Cx\frac{dy}{dx}+Dy=0 \rightarrow Am^3+(B-3A)m^2+(2A-B+C)m+D=0$
$Ax^4\frac{d^4y}{dx^4}+Bx^3\frac{d^3y}{dx^3}+Cx^2\frac{d^2y}{dx^2}+Dx\frac{dy}{dx}+Ey=0 \rightarrow$ $Am^4+(B-6A)m^3+(11A-3B+C)m^2+(2B-6A-C+D)m+E=0$
$Ax^5\frac{d^5y}{dx^5}+Bx^4\frac{d^4y}{dx^4}+Cx^3\frac{d^3y}{dx^3}+Dx^2\frac{d^2y}{dx^2}+Ex\frac{dy}{dx}+Fy=0 \rightarrow$ $Am^5+(B-10A)m^4+(35A-6B+C)m^3+(11B-50A-3C+D)m^2+(24A-6B+2C-D+E)m+F$

Example 1: Given the Cauchy-Euler equation: $5x\frac{dy}{dx}-3y=0$

Equation Characteristic: $5m-3=0$

Raízes da Equação Característica: $m=\frac{3}{5}$

Solution: $y=C_1x^{-\frac{3}{5}}$

Example 2: Given the Cauchy-Euler equation: $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-2y=0$

Equation Characteristic: $m^2+m+C=0$

Raízes da Equação Característica: $\begin{cases} m_1=-2 \\ m_2=1 \end{cases}$

Solution: $y=C_1x^{-2}+C_2x$

Example 3: Given the Cauchy-Euler equation: $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0$

Equation Characteristic: $m^3-m^2-m+1=0$

Raízes da Equação Característica:

$$\begin{cases} m_1=-1 \\ m_2=1 \\ m_3=1 \end{cases}$$

Solution: $y=C_1x^{-1}+C_2x+C_3x\ln x$

Example 4: Be $x^4\frac{d^4y}{dx^4}+3x^3\frac{d^3y}{dx^3}+4x^2\frac{d^2y}{dx^2}-8x\frac{dy}{dx}+8y=0$ an Equation of Cauchy - Euler.

Equation Characteristic: $m^4-3m^3+6m^2-12m+8=0$

Raízes da Equação Característica:

$$\begin{cases} m_1 = 1 \\ m_2 = 2 \\ m_3 = -2i \\ m_4 = 2i \end{cases}$$

Solution: $y = C_1 x + C_2 x^2 + C_3 \cos(\ln(x^2)) + C_4 sen(\ln(x^2))$

Example 5: Be $x^5 \dfrac{d^5 y}{dx^5} + 8x^4 \dfrac{d^4 y}{dx^4} + 23x^3 \dfrac{d^3 y}{dx^3} + 30x^2 \dfrac{d^2 y}{dx^2} + 10x \dfrac{dy}{dx} - 10y = 0$ an Equation of Cauchy - Euler.
Equation Characteristic: $m^5 - 2m^4 + 10m^3 - m^2 + 2m - 10 = 0$

Raízes da Equação Característica:

$$\begin{cases} m_1 = 1 \\ m_2 = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2} i \\ m_3 = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2} i \\ m_4 = 1 - 3i \\ m_5 = 1 + 3i \end{cases}$$

Solution: $y = C_1 x + x^{-\frac{1}{2}}[C_2 \cos(\ln(x^{\frac{\sqrt{3}}{2}})) + C_3 sen(\ln(x^{\frac{\sqrt{3}}{2}})] + x[C_4 \cos(\ln(x^3)) + C_5 sen(\ln(x^3))]$

V. Conclusion

This article was developed from a proposal to present the solution of a homogeneous Cauchy-Euler Equation from its associated characteristic equation. The development of the solution set of certain ordinary differential equations still remains the object of research, with attractive problems and high applicability in the phenomena of nature.

It is evident the difficulty encountered by students to establish a relation of interest with the area of calculation, particularly in Differential Equations, perhaps because they do not know the wide field of application that these equations make available. In the light of the above, it is expected that this work may significantly awaken other research on Cauchy-Euler Equation in order to minimize the lags between mathematical abstraction and its practice.

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