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Vanishing Viscosity Method And Nonlinear Hyperbolic Conservation Laws

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Abstract: *A conservation law states that a particular measurable property of an isolated physical system does not change as the system evolves. In particular, any change in such a conserved quantity can only occur as a result of an "influx" or an "outflow" of this quantity into or out of the system. In this paper, we study a class of first order PDEs that may serve as mathematical descriptions of physical conservation laws, such as the laws of gas dynamics and the laws of electromagnetism. We show how the viscosity method, known as the vanishing viscosity method, can be used to construct the entropy solution of the scalar conservation law as the limit of solutions of the parabolic equations with viscous term.*

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I. Introduction

The mathematical models for real-world problems occurring in Physics, Chemistry, Economics, Engineering and Biology are usually expressed in the form of partial differential equations (PDEs). However, it is well known that an initial value problem for a nonlinear PDE may fail to have a classical solution on the whole domain of definition of the equation. In fact, even a linear equation without initial conditions may fail to have a solution, as shown by the following example due to Lewy [68]. Lewy showed that there are C^∞-smooth functions f for which the equation

$$A(u) = f$$

Where $A(u) = u_x + iu_y - 2i(x+iy)u_t \quad (x,y,t) \in \mathrm{R}^3$

has no solution in D'-distributions in any neighborhood of any point.

In this paper, we study a class of first order PDEs that may serve as mathematical descriptions of physical conservation laws, such as the laws of gas dynamics and the laws of electromagnetism. A conservation law states that a particular measurable property of an isolated physical system does not change as the system evolves. In particular, any change in such a conserved quantity can only occur as a result of an "influx" or an "outflow" of this quantity into or out of the system.

The exact mathematical model for a single conservation law in one spatial dimension is given by the first order PDE

$$u_t + (f(u))_x = 0. \qquad (1)$$

Here u is the conserved quantity while f is the flux. Integrating equation (1) over some interval $[a,b]$ leads to

$$\frac{d}{dt}\int_a^b u(x,t)dx = \int_a^b u_t(x,t)dx = -\int_a^b f(u(x,t))_x dx = f(u(x,a)) - f(u(x,b))$$
$$= [\, in \;\; flow\, at\; a] - [\, out \;\; flow\, at\; b]$$

In other word, the quantity u is neither created nor destroyed. In particular, the total amount of u contained in the interval $[a,b]$ can only change due to the flow of u across the two endpoints. In general, if $u = (u_1, \cdots u_k)$ is a vector of conserved quantities, depending on time t and n independent variables $x_1, \cdots, x_n$, then the flux of u out of any bounded region $\Omega \subseteq \mathrm{R}^n$ is given by

$$\int_{\partial\Omega} F(u)\cdot n dS.$$

Here $F:\mathbb{R}^n \to \mathbb{M}^{n\times k}$ is the flux, n denotes the outward unit normal to $\partial\Omega$ and dS the surface element on $\partial\Omega.$ Since any change in u in such a domain Ω over time can only be due to the 'in flow' or 'out flow' of u into or out of $\Omega,$ it follows that

$$\frac{d}{dt}\int_{\Omega} u dx = -\int_{\partial\Omega} F(u)\cdot n dS. \qquad (2)$$

Note that the integral on the right of (2) measures the flow out of $\Omega,$ hence the minus sign. Assuming that F, u and $\partial\Omega$ are sufficiently smooth, we may apply the Divergence Theorem to equation (2) so that

$$\frac{d}{dt}\int_{\Omega} u dx = -\int_{\Omega} \nabla\cdot F(u) dx. \qquad (3)$$

Taking the derivatives with respect to t under the integral sign we obtain

$$\int_{\Omega} [u_t + \nabla\cdot F(u)] dx = 0. \qquad (4)$$

Assuming that u and F are sufficiently smooth, the Mean Value Theorem implies the differential form of conservation laws, which is given by

$$u_t + \nabla\cdot F(u) = 0. \qquad (5)$$

This report is a survey of the extensive literature on conservation laws. We shall be concerned mainly with the Cauchy problems for *strictly hyperbolic* systems in one spatial dimension. That is,

$$u_t + (F(u))_x = 0 \quad in \ \mathbb{R}\times(0,\infty) \qquad (6)$$

$$u(x,0) = u_0(x) \quad x\in\mathbb{R}, \qquad (7)$$

$u_0 = (u_0^1,\cdots,u_0^k)$ is the initial value of $u.$ If $A(u) = J_u F(u)$ is the $n\times n$ Jacobian matrix of the function F at the point $u,$ the system (6) can be written in the form

$$u_t + A(u)u_x = 0. \qquad (8)$$

We say that a system of conservation laws is *strictly hyperbolic* if the matrix $A(u)$ has n real, distinct eigenvalues, say

$$\lambda_1(u) < \cdots < \lambda_n(u). \qquad (9)$$

for every $u.$

1.1 Examples of Conservation laws

As mentioned, equation of conservation laws such as (6) may appear as mathematical models for certain real-world phenomena. In particular, such equations appears as precise mathematical description of physical conservation laws. In this section we mention a few examples of conservation laws that arise in applications.

Example 1.1: [Traffic Flow] Let $u(x,t)$ denote the density of cars on a highway at point x at time $t.$ For example, u may be the number of cars per kilometre. Assume that u is continuous and that the speed s of cars depend only on their density, that is, $s = s(u).$ We also assume that the speed s of the cars decreases as the density u increases that is $\frac{ds}{du} < 0.$ Given any two points a and b on the highway, the number of cars between a and b therefore varies according to the law

$$\frac{d}{dt}\int_a^b u(x,t) dx = -\int_a^b [s(u)u]_x dx. \qquad (10)$$

Since (10) holds for all $a, b\in\mathbb{R}$ this leads to the conservation law

$$u_t + [s(u)u]_x = 0 \qquad (11)$$

Here the flux is given by $f(u) = s(u)u.$ In practice the flux f is often taken to be

$$f(u) = a_1(\ln(\frac{a_2}{u}))u, \quad 0 < u < a_2,$$

for suitable constants a_1 and a_2.

Example 1.2 [The p-system] The p - system is a simple model for isentropic (constant entropy) gas dynamics. If v is the specific volume and u the velocity of the gas, then the equations are given as

$$v_t - u_x = 0 \qquad (12)$$

$$u_t + (p(v))_x = 0 \qquad (13)$$

The flux p is is given as

$$p(v) = kv^{-\lambda}, \; k \geq 0, \lambda \geq 1 \qquad (14)$$

where k and λ are constants. In applications λ is chosen such that $\lambda \in [1,3]$ for most gases; in particular $\lambda = \frac{7}{5}$ for air. In the region $v > 0,$ the system is strictly hyperbolic. Indeed

$$A = \mathrm{J}\, F = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}$$

has real distinct eigenvalues $\lambda = \pm\sqrt{-p'(v)}.$

Differentiating equation (12) with respect to t and equation (13) with respect to x gives

$$v_{tt} + p(v)_{xx}. \qquad (15)$$

Approximating p by a linear function $p(v) \approx p(v_0) - c^2(v - v_0)$ in a neighborhood of a given state v_0, equation (15) reduces to the wave equation

$$v_{tt} - c^2 v_{xx},$$

where c is a constant.

Example 1.3: [Gas dynamics] The Euler equations for the dynamics of a compressible, non-viscous gas is given by

$$v_t - u_x = 0 \quad (conservation\ of\ mass)$$

$$u_t + p_x = 0 \quad (conservation\ of\ momentum)$$

$$(v + \frac{u^2}{2})_t + (pu)_x = 0 \quad (conservation\ of\ energy).$$

Here $v = \rho^{-1},$ where ρ is the density and v is the specific volume. The velocity in the gas is $u,$ while ν is the internal energy and p the pressure. The system is closed by an additional equation $p = p(\nu, v)$ called the equation of state, which depend on the particular gas under consideration.

II. Scalar Conservation Laws

In this section we consider the initial value problem for scalar conservation laws in one spatial dimension

$$u_t + (F(u))_x = 0\, in \;\; \mathrm{R} \times (0, \infty) \; (16)$$

$$u(x,0) = u_0(x) \quad x \in \mathrm{R}. \; (17)$$

Here $u : \mathrm{R} \times [0, \infty) \to \mathrm{R}$ is the unknown conserved quantity, $F \in C^\infty(\mathrm{R})$ is the flux and $u_0 : \mathrm{R} \to \mathrm{R}$ is the initial condition.

When solving the Cauchy problem (16) - (17), one is typically confronted with the following difficulties: Even in the case of a C^∞ - smooth initial condition u_0, the initial value problem (16) - (17) may not have a classical solution on the whole domain of definition of the equation (16). Indeed solutions of (16) - (17) may develop discontinuities after a finite time.

2.1 Classical Solutions

A *classical solution* of the Cauchy problem (16) - (17) is a continuously differentiable function satisfying equation (16) - (17). One can obtain the classical solution of equation (16) - (17) by the method of characteristics. To do this, let the flux function $F \in C^{\infty}(\mathbb{R})$ be given, and assume that equation (16) - (17) is genuinely nonlinear. That is

$$F''(u) > 0 \quad \text{for all } u. \qquad (18)$$

If $u \in C^1(\mathbb{R} \times [0,\infty))$ is a solution of the Cauchy problem, then we define the characteristic curves in $\mathbb{R} \times [0,\infty)$ as the level curves of u. That is, for any $y \in \mathbb{R}$ the characteristic curve through the point $(y,0)$ consists of the set of points where $u(x,t) = u(y,0) = u_0(y)$. At every point (x,t) on the characteristic curve through $(y,0)$, (16) and (17) imply that

$$\nabla u(x,t) \cdot \langle F'(u_0(y)),1\rangle = 0.$$

Therefore

$$\langle 1, -F'(u_0(y))\rangle$$

is tangent to the curve at every point. So that, the characteristic through $(y,0)$ is a straight line with equation

$$x(t) = y - tF'(u_0(y)).$$

Since

$$u(x,t) = u_0(y)$$

for every point (x,t) on the curve, we may express the solution of (16) to (17) *implicitly* as

$$u = u_0(x + tF'(u)).$$

The Implicit Function Theorem may now be used to solve for u. The classical solution of (16) - (17) found above is unique, but may fail to exist for all $t > 0$ as the following theorem shows.

Theorem 2.1[90, Proposition 2.1.1]. Assume that $u_0 \in C^1(\mathbb{R})$, together with its derivative is bounded in $\mathbb{R}$. Set

$$T^* = \begin{cases} +\infty, & \text{if } F'(u_0) \text{ is an increasing function} \\ -(\inf \frac{d}{dx} F'(u_0))^{-1}, & \text{otherwise.} \end{cases} \qquad (19)$$

Then (16) - (17) has a unique solution $u \in C^1(\mathbb{R} \times (0,T^*))$. For $T > T^*$, (16) - (17) has no classical solutions on $\mathbb{R} \times [0,T)$.

Example 2.1: Consider the initial value problem for Burger's equation

$$u_t + (\frac{u^2}{2})_x = 0 \text{ in } \mathbb{R} \times (0,\infty) \qquad (20)$$

$$u(x,0) = u_0(x) \quad x \in \mathbb{R}. \quad (21)$$

Using the method of characteristics discussed above we see that for a C^1-smooth function u_0, a classical solution u is given by the implicit equation

$$u(x,t) = u_0(x - tu(x,t)), \ t \geq 0, \ x \in \mathbb{R}. \qquad (22)$$

By the Implicit Function Theorem, we can obtain $u(y,s)$ from (22) for y and s in suitable neighborhoods of x and t respectively, whenever

$$1 + tu_0'(x - tu(x,t)) \neq 0. \qquad (23)$$

If $u_0'(x) \geq 0$ for all $x \in \mathbb{R}$, then condition (23) is clearly satisfied for all (x,t), so that the Cauchy problem (20) - (21) has a unique solution on $\mathbb{R} \times (0,\infty)$. However, if for some point $x_0 \in \mathbb{R}$

$$u_0'(x_0) < 0,$$

then for certain values of $t > 0$, the condition (23) may fail, irrespective of the domain or degree of smoothness of u_0. This violation of condition (23) implies that the classical solution u fails to exists for the respective values of t and x. Thus, for certain $x \in \mathbb{R}$ the solution $u(x,t)$ does not exits for sufficiently large $t > 0$.

If we take

$$u_0(x) = \begin{cases} 1 & if\ x \le 0 \\ 1-x & if\ 0 \le x \le 1 \\ 0 & if\ x \ge 1. \end{cases} \quad (24)$$

then the unique classical solution of (20) - (21) is given by

$$u(x,t) = \begin{cases} 1 & if\ x < t \\ \dfrac{1-x}{1-t} & if\ t \le x \le 1, \qquad t < 1 \\ 0 & if\ x \ge 1. \end{cases}$$

Thus the classical solution of (20) - (21), with u_0 as in (24), breaks down at $t = 1.$ W

It should be noted that the breakdown of the solution $u(x,t)$ at $t = 1$ for initial data u_0 given in (24) is not due to the lack of smoothness of $u_0,$ but to the fact that $u_0' = -1 < 0$ for $x \in [0,1].$ In view of the nonexistence of global classical solutions, one is forced to consider suitable generalized solutions of (16) - (17).

2.2 Weak solutions and non-uniqueness

One well known and much studied generalized formulation of (16) - (17) is the weak form of (16) - (17). Let us assume temporarily that u is a smooth solution of (16) - (17). The idea is to multiply equation (16) with a smooth function ϕ and integrate by parts. More precisely, let ϕ be a test function, that is,

$$\phi : \mathrm{R} \times [0,\infty) \to \mathrm{R} \quad (25)$$

has compact support and is C^∞ - smooth. We denote the set of all such test functions by $C_0^\infty(\mathrm{R} \times [0,\infty)).$
Multiply equation (16) by ϕ and integrate by parts to get

$$0 = \int_0^\infty \int_{-\infty}^\infty (u_t + (F(u))_x)\phi dxdt = -\int_0^\infty \int_{-\infty}^\infty u\phi_t dxdt - \int_0^\infty \int_{-\infty}^\infty F(u)\phi_x dxdt - \int_{-\infty}^\infty u\phi\,|_{t=0}\ .$$

In view of the initial condition u_0, we obtain

$$\int_0^\infty \int_{-\infty}^\infty u\phi_t + F(u)\phi_x dxdt + \int_{-\infty}^\infty u_0\phi\,|_{t=0}\ dx = 0. \quad (26)$$

In contradistinction with equations (16) - (17), equation (26) does not involve any derivative of $u,$ thus equation (26) make sense not only for smooth functions, but also for bounded and measurable functions u and u_0. We thus have the following definition of a weak solution of (16) -(17).

Definition 2.2: We say that $u \in L_\infty(\mathrm{R} \times (0,\infty))$ is a *weak solution* of (16)-(17) if the equation (26) holds for each test function $\phi \in C_0^\infty(\mathrm{R} \times [0,\infty)).$

If u is a weak solution and $u \in C^1(\mathrm{R} \times (0,\infty))$ then u satisfies (16) - (17). That is, a regular weak solution is also a classical solution of equation (16) - (17). Thus the concept of weak solution of (16) - (17) is a generalization of the classical notion of solution.

Remark: Equation (16) can also be written in the form:

$$u_t + a(u)u_x = 0, \qquad with\ a(u) = F'(u). \quad (27)$$

At the level of classical solutions, equations (16) and (27) are equivalent. That is, $u \in C^1(\mathrm{R} \times [0,\infty))$ is a solution of (16) if and only if u is a solution of (27). However, if u has a discontinuity, then the left hand side of equation (27) may contain a product of a discontinuous function $a(u)$ with the distributional derivative u_x. Such a product is typically not well defined, see for instance [88]. Working with the equation in the form of (16) allows for the consideration of weak solutions as defined in Definition 2.2.

One difficulty that arises in the study of weak solutions of (16) - (17) is related to the uniqueness of such solutions. In contradistinction with classical solution of (16) - (17), weak solutions are not unique as shown in the following example:

Example 2.3 Consider the Burgers equation (20) with the initial data

$$u_0(x) = \begin{cases} 0 & if \;\; x < 0 \\ 1 & if \;\; x > 0. \end{cases} \quad (28)$$

For every $\alpha \in [0,1]$ the function

$$u_\alpha(x,t) = \begin{cases} 0 & if \;\; x < \frac{\alpha}{2}t \\ \alpha & if \;\; \frac{\alpha}{2}t \le x \le \alpha t \\ \frac{x}{t} & if \;\; \alpha t \le x \le t \\ 1 & if \;\; x > t \end{cases} \quad (29)$$

is a weak solution of (20) - (21) with u_0 as in (28).

The underlying physical laws that are modeled as mathematical conservation laws are deterministic in nature. That is, the future state of a system that evolves according to (16) is completely determined by the initial condition (17). From this point of view, the non uniqueness of weak solutions of (16) - (17), as demonstrated in Example 2.2, is unacceptable. In particular, in the context of physical systems that may be modeled through (16) - (17), the non uniqueness of weak solutions of the Cauchy problem may be interpreted as follows: The state of the system at time $t > 0$ is not completely determined by the weak formulation of (16) - (17) alone. Therefore further additional condition, motivated by physical consideration, must be imposed on the weak solutions of (16) - (17) in order to obtain the unique solution that describes the evolution of the underlying physical system.

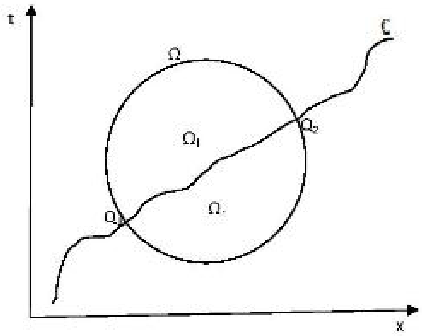

Figure 1:

In this regard, let u be a weak solution of (16) - (17). Assume that u has continuous first order partial derivatives everywhere in the open set $\Omega \subseteq \mathrm{R} \times [0,\infty)$ except on a smooth curve C in Ω with equation $x = x(t)$. That is u has a jump discontinuity across C. Let Ω_l and Ω_r be the parts of Ω on the left and on the right of curve C respectively, see Figure 1.

Furthermore, since u is smooth on either side of the curve C, it is smooth in Ω_l and Ω_r. u is a weak solution of (16) - (17), we have

$$\iint_\Omega u\phi_t + (F(u))\phi_x dxdt = 0, \quad (30)$$

for all $\phi \in C_0^\infty(\Omega)$. Thus, if $supp\phi \subset \Omega_r$, then

$$0 = \iint_\Omega u\phi_t + (F(u))\phi_x dxdt = -\iint_{\Omega_r} [u_t + (F(u))_x]\phi dxdt. \quad (31)$$

which implies

$$u_t + (F(u))_x = 0 \quad in\, \Omega_r. \quad (32)$$

Similarly,

$$u_t + (F(u))_x = 0 \quad in\, \Omega_l. \quad (33)$$

From (30) we get

$$0 = \int_{\Omega}\int u\phi_t + F(u)\phi_x dxdt = \int_{\Omega_l}\int u\phi_t + F(u)\phi_x dxdt + \int_{\Omega_r}\int u\phi_t + F(u)\phi_x dxdt. \quad (34)$$

Now using the fact that u is C^1 in Ω_r and Green's Theorem we find that

$$\int_{\Omega_r}\int (u\phi_t + F(u)\phi_x)dxdt = \int_{\Omega_r}\int [(u\phi)_t + (F(u)\phi)_x]dxdt = \int_{\partial\Omega_r} (-u\phi)dx + (F(u)\phi)dt$$

$$= \int_{\partial\Omega} (-u\phi)dx + (F(u)\phi)dt + \int_{C} (-u\phi)dx + (F(u)\phi)dt$$

Since $\phi = 0$ on $\partial\Omega,$ then denoting by u_r the right limit of u on the curve $C,$ we have

$$\int_{\Omega_r}\int (u\phi_t + F(u)\phi_x)dxdt = \int_{C} (-u_r\phi)dx + (F(u_r)\phi)dt. \quad (35)$$

Similarly,

$$\int_{\Omega_l}\int (u\phi_t + F(u)\phi_x)dxdt = -\int_{C} (-u_l\phi)dx + (F(u_l)\phi)dt. \quad (36)$$

Here, u_l is the left limit of u on the curve $C.$ Substituting equations (35) and (36) into equation (34) we have

$$0 = \int_{C} (-u_l + u_r)\phi dx + (F(u_l) - F(u_r))\phi dt = \int_{C} \phi[-(u_l - u_r)dx + (F(u_l) - F(u_r))dt] \quad (37)$$

which further implies

$$[-(u_l - u_r)dx + (F(u_l) - F(u_r))dt] = 0.$$

This implies

$$(u_l - u_r)\frac{dx}{dt} = (F(u_l) - F(u_r))$$

in Ω along the curve $C,$ which may be expressed as

$$(F(u_l) - F(u_r)) = \dot{x}(u_l - u_r).$$

We write this as

$$\rho[[u]] = [[F(u)]], \quad (38)$$

where $[[u]] = u_l - u_r$ is the jump in u across the curve $C,$ $[[F(u)]] = F(u_l) - F(u_r)$ is the jump in $F(u)$ and $\rho = \frac{dx}{dt}$ is the speed of curve $C.$ Relation (38) is known as the *jump condition*.

Example 2.4 Applying the jump condition to the Burgers' equation (20) where $F(u) = \frac{1}{2}u^2,$ we find that the speed of propagation of the discontinuities is $\frac{dx}{dt} = \rho = \frac{1}{2}(u_l + u_r).$ Again, applying the jump condition to the solutions u_α of Example 2.2, we see that only the solution for which $\alpha = \frac{1}{2}$ satisfies the jump condition.

A natural question is whether this is the unique solution of equation (27) with initial condition (28) that satisfies equation (38). More generally, can there be more than one weak solution of the Cauchy problem (16) - (17) satisfying the jump condition (38)? To give an answer to this question we consider the following example.

Example 2.5: Consider the initial value problem of the Burgers equation (20) - (21) with initial condition

$$u_0(x) = \begin{cases} 0 & if \ x < 0 \\ 1 & if \ x \geq 0. \end{cases} \quad (39)$$

For every $\alpha \in (0,1),$ the function u_α defined as

$$u_\alpha(x,t) = \begin{cases} 0 & \text{if } x < \frac{\alpha t}{2}, \\ \alpha & \text{if } \frac{\alpha t}{2} \le x < \frac{(1+\alpha)t}{2} \\ 1 & \text{if } x \ge \frac{(1+\alpha)t}{2}, \end{cases}$$

is a weak solution of the initial value problem. By the jump condition, $\rho = \frac{\alpha}{2}$ and $\rho = \frac{(1+\alpha)}{2}$ along the lines of discontinuity $x = \frac{\alpha t}{2}$ and $x = \frac{(1+\alpha)t}{2}$ respectively for each $\alpha \in (0,1)$. Thus, the jump condition alone is not sufficient to determine the unique, physically relevant solution of the Cauchy problem (16) - (17).

2.3 Admissibility conditions and the Entropy Condition

From Example 1, it is clear that the class of weak solutions may include various solutions which we are not physically relevant or desirable. In order to single out a *unique* solution that is of physical and/or mathematical relevance, suitable additional requirements, which we shall call *admissibility conditions*, are imposed on such solutions, see for instance [38], [64]. These admissibility conditions, such as *entropy conditions*, are typically motivated by some physical considerations. In the literature, various admissibility conditions have been introduced. In this section, we introduce some of these conditions.The main results in which these admissibility conditions are employed to single out the unique, physically relevant solution to the Cauchy problem (16) -(17) is also discussed.

2.3.1 Admissibility condition 1 (The Oleinik inequality)

Oleinik [82] introduced the Lipschitz (with respect to x) condition for a genuinely nonlinear single conservation law (16) given by

$$\frac{u(x+a,t)-u(x,t)}{a} \le \frac{E}{t}. \qquad a > 0, \qquad t > 0. \qquad (40)$$

Here $E = \frac{1}{\inf F''}$ is independent of $x, t,$ and $a.$ Using the Lax-Friedreich finite difference scheme, Oleinik showed that if F is convex (which implies $F'' > 0$), there exists precisely one weak solution of the Cauchy problem (16) - (17) satisfying (40). Note that the weak solutions of (16) - (17) that satisfies (40) will, for $t > 0$ have x -difference quotient bounded from above. Note that as t tends to zero, the upper bound may tend to plus infinity.

The Oleinik inequality (40) was motivated by the fact that if $u_0' \ge 0,$ a classical solution u of (16) - (17) exists with

$$u_x = \frac{u_0'}{1+tF''(u_0)u_0'} \qquad .$$

So that

$$u_x \le \frac{1}{tF''(u_0)} \le \frac{K}{t}, \ t > 0 \ \textit{for some constant } K > 0$$

which is an infinitesimal version of the Oleinik inequality (40). It is therefore reasonable for a solution of the Cauchy problem (16) - (17) to satisfy the inequality (40). However, typical nonclassical weak solutions of the Cauchy problem (16)-(17) may not have usual partial derivatives at all points $(x,t).$ To avoid the derivatives of weak solution and still show that u satisfies (40), one make use of finite difference scheme. The basic idea of the finite difference scheme in PDE is to replace derivatives with appropriate finite differences. The main result, concerning solutions satisfying (40), which is also found in [92], is given below.

Theorem 2.6 [[92], Theorem 16.1] Let $u_0 \in L_\infty(\mathbb{R}),$ and let $F \in C^2(\mathbb{R})$ with $F'' > 0$ on $\{u : |u| \le \|u_0\|_\infty\}.$ Then there exists exactly one weak solution u of (16)-(17) satisfying the following:

there exists a constant $E > 0,$ depending only on M, μ and $A,$ such that for every $a > 0,$ $t > 0,$ and

$x \in \mathbb{R},$ the inequality

$$\frac{u(x+a,t)-u(x,t)}{a} < \frac{E}{t}. \quad (41)$$

holds. Here, $M \equiv \|u_0\|_{L_\infty}$, $\mu = \min\{F''(u) : |u| \le \|u_0\|_\infty\}$ and $A = \max\{|F'(u)| : |u| \le \|u_0\|_\infty\}$. Furthermore,

1. $|u(x,t)| \le M, (x,t) \in \mathbb{R} \times [0,\infty).$
2. u is stable and depends continuously on u_0 in the following sense: If $u_0, v_0 \in L_\infty(\mathbb{R}) \cap L_1(\mathbb{R})$ with $\|v_0\|_\infty \le \|u_0\|_\infty$, and v is the solution of (16) with initial data v_0 satisfying (41), then for every $x_1, x_2 \in \mathbb{R}$, with $x_1 < x_2$ and every $t > 0$,

$$\int_{x_1}^{x_2} |u(x,t)-v(x,t)|\, dx \le \int_{x_1-At}^{x_2-At} |u_0(x)-v_0(x)|\, dx. \quad (42)$$

Remark 2.7 (i) An immediate consequence of (41) is that for any $t > 0,$ the solution $u(\cdot,t)$ is of locally bounded total variation, that is $u \in BV_{loc}$, which means the total variation of u is bounded in every compact subset of $\mathbb{R} \times [0,\infty).$ To see this, let us define a function

$$v(x,t) = u(x,t) - \frac{E}{t}x.$$

Then if $a > 0$ (41) implies

$$v(x+a,t)-v(x,t) = u(x+a,t)-u(x,t)-\frac{E}{t}a < 0.$$

That is, v is a decreasing function and thus has locally bounded total variation. Hence, u is of locally bounded total variation since the function cx is also of locally bounded total variation. Thus even though u_0 is only L_∞, the solution $u(\cdot,t)$ is fairly regular. In fact, we can conclude that it has at most a countable number of jump discontinuities, and it is differentiable almost everywhere.

(ii) Oleinik result, Theorem 2.3.1, is limited to single conservation laws in one spatial dimension. An analogue of the Oleinik inequality (41) has not been found for systems of conservation laws.

(iii) The Oleinik inequality (41) implies that $u_l > u_r$ as we move across a curve of discontinuity. To see this, note that the function $v(x,t) = u(x,t) - \frac{E}{t}x$ is bounded in a domain $(x_1, x_2) \times [0,\infty)$ containing the line of discontinuity. Then v has left and right limits at each point since it is decreasing as noted earlier. Consequently $u(x,t)$ has left and right limits at each point. For any point c on the line of discontinuity we have

$$u_r - u_l = \lim_{x\to c^+} u(x,t) - \lim_{x\to c^+} \frac{E}{t}x - \lim_{x\to c^-} u(x,t) + \lim_{x\to c^-} \frac{E}{t}x = \lim_{x\to c^+} v(x,t) - \lim_{x\to c^-} v(x,t) < 0.$$

Which implies $u_l > u_r$ as we move across a curve of discontinuity.

2.3.2 Admissibility condition 2 (The Lax inequality)

The inequality

$$F'(u_l) > \rho > F'(u_r) \qquad \text{for all } t > 0. \quad (43)$$

was introduced by Lax [63]. The inequality (43) implies that the characteristics starting on either sides of the curve of discontinuity should intersect each other on the curve. At this point of intersection, u has two values which is impossible, so that there is a jump discontinuity at that point. Indeed, if $u_0' < 0,$ there are two points $y_1, y_2 \in \mathbb{R}$ such that $y_1 < y_2$ and $u_1 = u_0(y_1) > u_0(y_2) = u_2.$ If (43) holds then $F'(u_0(y_1)) > F'(u_0(y_2))$ so that the characteristics drawn from points $(y_1, 0)$ and $(y_2, 0)$ intersect at the point when $t = \dfrac{y_2 - y_1}{F'(u_0(y_1)) - F'(u_0(y_2))}$ with u having values u_1 and u_2 at that point.

The Lax inequality can be obtain from the Jump condition. To see this, let F be a convex function, then $F'' > 0$ which implies F' is increasing so that if $u_l > u_r$, then

$$F'(u_l) > F'(u_r).$$

By the Mean Value Theorem there exists $\zeta \in [u_r, u_l]$ such that

$$F'(\zeta) = \frac{F(u_l) - F(u_r)}{u_l - u_r} = \rho.$$

Since F' is increasing we have that

$$F'(u_l) > F'(\zeta) > F'(u_r),$$

which leads to the Lax inequality (43).

If all the discontinuities of a weak solution satisfy condition (43), then no characteristics drawn will intersects the curve of discontinuity, see Figure 2. A discontinuity satisfying the jump condition (38) and the Lax inequality (43) is called a *shock*. A weak solution having only shocks as discontinuity is called shock wave solution. Lax showed that there is exactly one shock wave solution u of the equations (16) - (17), which is expressed explicitly as

$$u(x,t) = \begin{cases} u_1 & \text{if } x < \rho t, \\ u_2 & \text{if} x > \rho t, \end{cases}$$

if we take the initial value to be

$$u_0(x) = \begin{cases} u_1 & \text{if } x < 0, \\ u_2 & \text{if} x > 0. \end{cases}$$

Here, $u_1, u_2 \in \mathrm{R}$ are the left and right initial states with $u_1 > u_2$. The jump condition $\rho(u_1 - u_2) = F(u_1) - F(u_2)$ and the Lax condition $F'(u_1) > \rho > F'(u_2)$ are satisfied.

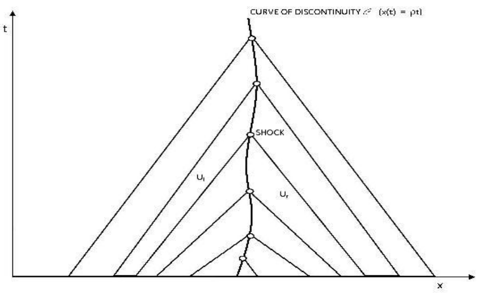

Figure 2:

Example 2.8 [The Riemann Problem] The Riemann problem is the Cauchy problem

$$u_t + (F(u))_x = 0 \; in \; \mathrm{R} \times (0, \infty)$$

$$u(x,0) = u_0(x) = \begin{cases} u_l & if \;\; x < 0 \\ u_r & if \;\; x > 0. \end{cases}$$

Here, $u_l, u_r \in \mathrm{R}$ are the left and right initial states of u. Note that $u_l \neq u_r$. If $u_l > u_r$ the shock wave solution to the Riemann's problem is

$$u(x,t) := \begin{cases} u_l & if \;\; x < \rho t, \\ u_r & if \;\; x > \rho t. \end{cases}$$

The main result by Lax is given below

Theorem 2.9 Let $F:\mathrm{R}\rightarrow\mathrm{R}$ be a C^2- smooth and convex function. If $u_0 \in L_\infty(\mathrm{R})$ then there exists a weak solution u of the Cauchy problem (16) - (17) given by the formula

$$u(x,t) = b\left(\frac{x-y_0}{t}\right) \qquad for \;\; each \quad t>0 \;\; and \; a.e. \; x\in\mathrm{R} \quad (44)$$

where $y_0 = y_0(x,t)$ is the the value of y that minimizes

$$K(x,y,t) = U_0(y) + tG(\frac{x-y}{t}).$$

Here the function $b(s)$ is defined as $b(s) = (F'(s))^{-1}$ and $G(s)$ is defined as the solution of

$$\frac{dG(s)}{ds} = b(s), \quad G(c)=0, \, with \; c = F'(0),$$

and

$$U_0(y) = \int_{-\infty}^{y} u_0(s)ds.$$

The discontinuity of the solution u constructed in Theorem 2 are shocks, which means u satisfied the Lax inequality (43) and has the semi group property. The semigroup property means that if $u(x,t_1)$ is taken as a new initial value, the solution $u(x,t_2)$ at $t_2 > t_1$ corresponding to the initial condition $u(x,t_1)$ equals $u(x,t_1+t_2)$. **Remark 2.10** (i) For fixed $t>0,$ the function $y_0(x,t)$ is an increasing function of $x,$ see [64, Lemma 3.3].

(ii)The shock wave solution constructed in Theorem 2 satisfies the Oleinik inequality (40). Indeed, since b and $y(x,t)$ are increasing functions, then for $x_2 > x_1$ we have that

$$u(x_1,t) = b\left(\frac{x_1 - y_0(x_1,t)}{t}\right) \geq b\left(\frac{x_1 - y_0(x_2,t)}{t}\right) \geq b\left(\frac{x_2 - y_0(x_2,t)}{t}\right) - \alpha\frac{x_2 - x_1}{t}$$

$$= u(x_2,t) - \alpha\frac{x_2-x_1}{t};$$

which implies

$$\frac{u(x_2,t)-u(x_1,t)}{x_2-x_1} \leq \frac{\alpha}{t},$$

here $\alpha > 0$ is a Lipschitz constant for the function $b.$

A generalized form of the Lax condition (43) was given by Oleinik [83]. For $0 \leq \alpha \leq 1,$

$$F(\alpha u_r + (1-\alpha)u_l) \leq \alpha F(u_r) + (1-\alpha)F(u_l) \qquad if \; u_l > u_r, \quad (45)$$

$$F(\alpha u_r + (1-\alpha)u_l) \geq \alpha F(u_r) + (1-\alpha)F(u_l) \qquad if \; u_l < u_r. \quad (46)$$

The inequality (45) implies that F is convex. Geometrically this means that the graph of F over the interval $[u_r, u_l]$ lies below the chord of F drawn from the point $(u_l, F(u_l))$ to the point $(u_r, F(u_r)).$ On the other hand, the inequality (46) implies that F is concave, which means that the graph of F over the interval $[u_l, u_r]$ lies above the chord of F drawn from the point $(u_l, F(u_l))$ to the point $(u_r, F(u_r)).$

We now discuss the relationship between the Lax inequality (43) and the generalized Oleinik inequality (45). To start with, the convexity of F implies that the inequality (45) is equivalent to

$$\frac{F(u^*)-F(u_l)}{u^*-u_l} \geq \frac{F(u_r)-F(u^*)}{u_r-u^*}. \quad (47)$$

for every $u^* = \alpha u_r + (1-\alpha)u_l,$ with $0<\alpha<1.$ Combining the inequality (47) with the Mean Value Theorem, we have that there exists $\zeta \in [u_r, u_l]$ such that

$$F'(\zeta) = \frac{F(u_l)-F(u_r)}{u_l-u_r} = \rho$$

and

$$\frac{F(u^*)-F(u_l)}{u^*-u_l} \geq F'(\zeta) \geq \frac{F(u_r)-F(u^*)}{u_r-u^*}. \qquad (48)$$

Now Taking limits as $u^* \to u_l$ and $u^* \to u_r$ in (48), we have

$$F'(u_l) \geq \rho \geq F'(u_r),$$

which is the Lax inequality. Thus, the generalized inequality (45) implies the Lax inequality. On the other hand, if the flux function F, is a convex function, which implies $F'' > 0$, then the Lax inequality (43) would implies the inequality (45).

Essentially, all the conditions considered so far require that the flux function F be convex or concave. Krushkov [59] introduced a more general admissibility condition for a flux function F that is not necessarily convex or concave. One advantage of the Kruzkov condition is that it is also applicable to systems of conservation laws in more than one dimension, whereas the Oleinik condition is limited to single conservation laws in one dimension. Although the Lax inequality is applicable to systems of conservation laws, it still requires the convexity of the flux function F, moreover the Lax inequality is limited to systems in one spatial dimension. Kruzkov's admissibility condition is given below.

2.3.3 Admissibility condition 3 (The Entropy/Entropy flux pair)

The admissibility condition discussed in this section was first introduced by Kruzhkov [59], in terms of entropy/entropy flux pairs which we define below

Definition 2.11 [Entropy/entropy flux] The pair (Φ, Ψ) is called an entropy/entropy flux pair for the conservation law (16), if for every C^∞-smooth and convex function $\Phi : \mathrm{R} \to \mathrm{R}$ there is a C^∞-smooth function $\Psi : \mathrm{R} \to \mathrm{R}$ such that

$$\Psi'(z) = \Phi'(z)F'(z), \quad z \in \mathrm{R}. \qquad (49)$$

Ψ is called an entropy flux function for the entropy function Φ. For every convex function Φ we can find a corresponding entropy flux function Ψ given by

$$\Psi(z) = \int_{z_0}^{z} \Phi'(z)F'(z), \quad z \in \mathrm{R}. \quad (50)$$

For each entropy/entropy flux pair (Φ, Ψ), the admissibility condition, known as the *entropy condition* is given by

$$\begin{cases} \iint\limits_{\Omega} \Phi(u)\phi_t + \Psi(u)\phi_x dxdt \geq 0, \\ \textit{for each } \phi \in C_c^\infty(\mathrm{R} \times (0,\infty)), \phi \geq 0 \end{cases} \qquad (51)$$

Definition 2.12: [*Entropy solution*] *The function* $u \in C([0,\infty), L^1(\mathrm{R})) \cap L_\infty(\mathrm{R} \times (0,\infty))$ *is called an entropy solution of the Cauchy problem (16) - (17) if it satisfies the entropy condition (51) for each entropy/entropy-flux pair* (Φ, Ψ), *and* $u(.,t) \to u_0$ *in* L^1 *as* $t \to 0$.

We now explore how the entropy condition relates to the other admissibility conditions considered in this section. In this regard, assume u is C^1 - smooth in the left subregion Ω_l and right subregion Ω_r of some region $\Omega \subseteq (\mathrm{R} \times [0,\infty))$ divided by a smooth curve C. Let u also satisfy the entropy condition. If we take $\Phi(u) = \pm u$ and $\Psi(u) = F(u)$ in (51) we see that

$$u_t + F(u)_x = 0 \quad \text{in } \Omega_l, \Omega_r.$$

Integrating (51) by parts we get

$$\iint\limits_{\Omega_l} \Phi(u)\phi_t + \Psi(u)\phi_x dxdt + \iint\limits_{\Omega_r} \Phi(u)\phi_t + \Psi(u)\phi_x dxdt \geq 0$$

From whence we deduce

$$\int_{\mathrm{C}} \phi[(\Phi(u_l) - \Phi(u_r))n_2 + (\Psi(u_l) - \Psi(u_r))n_1]dS \geq 0 \quad (52)$$

where $n = (n_1, n_2)$ is the unit normal to C pointing from Ω_l to Ω_r. Suppose the curve C is represented in parametric form as $\{(x,t) : x = s(t)\}$ for some smooth function $s : [0, \infty) \to \mathrm{R}$, then $n = (n_1, n_2) = \dfrac{(1, -\dot{s})}{\sqrt{1 + \dot{s}^2}}$. Consequently (52) becomes

$$\dot{s}(\Phi(u_r) - \Phi(u_l)) \geq \Psi(u_r) - \Psi(u_l) \quad along \;\; \mathrm{C}, \qquad (53)$$

which leads to the jump condition

$$\dot{s}[[u]] = [[F(u)]]. \qquad (54)$$

Thus the entropy condition (51) is stronger than the jump condition.

Suppose $u_l > u_r$. Fix u^* such that $u_l > u^* > u_r$ and define the entropy/entropy flux pair as

$$\Phi(z) = \begin{cases} (z - u^*) & \text{if} z - u^* > 0 \\ 0 & otherwise \end{cases}$$

and

$$\Psi(z) = \int_{u_l}^{z} sgn_+(v - u) F'(v) dv.$$

Then

$$\Phi(u_r) - \Phi(u_l) = u^* - u_l$$

and

$$\Psi(u_r) - \Psi(u_l) = F(u^*) - F(u_l).$$

Consequently (53) implies

$$\dot{s}(u^* - u_l) \geq F(u^*) - F(u_l)$$

which, when combined with (54), gives

$$\dot{s} \leq \frac{F(u^*) - F(u_l)}{u^* - u_l} \qquad (55)$$

Similarly, if $u_r > u_l$ and $u_r > u^* > u_l$ then

$$\dot{s} \geq \frac{F(u_r) - F(u^*)}{u_r - u^*} \qquad (56)$$

Conditions (55) and (56) gives condition (47). Thus the entropy condition (51) implies the Lax condition.

Remark 2.13(i) Note that any entropy solution of (16) - (17) satisfying definition 2.3.3 is also a weak solution of (16). This follows if we set $\Phi(z) = z, z \in \mathrm{R},$ in which case $\Psi = F.$

(ii) There is at most one entropy solution to the Cauchy problem (16) - (17).

(iii) If $u \in C^1(\Omega)$ is a classical solution of the initial value problem, then

$$\Phi'(u)(u_t + (F(u))_x) = 0$$

for any convex function $\Phi.$ This further implies

$$0 = \Phi'(u)u_t + \Phi'(u)F'(u)u_x = \Phi'(u)u_t + \Psi'(u)u_x,$$

with Ψ any entropy flux associated with $\Phi.$ This verifies that a classical solution is also an entropy solution. In the next section we discuss the vanishing viscosity method for conservation laws.

III. Solutions of Scalar Conservation Laws via Vanishing Viscosity

The role of the entropy condition in conservation laws is to distinguish between the physically relevant weak solution and other, possibly irrelevant weak solutions. One method for obtaining and analyzing solutions to hyperbolic conservation laws is to modify the given conservation law by adding a small perturbation term to the right-hand side of the equation, for example, $\varepsilon u_{xx}, \text{with}\, \varepsilon << 1,$ to obtain from (16) a regularized equation

$$u_t^{\varepsilon} + (F(u^{\varepsilon}))_x - \varepsilon u_{xx} = 0 \quad (57)$$

The motivation that is often given for the study of the Cauchy problem (16) - (17) through the regularized problem such as

$$u_t + (F(u))_x = \varepsilon u_{xx} \text{ in } \mathbb{R}\times(0,\infty), \quad \varepsilon > 0 \qquad (58)$$

$$u(x,0) = u_0(x) \quad x \in \mathbb{R}. \qquad (59)$$

is that physically and mathematically correct solutions of (16) - (17) should arise as the limit of solutions of u^{ε} of (58) - (59), as the parameter ε tends to zero. This method is generally known as viscosity method [38], [92], [90].

In this regard we may recall that the model for thermoelastic materials under adiabatic conditions is a first order system of hyperbolic PDEs, while that of thermoviscoelastic, heat-conducting materials is a second order PDE, containing a diffusive term, see [38]. Every material has a degree of viscous response and conducts heat. Classifying a material as an elastic nonconductor of heat simply means that viscosity and heat conductivity are negligible, but not totally absent. The consequence of this is that the theory of adiabatic thermoelasticity may be physically meaningful only as a limiting case of thermoviscoelasticity, with viscosity and heat conductivity tending to zero see [38], [92]. In the same way the theory of hyperbolic conservation laws is considered to be physically meaningful as a limiting case of the parabolic equation (58).

Here we should note that solutions of nonlinear PDEs are in general highly unstable with respect to small perturbations of the equation. Thus in spite of the physical intuition underlying such viscosity methods, the rigorous mathematical analysis of the limiting behavior of solutions of equations like (58) - (59) as ε tends to $0,$ is highly non trivial.

One particular example of the viscosity method which we discuss in this section is the vanishing viscosity method. The aim is to construct the entropy solution of the scalar conservation law (16) as the limit of solutions of the parabolic equations (58) - (59). The artificial viscosity term εu_{xx} added to the right side of (16) is supposed to provide a small viscous effect, which 'smear out' sharp shocks.

It is well known that for any $\varepsilon > 0,$ and for bounded and measurable initial data, there exists a unique classical solution u^{ε} of the parabolic equation (58) - (59), see [82], [107]. This unique solution u^{ε} of equation (58) - (59) is called a viscosity solution of (58) - (59). The following general theorem guarantees the existence of a sequence of solution to the parabolic problem (58) - (59).

Theorem 3.1 *([107]) (i) For any fixed $\varepsilon > 0,$ the Cauchy problem (58) - (59) with $u_0 \in L_\infty$ always has a local classical solution $u^{\varepsilon}(x,t) \in C^{\infty}(\mathbb{R}\times(0,\tau))$ for a small time $\tau,$ which depends only on the L_∞ norm of the initial data $u_0(x)$.*

(ii) If the solution u^{ε} has an a priori L_∞ bound or estimate $\|u^{\varepsilon}(\cdot,t)\|_{L_\infty} \le M(\varepsilon,T)$ for any $t \in [0,T],$ then the solution exists on $\mathbb{R}\times[0,T].$

(iii) The solution u^{ε} satisfies:

$$\lim_{|x|\to\infty} u^{\varepsilon}(x,t) = 0, \quad if \lim_{|x|\to\infty} u_0(x) = 0.$$

Following standard theory for parabolic equations, the local existence of a solution can easily be obtained by applying the contraction mapping principle to an integral representation of the solution. Whenever there is a local solution with a priori L_∞ bound, the time τ can be extended, step by step, to a further time T since the step time depend on the L_∞ norm. Details of the proof can be found in [62], [92]. Two fundamental questions concerning the solution u^{ε} of (58) - (59) are the following.

- In what sense does the sequence of functions u^{ε} converge to a limit function u as ε tends to 0?
- Given that u^{ε} converges to some u in a specified way, in what sense can we interpret u as a solution of the Cauchy problem (16) - (17)? In particular, if u^{ε} is the unique classical solution of (58)-(59) and u^{ε} converges to some function u as ε tends to $0,$ is u an entropy solution of the Cauchy problem (16) - (17)?

An interesting problem related to first question concerns the behavior of u^{ε} in the neighborhood of a discontinuity of $u.$ A partial answer to the above questions is given in the following Theorem, see [38], [49], [60]. **Theorem 3.2** *[38, Theorem 6.3.1] Suppose u_{ε} is the solution of (58), (59), and assume that for some sequence $\{\varepsilon_n\},$ with $\varepsilon_n \to 0$ as $n\to\infty,$ we have that $u_{\varepsilon_n} \to u$ boundedly a.e on $\mathbb{R}\times[0,\infty).$ Then u is* *an entropy solution of (16)-(17) on* $\mathbb{R}\times[0,\infty)$.

Remark 3.3 $u_{\varepsilon_n}\to u$ boundedly a.e means that u_{ε_n} is norm bounded in L_∞ and $u_{\varepsilon_n}(x,t)\to u(x,t)$ as $\varepsilon_n\to 0$ in $\mathbb{R}$ for almost all $(x,t)\in\mathbb{R}\times[0,\infty)$. Since the weak solutions of (16)-(17) are in L_∞, and are typically not continuous, it may happen that as the smooth function u^ε approaches u the functions u_x^ε and u_{xx}^ε may become unbounded, in a neighborhood of a point of discontinuity of u. Thus establishing the convergence $u^\varepsilon\to u$ is a nontrivial issue.

The rigorous mathematical theory of scalar conservation laws via vanishing viscosity was initiated by E. Hopf in his paper 1950 [57]. There Hopf considered the viscous Burgers equation

$$u_t+uu_x=\varepsilon u_{xx}\ in\ \mathbb{R}\times(0,\infty)\quad (60)$$

$$u(x,0)=u_0(x)\ in\ \mathbb{R}.\quad (61)$$

Using the transformation

$$v=e^{-(\frac{1}{2\varepsilon}\int u dx)}\quad with\ inverse\quad u=-2\varepsilon(\log v)_x=-2\varepsilon(\frac{v_x}{v}),\quad (62)$$

he first transformed the viscous Burgers equation (60)-(61) into the linear heat equation

$$v_t-\varepsilon v_{xx}=0,\quad (63)$$

$$v(x,0)=v_0(x)=e^{-\frac{1}{2\varepsilon}\int_0^x u_0(\nu)d\nu}.\quad (64)$$

The solution of (63) - (64) is

$$v(x,t)=\frac{1}{\sqrt{4\pi\varepsilon t}}\int_{-\infty}^{\infty}v_0(y)e^{-\frac{(x-y)^2}{4\varepsilon t}}dy,\quad (65)$$

which may be obtained by standard methods for solving parabolic PDEs, for example using the Fourier transform. Substituting the expression (65) for $v(x,t)$ in (62) one obtains an explicit formula for the unique solution of equation (60) - (61). This result is stated below:

Theorem 3.4 *[Hopf E., [57]] Suppose* $u_0\in L^1_{loc}(\mathbb{R})$ *is such that*

$$\int_0^x u_0(\xi)d\xi=o(x^2)\quad for\ |x|\,large.\quad (66)$$

Then there exists a unique classical solution of equation (60)-(61) given by

$$u^\varepsilon(x,t)=\frac{\int_{-\infty}^{\infty}\frac{x-y}{t}e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy}{\int_{-\infty}^{\infty}e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy}\quad (67)$$

where

$$K(x,y,t)=\int_0^y u_0(\eta)d\eta+\frac{(x-y)^2}{2t}.\quad (68)$$

The solution u^ε *has the following properties*

1. *For all* $a\in\mathbb{R}$,

$$\int_0^x u^\varepsilon(\xi,t)d\xi\to\int_0^a u_0(\xi)d\xi\ \text{ as }\ x\to a,\ t\to 0,\quad (69)$$

2. *If* $u_0(x)$ *is continuous at* $x=a$ *then*

$$u(x,t)\to u_0(a)\ \text{ as }\ x\to a,\ t\to 0.\quad (70)$$

A solution of (60) - (61) which is C^2*-smooth in the interval* $0<t<T$ *and satisfies (69) for each value of* a *necessarily coincides with (67) in the interval.*

The condition (66) on the initial value is necessary to guarantee the convergence of the definite integral (65),

thus also those in the expression (67) for the solution of (60) - (61).
Hopf also studied the convergence of u^{ε}, as ε tends to 0. In this regard, he made use of the functions y_{min} and y_{max}, which are defined as follows:

$$y_{min}(x,t) := \min\{y : K(x,y,t) \text{ attains its minimum value}\}$$

and

$$y_{max}(x,t) := \max\{y : K(x,y,t) \text{ attains its minimum value}\}.$$

That is, the minimum of $K(x,y,t)$ for fixed x,t is $m(x,t) = \min K = K(x, y_{min}, t) = K(x, y_{max}, t)$. $m(x,t)$ is a continuous function of x,t, [57, Lemma 2]. Obviously,

$$y_{min}(x,t) \le y_{max}(x,t).$$

However, it was shown that

$$y_{max}(x_1,t) \le y_{min}(x_2,t) \ \forall\ x_1 < x_2,$$

thus y_{min} and y_{max} are monotone in x. Since a monotone function has only a denumerable number of discontinuities one can infer for any $t > 0$, $y_{min} = y_{max}$ holds for all x with the possible exception of a denumerable set of values of x where $y_{min} < y_{max}$. Note that $y_{min}(x,t)$ is lower semi-continuous while $y_{max}(x,t)$ is upper semi-continuous [57, Lemma 3]. Both functions are continuous at a point where $y_{min} = y_{max}$. The convergence theorem for the solution u^{ε} of (60) - (61) may now be stated as follows

Theorem 3.5 *Let* $u^{\varepsilon}(x,t), t > 0$, *be the solution of (60) - (61) with* u_0 *satisfying (66). Then for all* x *and* $t > 0$,

$$\frac{x - y_{max}(x,t)}{t} \le \liminf_{\varepsilon \to 0} u^{\varepsilon}(x,t) \le \limsup_{\varepsilon \to 0} u^{\varepsilon}(x,t) \le \frac{x - y_{min}(x,t)}{t}.$$

In particular,

$$\lim_{\varepsilon \to 0} u^{\varepsilon}(x,t) = \frac{x - y_{max}(x,t)}{t} = \frac{x - y_{min}(x,t)}{t}$$

holds at every point (x,t), $t > 0$, *in which* $y_{max} = y_{min}$.

Define the function u as

$$u(x,t) := \lim_{\varepsilon \to 0} u^{\varepsilon}(x,t) \ \ a.e.$$

in every point $(x,t), t > 0$ where this limit exists. By Theorem 2(`)@, this limit will exist at every point in which $y_{min} = y_{max}$. Thus the convergence of u^{ε} to u is almost everywhere. At the point where the limit exists, $u(x,t)$ is defined and is continuous in both variables.

It was further shown that the inequality

$$u_l = u(x^-,t) \ge u(x^+,t) = u_r$$

holds for all $t > 0$ and by the method of characteristics, that the equation

$$\lim_{t \to t_1} \frac{x(t) - x(t_1)}{t - t_1} = \frac{1}{2}[u_l + u_r] \quad (71)$$

holds. Here $x = x(t)$ is the curve of discontinuity for $u(x,t)$. From (71) we have that

$$\dot{x} = \frac{1}{2}[u_l + u_r],$$

which is the jump condition for inviscid Burgers equation

$$u_t + uu_x = 0. \quad (72)$$

To show that the limit function $u(x,t)$ obtained above is a weak solution of the inviscid Burgers equation, we note firts that a solution u^{ε} of (58) - (59) satisfies the equation

$$\int_0^\infty\int_{-\infty}^\infty \{u^\varepsilon \phi_t + \frac{(u^\varepsilon)^2}{2}\phi_x\}dxdt + \varepsilon\int_0^\infty\int_{-\infty}^\infty u^\varepsilon \phi_{xx} dxdt = 0, \quad (73)$$

for each test function $\phi \in C_0^\infty(\Omega)$. From the convergence Theorem 2(`)@ it follows that every point (x,t) has a neighborhood in which the solutions u^ε of (58) - (59) are uniformly bounded as ε tends to 0. As such we can pass limit $\varepsilon \to 0$ in (73) with ϕ being fixed. Thus we have

$$\int_0^\infty\int_{-\infty}^\infty \{u\phi_t + \frac{u^2}{2}\phi_x\}dxdt = 0,$$

which shows that $u(x,t)$ is a weak solution of the inviscid Burgers equation (72).
However, the solution u that satisfies the jump condition (71) that was constructed here was not shown to be the unique solution that satisfies this condition.
Lax [63] obtained a result similar to that of Hopf by showing that the weak solution

$$u(x,t) = b\left(\frac{x-y_0}{t}\right) \qquad for\ \ each\ \ \ t>0\ \ and\ a.e.\ x \in \mathrm{R}$$

obtained in Theorem 2 can be written as the limit of the solution u^ε of the inviscid Burgers equations (60) - (61), that is

$$u = \lim_{n\to\infty} u_n. \qquad (74)$$

To see this, consider the equation (60) in the form

$$u_t + (F(u))_x = \frac{1}{2n}u_{xx},\ \ F(u) = \frac{1}{2}u^2. \qquad (75)$$

Defined $F(u)$ as

$$F(u) = \lim_{n\to\infty} F_n,$$

where

$$F_n = \frac{\int_{-\infty}^\infty F(b(\frac{x-y}{t}))e^{-nK}dy}{\int_{-\infty}^\infty e^{-nK}dy}. \quad (76)$$

Then the function u_n defined as

$$u_n = \frac{\int_{-\infty}^\infty b(\frac{x-y}{t})e^{-nK(x,y,t)}dy}{\int_{-\infty}^\infty e^{-nK(x,y,t)}dy} \quad (77)$$

is a solution to the equation (75) - (61). Here,

$$K(x,y,t) = U_0(y) + tG(\frac{x-y}{t}),$$

the function $b(s)$ is defined as $b(s) = (F'(s))^{-1}$, $G(s)$ is defined as the solution of

$$\frac{dG(s)}{ds} = b(s), \quad G(c) = 0, \text{ with } F'(0) = c,$$

and

$$U_0(y) = \int_0^y u_0(s)ds.$$

The convergence of u_n to u as n tends to ∞ is almost everywhere and the function u is a weak solution of (16) - (17) [63, Theorem 2.1].
If we denote by V_n the function

$$V_n = \log\int_{-\infty}^\infty e^{-nK}dy$$

then

$$u_n = -\frac{1}{n}\frac{\partial}{\partial x}V_n$$

and

$$F_n(x,t) = -\frac{1}{n}\frac{\partial}{\partial t}V_n$$

provided that $F(b(z)) = zb(z) - G(z).$ It then follows that

$$(u_n)_t + (F_n)_x = 0. \quad (78)$$

Multiply equation (78) by a test function $\phi \in C_0^\infty(\Omega)$ and integrate to get

$$\iint_\Omega u_n\phi_t + F_n\phi_x = 0,$$

letting $n \to \infty$ we obtain the limit relation

$$\iint_\Omega u\phi_t + F(u)\phi_x = 0,$$

This shows that u is a weak solution of equations (16) - (17).

In addition, Lax [65] introduced a finite difference scheme for the scalar conservation law and showed, using some numeric examples, that both the viscosity method and the finite difference scheme, when applied to the cauchy problem (16) - (17) with $u_0(x) \in L_\infty,$ converge to the same limit $u(x,t),$ given by the explicit formula (44). He also applied the finite difference scheme to the case where $F(u) = -\log(a + be^u)$ and obtain the following result:

Theorem 3.6 *[Lax, [66]] Consider the single conservation law (16) with*

$$F(u) = -\log(a + be^u) \quad a + b = 1. \quad (79)$$

Replace in (16) the time derivative by a forward difference and the space (x) derivative by a left difference:

$$u(x,t+\Delta) = u(x,t) - \{F(u(x,t)) - F(u(x-\Delta,t))\}. \quad (80)$$

Let u_Δ be the solution of this difference equation with bounded and measurable initial value $u(x,0) = u_0(x).$ Then

$$\lim_{\Delta \to 0} u_\Delta = u(x,t) \qquad (81)$$

exists for fixed t for almost all $x.$ Furthermore the limit u is given by (44).

The following theorem gives the properties of the limit function u obtained through the finite difference scheme.

Theorem 3.7 *The function $u,$ defined by (44) is a weak solution of (16)-(17) with F given by (79), and satisfies the following properties.*

1. *It depends continuously on the initial data u_0.*
2. *The dependence of u on u_0 is completely continuous in the following sense: For $u_0 \in L_\infty(\mathrm{R},$ denote by $g(u_0)$ the solution (44) of (16) - (17), with F given by (79). If $A \subset L_\infty(\mathrm{R})$ is bounded in $L_\infty(\mathrm{R})$ and $suppA$ is compact, then $g(A)$ is compact in the L_1 topology.*
3. *It has a semigroup property. That is if $u(x,t_1)$ is taken as a new initial value, the corresponding solution at time t_2 equals $u(x,t_1+t_2).$*

For an arbitrary function $F,$ there are no explicit formulas for the solution to the parabolic equation. However, Oleinik [80] proved that for a general convex or concave function, the solutions of the parabolic problem (58) - (59) tends to the weak solution of (16). A simpler proof was given by Ladyzhenskaya in [61].

We remark that if u_ε converges to u in the weak sense only; the sequence $F(u_\varepsilon)$ will converge in the weak sense but not to $F(u).$ In this regard, we have the following theorem by Lax:

Theorem 3.8 *([63], [65]) If the sequence of functions u_n converges in the weak sense to a limit $u,$ then $F(u_n)$ converges in the weak sense to $F(u)$ if and only if $u_n \to u$ strongly in $L_1.$*

As mentioned in section 2.3.1, see in particular Theorem 2.6, Oleinik [82] showed that there exists a unique solution of (16)-(17) that satisfies the admissibility condition (40), provided that the flux function F is convex. This solution is constructed as a limit of solutions u^{ε} of equation (58) -(59) obtained through a *finite difference scheme* introduced by Lax in [65]. It was subsequently shown that u is in fact the unique solution of (16) - (17) satisfying (40), see [92, Lema 16.9 through Theorem 16.11].

Kruzhkov [59], [60] introduced a new method to apply the vanishing viscosity method to a larger class of equations. For initial data $u_0 \in L_\infty,$ he proved existence and uniqueness of the classical solution $u^{\varepsilon}(x,t)$ of (58)-(59). Using a family of entropy-entropy flux pairs $(\Phi_k, \Psi_k)_{k\in \mathrm{R}}$ where

$$\Phi_k(u) = |u-k| \, and \, \Psi_k(u) := sgn(u-k)(f(u)-f(k)),$$

it was shown that the solution $u^{\varepsilon}(x,t)$ of equations (58) - (59) converges as ε tends to 0 almost everywhere to a weak solution $u(x,t)$ of the Cauchy problem (16) -(17).

Theorem 3.9 *[Kruzhkov, [60]] Let $u_0 \in L_\infty(\mathrm{R}).$ Then the solution $u^{\varepsilon}(x,t)$ of problem (58) - (59) converges as $\varepsilon \to 0$ almost everywhere in $\mathrm{R}\times[0,T)$ to a function $u(x,t)$ which is a weak solution of the problem (16) - (17).*

In the proof of the above theorem, a priori bounds (independent of ε) were obtained for the solutions $u^{\varepsilon}(x,t)$ which ensures the compactness of the family of functions $\{u^{\varepsilon}(x,t)\}$ with respect to the L_1 - norm. This in turn guarantees the existence of a subsequence u^{ε_n} of u^{ε} that converges to the weak solution $u(x,t).$ Thus a weak solution of the Cauchy problem (16) - (17) is constructed as the limit of solution u^{ε} of the parabolic problem (58) - (59). The following theorem shows that the weak solution constructed above is an entropy solution.

Theorem 3.10 *Let $u_0 \in L_\infty(\mathrm{R}).$ If $u^{\varepsilon}(x,t)$ converges to a function $u(x,t)$ almost everywhere as $\varepsilon \to 0$ in $\mathrm{R}\times[0,T).$ Then the solution u also satisfies the following inequality, for every entropy/entropy flux pair (Φ,Ψ)*

$$\iint_{\Omega} \Phi(u)\phi_t + \Psi(u)\phi_x dxdt + \int_{\mathrm{R}} \Phi(u_0(x))\phi(x,0)dx \geq 0, \qquad (82)$$

for all $\phi \in C_0^{\infty}(\mathrm{R}\times(0,\infty)), \phi \geq 0.$

The issue of uniqueness the solution $u(x,t)$ of the problem (16) - (17) is addressed in the following result, [59], [60].

Theorem 3.11 *[90, Theorem 2.3.5] For every function $u_0 \in L_\infty(\mathrm{R}),$ there exists one and only one entropy solution $u \in L_\infty(\mathrm{R}\times[0,T)) \bigcap C([0,T); L^1_{loc}(\mathrm{R}))$ of (16) -(17). The entropy solution u satisfies the maximum principle*

$$\|u\|_{L_\infty(\mathrm{R}\times[0,T))} = \|u_0\|_{L_\infty(\mathrm{R})}.$$

Furthermore, let $u_0, v_0 \in L_\infty$ and u and v the entropy solutions of (16) -(17) associated with u_0 and v_0 respectively. Let

$$M = \sup\{|F'(s)| : s \in [\inf(u_0,v_0), \sup(u_0,v_0)]\}.$$

Then the entropy solution u satisfies the following:

1. *For all $t > 0$ and every interval $[a,b],$ we have*

$$\int_a^b |v(x,t)-u(x,t)|\,dx \leq \int_{a+Mt}^{b+Mt} |v_0(x)-u_0(x)|\,dx.$$

2. *In particular, if u_0 and v_0 coincide on $[x-\delta, x_0+\delta]$ for some $\delta >, 0$ then u and v coincide on the triangle $\{(x,t) : |x-x_0| + Mt < \delta\}.$*
3. *If $u_0 - v_0 \in L^1(\mathrm{R}),$ then $u(t)-v(t) \in L^1(\mathrm{R}),$ where $u(t) := u(\cdot,t),$ and*

$$\|v(t)-u(t)\|_{L^1(\mathrm{R})} \leq \|v_0 - u_0\|_{L^1(\mathrm{R})},$$

$$\int_{\mathrm{R}}(v(x,t)-u(x,t))dx = \int_{\mathrm{R}}(v_0(x)-u_0(x))dx.$$

4. *If* $u_0 \in L^1(\mathrm{R}),$ *then* $u(t) \in L^1(\mathrm{R}),$ *for all* $t > 0,$ *and*

$$\|u(t)\|_{L^1(\mathrm{R})} \leq \|u_0\|_{L^1(\mathrm{R})}, \qquad \int_{\mathrm{R}} u(x,t)dx = \int_{\mathrm{R}} u_0(x)dx.$$

5. *If* $u_0(x) \leq v_0(x)$ *for almost all* $x \in \mathrm{R},$ *then* $u(x,t) \leq v(x,t)$ *for almost all* $(x,t) \in \mathrm{R}\times[0,\infty).$
6. *If* u_0 *has bounded total variation, then* $u(t)$ *has bounded total variation for all* $t > 0$ *and*

$$TV(u(t)) \leq TV(u_0).$$

Remark 3.12: Theorem 3.11, is valid in several spatial dimension, [60], [90]. By Theorem 3(`)@, one can construct a semi group operator $S(t),$ associated with the entropy solution $u(x,t)$ with respect to the initial data u_0 and time $t > 0$ written as,

$$u(x,t) = S_t u_0(x).$$

The semi group $S : \mathrm{D}\times[0,\infty) \to \mathrm{D}$ with $\mathrm{D} \subset L_1(\mathrm{R})$ a closed domain containing all functions with bounded total variation, has the following properties [16]:

1. $S_0 u = u,\ S_{t+s}u = S_t S_s u,$
2. S_t is uniformly Lipschitz continuous w.r.t time and initial data: There exists $L, L' > 0$ such that

$$\|S_t u_0 - S_s v_0\| \leq L\|u_0 - v_0\| + L'|t-s|.$$

The proof of the above uniqueness Theorem 3(`)@ is based on the fact that the semigroup operator S_t of (16) is a contraction in $L^1(\mathrm{R})\cap L_\infty(\mathrm{R})$ with respect to the L^1 -norm. This fact is expressed in property (P3), which implies that if $u_0 \in BV,$ then $u \in BV$ for all $t > 0$ as stated in property (P6). Property (P4) is a consequence of property (P3), which leads to property (P5).
Panov [84] has proved that it is not necessary to consider the whole family of entropy/entropy flux pair. A single pair of entropy/entropy flux pair (Φ,Ψ) is sufficient to characterize entropy solutions of (16) - (17).

IV. Conclusion

The theory of hyperbolic conservation laws has developed in a number of directions. One major approach, as discussed, consists of considering weak solutions in suitable spaces of functions with bounded variation (BV functions). Actually, the problem, and a very difficult one, is to prove that various approximating schemes such as the vanishing viscosity, Glimm scheme, wave front tracking etc, converge to the entropy solution. The BV approach then consists of proving convergence of these schemes under assumption on the initial condition u_0 related to it's total variation. Typically, one assumes that the total variation satisfies a smallness condition, see [10]. Another approach is to construct weak solutions through weak convergence and compensated compactness arguments, see for instance [78], [97], [107]. Recently, the ordercompletionmethod wa applied to prove existence of solution to conservation. The method consist of constructing solutions to conservations in a space of Hausdorff continuous functions, see [1], [5].

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