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Another Special Differential Equation and Polynomials

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Abstract: *Special differential equations and polynomials are very popular in the field of mathematics and serve as important tools in the solution of some engineering problems. Examples of these equations are Legendre, Hermite, Laguerre, Bessel, Gegenbaur differential equations. In this paper, we established a new special differential equation and its polynomial which we called Legendre subsidiary equation and polynomial. The Rodrigue formula, generating function and recurrence relations of the polynomial are given. We also gave the orthogonality properties of the polynomials, and our results are entering the literature for the first time.*

Keywords: *Rodrigue formula, generating function, recurrence relations and orthogonality.*

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I. Introduction

The new special differential equation is a second order homogeneous ODE with variable coefficient of the form

$$\left(1+x^2\right)y''+2xy'-n\left(n+1\right)y=0 \tag{1.1}$$

The general solution of the equation is given as

$$y\left(x\right)=c_1A_n\left(x\right)+c_2H_n\left(x\right) \tag{1.2}$$

Where $A_n\left(x\right)$ represents the polynomials and $H_n\left(x\right)$ are functions of order *n*. The equation has an ordinary point at $x=0$ and a regular singular point at $x=\pm i=\pm e^{i\pi/2}$.

This differential equation and the polynomials are related to the well known Legendre equation and polynomials which serves as an eye opener to our new special differential equation.

II. Deduction of the Equation

Let $z=\left(1+x^2\right)^n \qquad \Rightarrow\left(1+x^2\right)\dfrac{dz}{dx}=2nxz$

Differentiating $\left(n+1\right)$ times using Leibnitz formula and simplifying gives

$$\left(1+x^2\right)\frac{d^{n+2}z}{dx^{n+2}}+2x\frac{d^{n+1}z}{dx^{n+1}}-n\left(n+1\right)\frac{d^nz}{dx^n}=0 \tag{2.1}$$

On letting $y=\dfrac{d^nz}{dx^n}$ (2.1) becomes (2.2)

$$\left(1+x^2\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-n\left(n+1\right)y=0$$

and this leads to our result.

III. Solution of the Differential Equation

To solve this, we apply Frobenius method of the form of descending power series.

$$y(x)=\sum_{r=0}^{\infty}a_r x^{k-r}; \qquad y'(x)=\sum_{r=0}^{\infty}a_r(k-r)x^{k-r-1};$$

$$y''(x)=\sum_{r=0}^{\infty}a_r(k-r)(k-r-1)x^{k-r-2}$$

Plugging these results into (1.1) and simplifying, we have

$$\sum_{r=0}^{\infty}(k-r)(k-r-1)a_r x^{k-r-2}+\sum_{r=0}^{\infty}(k-r-n)(k-r+n+1)a_r x^{k-r}=0 \tag{3.1}$$

Putting $r=0$ in the second summation and equating to zero, we get

$$(k-n)(k+n+1)a_0=0 \qquad \Rightarrow (k-n)(k+n+1)=0 \ \text{ since } a_0\neq 0.$$

Hence the indicial root is $k=n$ or $k=-(n+1)$, and the difference is an integer.

Replacing *r* by $r-2$ in the first summation of (3.1) and simplifying, we get the recurrence relation as

$$a_r=-\frac{(k-r+1)(k-r+2)}{(k-r-n)(k-r+n+1)}a_{r-2} \tag{3.2}$$

Case 1: If $k=n$; putting $k=n$ in (3.2) gives

$$a_r=\frac{(n-r+1)(n-r+2)}{r(2n+1-r)}a_{r-2}$$

If $r=1,\ a_1=a_3=\ldots=a_{2n-1}=0$, so if $r=2,\ 4,\ 6,\ \ldots,2n$, we obtain the following

$$a_2=\frac{n(n-1)a_0}{2(2n-1)};\ a_4=\frac{n(n-1)(n-2)(n-3)a_0}{2.4(2n-1)(2n-3)};\ a_6=\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)a_0}{2.4.6(2n-1)(2n-3)(2n-5)}$$

$$\therefore a_{2r}=\frac{n(n-1)\ldots(n-2r+1)a_0}{2.4\ldots 2r(2n-1)\ldots(2n-2r+1)}=\frac{n(n-1)\ldots(n-2r+1)a_0}{2^r r!(2n-1)\ldots(2n-2r+1)}\frac{(n-2r)!}{(n-2r)!}$$

$$=\frac{n!a_0}{2^r r!(n-2r)!(2n-1)\ldots(2n-2r+1)}=\frac{n!a_0 2n(2n-2)\ldots(2n-2r+2)}{2^r r!(n-2r)!2n(2n-1)\ldots(2n-2r+2)(2n-2r+1)}$$

$$=\frac{n!a_0 2n(2n-2)\ldots(2n-2r+2)}{2^r r!(n-2r)!2n(2n-1)\ldots(2n-2r+2)(2n-2r+1)}\frac{(2n-2r)!}{(2n-2r)!}$$

$$=\frac{n!a_0(2n-2r)!2^r n(n-1)\ldots(n-r+1)}{2^r r!(n-2r)!(2n)!}=\frac{(n!)^2(2n-2r)!a_0}{(2n)!r!(n-2r)!(n-r)!}$$

Here we intend to eliminate all the expressions independent of *r* and therefore we let

$$a_0=\frac{(2n)!}{2^n(n!)^2}$$

$$\therefore a_{2r}=\frac{(n!)^2(2n-2r)!a_0}{(2n)!r!(n-2r)!(n-r)!}=\frac{(2n-2r)!}{r!(n-2r)!(n-r)!} \tag{3.3}$$

Hence the first solution is therefore $y_1(x)=\sum_{r=0}^{\infty}a_{2r}x^{k-2r}$ and the polynomial becomes

$$\left.\begin{aligned} A_n(x) &= \sum_{r=0}^{\infty} \frac{(2n-2r)!\,x^{n-2r}}{2^n r!(n-2r)!(n-r)!} \\ &= \sum_{r=0}^{\frac{1}{2}n} \frac{(2n-2r)!\,x^{n-2r}}{2^n r!(n-2r)!(n-r)!} \qquad (for\ even\ n) \\ &= \sum_{r=0}^{\frac{1}{2}(n-1)} \frac{(2n-2r)!\,x^{n-2r}}{2^n r!(n-2r)!(n-r)!} \qquad (for\ odd\ n) \end{aligned}\right\} \tag{3.4}$$

The polynomial can also be defined in an explicit form as

$$y_1(x) = a_0\left\{x^n + \frac{n(n-1)x^{n-2}}{2(2n-1)} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} + \ldots + \frac{n(n-1)\ldots(n-2r+1)x^{n-2r}}{2^r r!(2n-1)\ldots(2n-2r+1)}\right\}$$

$$\therefore A_n(x) = \frac{(2n)!}{2^n (n!)^2}\left\{x^n + \frac{n(n-1)x^{n-2}}{2(2n-1)} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} + \ldots\right\}$$

(3.5)

The first few polynomials are generated as follows

$A_0(x) = 1$ $\qquad$ $A_4(x) = \frac{1}{8}\left(35x^4 + 30x^2 + 3\right)$

$A_1(x) = x$ $\qquad$ $A_5(x) = \frac{1}{8}\left(63x^5 + 70x^3 + 15x\right)$

$A_2(x) = \frac{1}{2}\left(3x^2 + 1\right)$ $\qquad$ $A_6(x) = \frac{1}{16}\left(231x^6 + 315x^4 + 105x^2 + 5\right)$

$A_3(x) = \frac{1}{2}\left(5x^3 + 3x\right)$ $\qquad$ $A_7(x) = \frac{1}{16}\left(429x^7 + 693x^5 + 315x^3 + 35x\right)$

Case 2: If $k = -(n+1)$; putting $k = -(n+1)$ in (3.2), the recurrence relation becomes

$$a_r = \frac{-(n+r)(n+r+1)}{r(2n+r+1)}a_{r-2}$$

If $r = 2,\ 4,\ 6,\ \ldots, 2n$, we obtain the following

$$a_2 = \frac{-(n+1)(n+2)a_0}{2(2n+3)}; \qquad a_4 = \frac{(n+1)(n+2)(n+3)(n+4)a_0}{2.4(2n+3)(2+5)}\ldots$$

$$\therefore a_{2r} = \frac{(-1)^r (n+1)\ldots(n+2r)a_0}{2^r r!(2n+3)\ldots(2n+2r+1)}$$

Hence the second solution becomes

$$y_2(x) = a_0\left\{x^{-(n+1)} - \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2+5)}x^{-(n+5)} - \ldots\right\}$$

Let $a_0 = \frac{(n!)^2 2^n}{(2n+1)}$, the second solution becomes

$$H_n(x)=\frac{(n!)^2\,2^n}{(2n+1)}\left\{x^{-(n+1)}-\frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2+5)}x^{-(n+5)}-\dots\right\} \tag{3.6}$$

$$=\frac{(n!)^2\,2^n}{(2n+1)}\sum_{r=0}^{\infty}\frac{(-1)^r(n+1)\dots(n+2r)\,x^{-(n+2r+1)}}{2^r\,r!(2n+3)\dots(2n+2r+1)}$$

IV. Generating Function Of $A_n(x)$

The generating function of the polynomial $A_n(x)$ is given as

$$\frac{1}{\sqrt{1-2xt-t^2}}=\sum_{n=0}^{\infty}t^nA_n(x) \tag{4.1}$$

Proof:

$$\frac{1}{\sqrt{1-2xt-t^2}}=\left[1-\left(2xt+t^2\right)\right]^{-\frac{1}{2}}$$

$$=1+\frac{1}{2}\left(2xt+t^2\right)+\frac{-\tfrac{1}{2}\left(-\tfrac{3}{2}\right)}{2!}\left(2xt+t^2\right)^2-\frac{-\tfrac{1}{2}\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)}{3!}\left(2xt+t^2\right)^3+\dots$$

$$=1t^0+xt^1+\tfrac{1}{2}\left(3x^2+1\right)t^2+\tfrac{1}{2}\left(5x^3+3x\right)t^3+\dots$$

$$=A_0(x)t^0+A_1(x)t^1+A_2(x)t^2+A_3(x)t^3+\dots=\sum_{n=0}^{\infty}A_n(x)t^n$$

This completes the proof.

Corollary; putting $x=i$ in (4.1), we get

$$\sum_{n=0}^{\infty}t^nA_n(i)=\frac{1}{\sqrt{1-2it-t^2}}=\frac{1}{1-it}=\sum_{n=0}^{\infty}i^n(t)^n$$

Equating the coefficient of t^n we have that

$$A_n(i)=i^n \tag{4.2}$$

V. Rodriguez Formula For $A_n(x)$

The Rodrigue formula for the polynomial $A_n(x)$ is given as

$$A_n(x)=\frac{1}{2^n n!}D^n\left(1+x^2\right)^n \qquad where\ \ D^n=\frac{d^n}{dx^n}$$

Proof:

From (2.2), we have that $y=\frac{d^nz}{dx^n}=\frac{d^n}{dx^n}\left(1+x^2\right)^n$ and *y* is a solution of (1.1), that is

$$y=c_1A_n(x)=\frac{d^n}{dx^n}\left(1+x^2\right)^n=D^n\left(1+x^2\right)^n=D^n(x-i)^n(x+i)^n$$

$$\Rightarrow c_1A_n(x)=D^n(x-i)^n(x+i)^n \qquad (*)$$

Applying Leibnitz formula, we get

$$c_1 A_n(x) = (x-i)^n D^n (x+i)^n + {}^n C_1 n (x-i)^{n-1} D^{n-1}(x+i)^n + {}^n C_2 n(n-1)(x-i)^{n-2} D^{n-2}(x+i)^n$$

$$+ {}^n C_3 n(n-1)(n-2)(x-i)^{n-3} D^{n-3}(x+i)^n + \ldots + {}^n C_n n! D^{n-n}(x+i)^n$$

Putting $x = i$ and using (4.2) gives

$$c_1 A_n(i) = 0 + 0 + \ldots + 0 + n!(i+i)^n = n! i^n 2^n$$

$$\Rightarrow c_1 i^n = n! i^n 2^n \qquad \therefore c_1 = 2^n n!$$

Hence we have

$$n! 2^n A_n(x) = D^n (1+x^2)^n \qquad \therefore A_n(x) = \frac{1}{2^n n!} D^n (1+x^2)^n$$

and the proof completes.

VI. Recurrence Relation for the Polynomial $A_n(x)$

The recurrence relations for polynomials $A_n(x)$ are given as follows

1. $(n+1) A_{n+1}(x) = (2n+1) x A_n(x) + n A_{n-1}(x)$ 2. $xA'(x)_n + A'_{n-1}(x) = nA_n(x)$

3. $A'_{n+1}(x) + A'_{n-1}(x) = (2n+1) A_n(x)$

Proof: From the generating function formula, let

1. $U = (1-2xt-t^2)^{-1/2} = \sum_{n=0}^{\infty} A_n(x) t^n.$

Differentiating partially w.r.t "t" and simplifying gives

$$\Rightarrow (1-2xt-t^2) U_t = (x+t) U \qquad \Rightarrow (1-2xt-t^2) \sum_{n=0}^{\infty} A_n(x) n t^{n-1} = (x+t) \sum_{n=0}^{\infty} A_n(x) t^n$$

Equating the coefficient of t^n from both sides, we obtain

$$(n+1) A_{n+1}(x) = (2n+1) x A_n(x) + n A_{n-1}(x) \quad \text{Proved.}$$

2. Differentiating partially w.r.t "x" and simplifying gives

$$(1-2xt-t^2) U_x = xU \quad \Rightarrow \frac{(1-2xt-t^2) U_t}{(1-2xt-t^2) U_x} = \frac{(x+t)U}{tU} \qquad \Rightarrow tU_t = (x+t) U_x$$

$$\Rightarrow \sum_{n=0}^{\infty} A_n(x) n t^n = x \sum_{n=0}^{\infty} A_n{}'(x) t^n + \sum_{n=0}^{\infty} A_{n-1}{}'(x) t^n$$

$$\Rightarrow nA_n = xA'_n + A'_{n-1}. \quad \text{The result follows.}$$

3. Differentiating the recurrence formula (1) w.r.t. *x*, we obtain

$$(n+1) A'_{n+1} = (2n+1) A_n + (2n+1) x A_n{}' + n A'_{n-1}$$

Applying recurrence formula (2) for xA'_n, we get

$$(n+1) A'_{n+1} = (2n+1) A_n + (2n+1)(nA_n - A'_{n-1}) + nA'_{n-1}$$

$$\Rightarrow A'_{n+1} = (2n+1) A_n - nA'_{n-1} \quad \text{or} \quad A'_{n+1} + A'_{n-1} = (2n+1) A_n. \text{ Hence the proof.}$$

VII. Orthogonality of the Polynomial $A_m(x)$

1. Any of two of the polynomials is orthogonal in the interval $(-i, i)$, that is

$$\int_{-i}^{i} A_m(x) A_n(x)\,dx = 0 \quad if \quad m \neq n. \tag{7.1}$$

Proof: Since A_m and A_n are solutions of (1.1), then we must have that

$$\left(1+x^2\right)A_m'' + 2xA_m' - m(m+1)A_m = 0 \qquad (i)$$

$$\left(1+x^2\right)A_n'' + 2xA_n' - n(n+1)A_n = 0 \qquad (ii)$$

Multiplying A_n by (*i*) and A_m by (*ii*) and subtracting, we get

$$\left(1+x^2\right)\left[A_m''A_n - A_n''A_m\right] + 2x\left[A_m'A_n - A_m'A_n\right] = \left[m(m+1) - n(n+1)\right]A_mA_n$$

$$\Rightarrow \frac{d}{dx}\left\{\left(1+x^2\right)\left[A_m'A_n - A_m'A_n\right]\right\} = \left[m(m+1) - n(n+1)\right]A_mA_n$$

Integrating both sides w.r.t. *x* along the boundary $-i < x < i$ to get

$$\int_{-i}^{i}\left[m(m+1) - n(n+1)\right]A_mA_n dx = \left\{\left(1+x^2\right)\left[A_m'A_n - A_m'A_n\right]\right\}_{-i}^{i} = 0$$

$$\sin ce\ m \neq n \quad \therefore \int_{-i}^{i} A_m(x)A_n(x)\,dx = 0.$$

2. $$\int_{-i}^{i} A_n^2(x)\,dx = \frac{2i(-1)^n}{2n+1} \tag{7.2}$$

Proof: Squaring both sides of the generating function, we have

$$\sum_{n=0}^{\infty} t^{2n} A_n^2(x) = \frac{1}{1-2xt-t^2}$$

Integrating both sides w.r.t. *x* along the boundary $-i < x < i$ to get

$$\sum_{n=0}^{\infty} t^{2n}\int_{-i}^{i} A_n^2(x)\,dx = \int_{-i}^{i}\frac{1}{1-2xt-t^2}\,dx = \left[-\frac{1}{2t}\ln\left(1-2xt-t^2\right)\right]_{-i}^{i}$$

$$= \frac{1}{2t}\ln\left(\frac{1+2it-t^2}{1-2it-t^2}\right) = \frac{1}{t}\ln\left(\frac{1+it}{1-it}\right) = \frac{1}{t}\sum_{n=0}^{\infty}\frac{2(it)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{2(-1)^n t^{2n} i}{2n+1}$$

Equating the coefficient of t^{2n} we get

$$\int_{-i}^{i} A_n^2(x)\,dx = \frac{2i(-1)^n}{2n+1}$$

VIII. Further Research

In our next research work, we intend to present the following

1. Application of the polynomial
2. Integral representation of the polynomial
3. Series of the type $f(x) = \sum_{n=0}^{\infty} c_n A_n(x)$
4. Confluent hypergeometric representation of (1.1) and lots more

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