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Cone Metric Spaces and Extension of Fixed Point Theorem for Contraction Mappings Applying C- Distance

S. K. Tiwari[1], Kaushik Das[2], Golu Kumar Sahu[3]
[1]*Department of Mathematics, Dr. C. V. Raman University Bilaspur (C.G.)-India*
[2]*B. Ed Department, Gobardanga Hindu college, North 24 Pargans,P.S.-Habra, West Bengal , India*
[3]*Department of Mathematics (M. Phil), Dr. C. V. Raman University Bilaspur (C.G.)-India*
Corresponding Author: S. K. Tiwari

Abstract: *A new concept of the c-distance in cone metric spaces has been introduced by Cho et al. [2] in 2011. Recently, Dubey, A.K. et al. [13] proved some fixed point results contractive conditions under c- distance in cone metric spaces. The purpose of this paper is to establish, extend and the generalization of fixed point theorems for contractive type mapping on complete cone metric spaces applying under c- distance. Our results generalize and extend some well known results in the literature [8].*

Keywords: *Cone metric space, complete cone metric space, c-distance, fixed point, contraction mapping.*

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I. Introduction

The concept of cone metric spaces which is generalization of the classical metric space was introduced by Huang and Zhang [1] replacing the set of real numbers by an order Banach space, and showed some fixed point theorems of different type of contractive mappings on cone metric spaces. Later, many authors generalized and studied fixed and common fixed point results in cone metric spaces for normal and non normal cones. Recently, Cho et al.[2] ,Wang and Guo [5]defined a concept of the c- distance in a cone metric space, which is a cone version of the w-distance Kada et al.[3] and proved some fixed point theorems in ordered cone metric spaces. Then Sintunavarat et al. [4] generalized the Banach contraction theorem on c- distance of Cho et al. [2]. After that, several authors studied the existence and uniqueness of the fixed point, common fixed point, coupled fixed point and common coupled fixed point problems using this distance in cone metric spaces and ordered cone metric spaces see for examples [6-14], [17].Quick recently, in 2017 Fadail et al. [16], studied some fixed point theorems of T-Reich contraction type mappings under the concept of c- distance in complete cone metric spaces depended on another function. In the same year, Tiwari, S. K., et al. [3-4] proved, Generalized and extended unique fixed point theorems applying this distance in cone metric spaces. Our results generalize and extend the respective results [13].

II. Preliminary Notes

First, we recall some standard notations and definitions in cone metric spaces with some of their properties [1].

Definition 2.1: Let E be a real Banach space and P be a subset of $E$and θdenote to the zero element in E, then P is called a cone if and only if :

(i) P is a non-empty set closed and $P \neq \{\theta\}$,

(ii If a, b are non-negative real numbers and $x, y \in P$,then $ax + by \in P$,

(iii) $x \in P \; and - x \in P \Longrightarrow x = \theta \Leftrightarrow P \cap (-P) = \{\theta\}$.

Given a cone P ⊂ E, we define a partial ordering ≤ on E with respect to P by $x \leq y$ if and only if $y - x \in P$.We shall write $x \ll y$ if $y - x \in intP$(where int P denotes the interior of P). If $intP \neq \emptyset$, then cone P is solid. The cone P called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \; => \| x \| \leq k \| y \|.$$

The least positive number k satisfying the above is called the normal constant of P.

Definition: 2.2: Let x be a non-empty set. Suppose the mapping $d: X \times X \longrightarrow E$ satisfies

(i) $\theta < d(x, y) for \; all \; x, y \in X \; and \; (x, y) = \theta$if and only if $x = y$,

(ii) $d(x, y) = d(y, x) for \; all \; x, y \in X$,

(iii) $d(x, y) \leq d(x, z) + d(z, y) for \; all \; x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space .The concept of cone metric space is more general than that of a metric space.

Example2.3: Let $E = R^2, P = \{(x,y) \in E: x,y \geq 0\}, X = R$ and $d: X \times X \to E$ defined by $d(x,y) = (\left|x - y\right|, \alpha\left|x - y\right|)$,where α ≥ 0 is a constant. Then (X,d) is a cone metric space.

Definition: 2.4[1]: Let (X,d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq1}$ be a sequence in X. then,

(1) $\{x_n\}_{n\geq1}$ Converges to x whenever for every $c \in E$ with $\theta \ll c$, if there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $lim_{n\to\infty} x_n = x$ or $x_n \to x, (n \to \infty$

(2) $\{x_n\}_{n\geq1}$is said to be a Cauchy sequence if for every $c \in E$ with θ<< c,if there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n.m \geq N$.

(3) (X,d) is called a complete cone metric space if every Cauchy sequence in X is Convergent.

Lemma 2.5 ([4]).

1. If E is a real Banach space with cone P and $a \leq \lambda a$ where $a \in P$ and $\theta \leq \lambda < 1$, then $a = \theta$

2. If $c \in intP, \theta \leq a_n$ and $a_n \to \theta$ then there a positive integer N such that $a_n \ll c$ for all $n \geq N$.

Next, we give the definition of c-distance on a cone metric space(X,d) which is generalization of w- distance of Kada et al. [3] with some properties.

Definition 2.6 ([2]): Let (X,d) be a cone metric space. A function $q: X \times X \to E$ is called a c- distance on X if the following conditions hold:

(q_1). $\theta \leq q(x,y)$ for all $x,y \in X$,

(q_2). $q(x,y) \leq q(x,y) + q(y,z)$ for all $x,y,z \in X$,

(q_3). for each $x \in X$ and $n \geq 1$, if $q(x,y_n) \leq u$ for some $u = u_x \in P$, then $q(x,y) \leq u$ Whenever$\{y_n\}$ is a sequence in X converging to a point $y \in X$,

(q_4). foe all $c \in E$with $\theta \ll c$, there exist $e \in E$ with $\theta \in e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll c$.

Example 2.7 ([2]): Let $E = R$ and $P = \{x \in E: x \geq 0\}$. Let $X = [0,\infty)$and define a mapping $d: X \times X \to E$ by $d(x,y) = |x - y|$ for all $x,y \in X$. Then $(X,d)$is a cone metric space. Define by $q: X \times X \to E$ by $q(x,y) = y$ for all $x,y \in X$. Then $q$is a c-distance on$X$.

Example 2.8([10, 11]): Let $E = R^2$ and $P = \{(x,y) \in E: x,y \geq 0\}$. Let $X = [0,1]$ and define a mapping $d: X \times X \to E$ by $d(x,y) = (|x - y|, |x - y|)$ for all $x,y \in X$. Then (X,d) is a complete cone metric space. Define a mapping $q: X \times X \to E$ by $q(x,y) = (y,y)$ for all $x,y \in X$. Then q is a $c-$ distance.

Example 2.9 ([16]): Let $X = C_{\frac{1}{R}}[0,1]$(the set of real valued functions on X which also have continuous derivatives on X),$P = \{\varphi \in E: \varphi(t) \geq 0\}$. A cone metric d on X is defined by $d(x,y)(t) \coloneqq |x - y|.\varphi(t)$ where $\emptyset \in P$ is an arbitrary function. This cone is non normal. Then $(X,d)$is a complete cone metric space. Define a mapping $q: X \times X \to E$ by $q(x,y)(t) = y.e^t$ for all $x,y \in X$. It is easy to see that q is a $c-$distance.

Lemma 2.10([2]): Let (X,d) be a cone metric space and q is c- distance on X. Let $\{x_n\}$ and $\{y_n\}$ be a sequences in X and $x,y,z \in X$.Suppose that u_n is sequence in P converging to 0. Then the following conditions hold:

(1) If $q(x_{n,}y) \leq u_n$ and $q(x_{n,}z) \leq u_{n,}$ then $y = z$.

(2) If $q(x_{n,}y_n) \leq u_n$ and $q(x_{n,}z) \leq u_{n,}$ then $\{y_n\}$ converges to z.

(3) If $q(x_{n,}x_m) \leq u_n$ for $m > n$ and $\{x_{n,}\}$ is a Cauchy sequence in X.

(4) If $q(y, x_{n,}) \leq u_n$then $\{x_{n,}\}$ is a Cauchy sequence in X.

Remark 2.11([2]):

(1) $q(x,y) = q(y,x)$ does not necessarily for all $x,y \in X$.

(2) $q(x,y) = \theta$ is not necessarily equivalent to $x - y$ for all $x,y \in X$.

III. Main Results.

The following results, which we will generalizes and extend the results of [13].

Theorem 3.1: Let (X,d)be cone metric spaces, P be a solid cone and q be a c-distance on$X$. Suppose that $T: X \to X$ be continuous and satisfies the contractive condition;

$$\boldsymbol{q}(Tx,Ty) \leq a_1 q(x,y) + a_2 q(x,Tx) + a_3 q(y,Ty) + a_4[(q(x,Tx) + q(y,Ty)] + a_5[q(y,Tx) + q(x,Ty)] \quad \ldots\ldots (3.1.1)$$

for all $x,y \in X$, where a_1, a_2, a_3, a_4, a_5 are non negative real numbers such that $a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1$. Then T has a fixed point $x^* \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point. If $u = Tu$. Then $(u,u) = \theta$. The fixed point is unique.

Proof: Choose $x_0 \in X$. Set $x_1 = Tx_{0,}$, $x_2 = Tx_1 = T^2 \ldots\ldots\ldots\ldots x_{n+1} = Tx_n = T^n x_0$

Then we have,

$$q(x_n, x_{n+1}) \leq q(Tx_{n-1,}Tx_n) \quad \ldots\ldots (3.1.2)$$

$\leq a_1q(x_{n-1},x_n)+a_2q(x_{n-1},Tx_{n-1})+a_3q(x_n,Tx_n)$
$+a_4[q(x_{n-1},Tx_{n-1})+q(x_n,Tx_n)]+a_5[q(x_n,Tx_{n-1})+q(x_{n-1},Tx_n)$
$=a_1q(x_{n-1},x_n)+a_2q(x_{n-1},x_n)+a_3q(x_n,x_{n+1})$
$+a_4[q(x_{n-1},x_n)+q(x_n,x_{n+1})]+a_5[q(x_n,x_n)+q(x_{n-1},x_{n+1})$
$q(x_n,x_{n+1}) \leq (a_1+a_2+a_3+a_4+a_5)q(x_{n-1},x_n)+(a_3+a_4+a_5)q(x_n,x_{n+1})$
So,$q(x_n,x_{n+1}) \leq \frac{(a_1+a_2+a_3+a_4+a_5)}{1-(a_3+a_4+a_5)}q(x_{n-1},x_n)$
$=hq(x_{n-1},x_n)$, where $h=\frac{(a_1+a_2+a_3+a_4+a_5)}{1-(a_3+a_4+a_5)}<1.$ …………...(3.1.3)

Let $m>n\geq 1$. Then it follows that
$q(x_{n,}x_m) \leq q(x_{n,}x_{n+1})+q(x_{n+1,}x_{n+2})+$……….$+q(x_{n-1,}x_n)$
$\leq (h^n+h^{n+1}+\cdots\ldots\ldots\ldots\ldots\ldots+h^{n-1})q(x_{0,}x_1)$
$\leq \frac{h^n}{1-h}q(x_{0,}x_1)\to\infty, h\to\infty.$ …………………………………… .(3.1.4)

Thus, Lemma 2.10 shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^*\in X$ such that $x_n\to x^*$. Since T is continuous, then $x^*=\lim_{x_{n+1}}=limT(x_n)=T(limx_n)=T(x^*)$. Therefore, x^* is a fixed point of T.Suppose that $u=Tu$.
Then we have
$q(u,u)\leq q(Tu,Tu)$
$\leq a_1q(u,u)+a_2q(u,Tu)+a_3q(u,Tu)+a_4[q(u,Tu)+q(u,Tu)]$
$+a_5[q(u,Tu)+q(u,Tu)$
$=(a_1+a_2+a_3+2a_4+2a_5)q(u,u).$ ……………………………… (3.1.5)

Since $a_1+a_2+a_3+2a_4+2a_5<1$, Lemma 2.5 shows that $q(u,u)=\theta$. Next we prove that the uniqueness of the fixed point. Suppose that, there is another fixed point ofy^* of T, then we have
$q(x^*y^*)\leq q(Tx^*,Ty^*)$
$\leq a_1q(x^*,y^*)+a_2q(x^*,Tx^*)+a_3q(y^*,Ty^*)+a_4[q(x^*,Tx^*)+q(y^*,Ty^*)]$
$+a_5[q(y^*,Tx^*)+q(x^*,Ty^*)$
$=(a_1+2a_5)q(x^*,y^*).$
$\leq (a_1+a_2+a_3+2a_4+2a_5)\,q(x^*,y^*).$ ………………………… (3.1.6)

Since $(a_1+a_2+a_3+2a_4+2a_5)<1$, then by Lemma 2.5 we have$q(x^*,y^*)=\theta$and also we have$(x^*,x^*)=\theta$. Hence by Lemma 2.10(1), $x^*=y^*$.Therefore the fixed point is unique.

Remark3.2

(1). Put $a_4=0$ and $a_4=a_5$ in theorem 3.1, we get the result of theorem 2.1 of Dubey, A. K.et al.[13].
(2). If we put $a_4=0$ and $a_5=0$ in theorem 3.1, we get the result of theorem 3.3 of Fadail,et al. [9].
(3). If we put $a_1=a_2=a_3=a_5=0$ and $a_2=a_4$in theorem 3.1, we get the result of Corollory3.4 of Fadail, et al. [9].

Theorem 3.3: Let (X,d)be cone metric spaces, P be a solid cone and q be a c-distance on$X$. Suppose that $T:X\to X$ be continuous and satisfies the contractive condition;
$\boldsymbol{q}(Tx,Ty)\leq a_1q(x,y)+a_2[q(x,Tx)+q(y,Ty)]+a_3[q(x,Ty)+q(y,Tx)]$
$+a_4[q(x,Tx)+q(x,y)]+a_5[q(y,Ty)+q(x,y)]$……………… (3.3.1)

for all $x,y\in X$, where a_1,a_2,a_3,a_4,a_5 are non negative real numbers such that $a_1+2a_2+2a_3+2a_4+2a_5<1$. Then T has a fixed point $x^*\in X$, iterative sequence $\{T^nx\}$ converges to the fixed point. If $u=Tu$. Then $(u,u)=\theta$. The fixed point is unique.
Proof: Choose $x_0\in X$. Set $x_1=Tx_{0,}$, $x_2=Tx_1=T^2\ldots\ldots\ldots\ldots.x_{n+1}=Tx_n=T^nx_0$
Then we have,
$q(x_n,x_{n+1})\leq q(Tx_{n-1,}Tx_n)$ ……………………………………………… (3.3.2)
$\leq a_1q(x_{n-1},x_n)+a_2[q(x_{n-1},Tx_{n-1})+q(x_n,Tx_n)]$
$+a_3[q(x_{n-1},Tx_n)+q(x_n,Tx_{n-1})]+a_4[q(x_{n-1},Tx_{n-1})+q(x_{n-1},x_n)]$
$+a_5[q(x_n,Tx_n)+q(x_n,x_n)]$
$=a_1q(x_{n-1},x_n)+a_2[q(x_{n-1},x_n)+q(x_n,x_{n+1})]$
$+a_3[q(x_{n-1},x_{n+1})+q(x_n,x_n)]+a_4[q(x_{n-1},x_n)+q(x_{n-1},x_n)]$
$+a_5[q(x_n,x_{n+1})+q(x_{n-1},x_n)]$
$q(x_n,x_{n+1}) \leq (a_1+a_2+a_3+2a_4+a_5)q(x_{n-1},x_n)+(a_2+a_3+a_5)q(x_n,x_{n+1})$
So,$q(x_n,x_{n+1}) \leq \frac{(a_1+a_2+a_3+2a_4+a_5)}{1-(a_3+a_4+a_5)}q(x_{n-1},x_n)$
$=hq(x_{n-1},x_n)$, where $h=\frac{(a_1+a_2+a_3+a_4+a_5)}{1-(a_3+a_4+a_5)}<1.$ …………...(3.3.3)

Let $m > n \geq 1$. Then it follows that

$$q(x_{n,} x_m) \leq q(x_{n,}x_{n+1}) + q(x_{n+1,}x_{n+2}) + \ldots\ldots\ldots + q(x_{n-1,}x_n)$$
$$\leq (h^n + h^{n+1} + \cdots \ldots \ldots \ldots \ldots \ldots + h^{n-1}) q(x_{0,}x_1)$$
$$\leq \frac{h^n}{1-h} q(x_{0,}x_1) \to \infty, h \to \infty. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad .(3.3.4)$$

Thus, Lemma 2.10 shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Since T is continuous, then $x^* = \lim_{x_{n+1}} = limT(x_n) = T(limx_n) = T(x^*)$. Therefore, x^* is a fixed point of T.Suppose that $u = Tu$.

Then we have

$$q(u,u) \leq q(Tu,Tu)$$
$$\leq a_1 q(u,u) + a_2[q(u,Tu) + q(u,Tu)] + a_3[q(u,Tu) + q(u,Tu)]$$
$$+ a_4[q(u,Tu) + q(u,u)] + a_5[q(u,Tu) + q(u,u)]$$
$$= [(a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5)]q(u,u) \quad \ldots\ldots\ldots\ldots\ldots\ldots \, (3.3.5)$$

Since $a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 < 1$, Lemma 2.5 shows that $q(u,u) = \theta$. Next we prove that the uniqueness of the fixed point. Suppose that, there is another fixed point ofy^* of T, then we have

$$q(x^* y^*) \leq q(Tx^*, Ty^*)$$
$$\leq a_1 q(x^*, y^*) + a_2[q(x^*, Tx^*) + q(y^*, Ty^*)] + a_3[q(x^*, Ty^*) + q(y^*, Tx^*)]$$
$$+ a_4[q(x^*, Tx^*) + q(x^*, y^*)] + a_5[q(y^*, Ty^*) + q(x^*, y^*)]$$
$$= (a_1 + 2a_3 + a_4 + a_5) q(x^*, y^*).$$
$$\leq (a_1 + a_2 + a_3 + 2a_4 + 2a_5) \, q(x^*, y^*). \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \, (3.3.6)$$

Since $(a_1 + a_2 + a_3 + 2a_4 + 2a_5) < 1$, then by Lemma 2.5 we have$q(x^*, y^*) = \theta$and also we have$(x^*, x^*) = \theta$. Hence by Lemma 2.10(1), $x^* = y^*$.Therefore the fixed point is unique.

Remark3.2

(1). Put $a_4 = 0$ and $a_4 = a_5$ in theorem 3.2, we get the result of theorem 2.2of Dubey, A. K.et al.[13].

IV. Conclusion.

In this attempt, we prove unique fixed point results in cone metric spaces with corollaries. These results generalizes and improves the recent results of Dubey, A.K. et al. [13] in the sense that employing c-distances and in contractive conditions, which extends the further scope of our results.

Conflict of Interests

The authors declare that there is no conflict of interests.

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