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The Present Value of Stochastic Perpetuityandinverse Gamma Distribution using the Scale Function

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Abstract: *We examine the probability density function of the present value ofperpetuity, subject to a stochastic Brownian motion rate of return of an investment and then show that its inverse is a gamma distribution. The derivation uses the scale function and martingale results from the theory of calculus.*
Keyword: *Stochastic calculus; scale function; perpetuity*

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I. Introduction

In a real world the expected occurrence of events in future cannot be predicted accurately without errors. The assumption of accurate prediction is only theoretical; hence the present value of a unit-currency in a deterministic world can be represented by a continuous perpetuity as;

$$A = \int_0^\infty \exp(-rt)dt = \frac{1}{r} \quad (1)$$

This impliesthat, an initial amount A invested at a rate of interest r will be enough to fund a unit-currency continuous perpetuity. Giaccotto (1989)concludes that the assumption, that future interest rates are known with certainty at the time of investment is not practicable; interestand investment rates of return are stochastic in real situations. Stochastic differential equations, according toShiryaev (2003), enrich modeling capacity to an astonishing degree. Hence the present value of a unit-currency continuous perpetuity becomes a random variable which can be presented as:

$$A^* = \int_0^\infty \exp(-r_t)tdt \quad (2)$$

where $r_t t$ is the rate of return on $(0,t)$. Thus the certainty of stating the value of A needed to fund the perpetuity is not guaranteed. One can only estimate the probability $P(A^* \le A)$, that the sum A is enough. The random quantity A^* isreferred to as the present value of a stochastic perpetuity (PVSP). The main objective of this paper is to show that the inverse of the present value, $(A^*)^{-1}$, of a stochastic perpetuity has a gamma distribution when the rate of return obeys Brownian motion with a drift term. A precise confidence interval can be obtained for the amount needed to sustain a fixed perpetual consumption level when rates of return on investment are stochastic(Karatzas& Shreve 1991).Boyles (1976) analyses the statistical properties of the present value of an insurance contract under stochastic returns in discrete time. Panjer and Bellhouse (1990) expand Boyle's results and apply them to contingency reserves and premium margins. Beekman and Fuelling (1991) used an Ornstein-Uhlenbeck process to model the force of interest accumulation function and derived the first two moments of both deterministic and contingent future cash flows.De Shepper et al (1994) derived approximate expressions for the present value of annuitiesfor stochastic interest rates.Christiansen(2013) argues that the class of Gaussian diffusions would be a good approximation of the real future development. Shilong et al (2017) construct a new class of interest models based on compound Poisson process.

II. Derivation

If the gamma distribution with parameters (α, β) is defined by the probability density function (PDF):

$$g(w \mid \alpha, \beta) = \frac{1}{\beta \Gamma(\alpha)} \exp\left(\frac{-w}{\beta}\right)\left(\frac{w}{\beta}\right)^{\alpha-1} \quad \text{for} \quad \alpha > 0 \quad (3)$$

where $\alpha > 0$, $\beta > 0$,with mean $E(W) = \alpha\beta$ and $\operatorname{var}(W) = \alpha\beta^2$. Let the cumulative distribution function (CDF) be denoted by $G(w \mid \alpha, \beta)$, and also let the inverse cumulative distribution function (ICDF) be denoted by $IG(q \mid \alpha, \beta)$ where $0 \le q \le 1$, this imply that $G(IG(q \mid \alpha, \beta) \mid \alpha, \beta) = q.$ Now, we defined the integral

$I_\infty = \int_0^\infty \exp(-(\mu t + \sigma B_t))dt$ (4) as the PVSP of a unit currency with an instantaneous interest rate μ and variance σ driven by a Brownian motion B_t. The aim of this paper is to show that the inverse of the PVSP obeys a gamma distribution; that is $P(I_\infty^{-1} \le w)$has afunctional form that is identical to $G\left(w \mid \frac{2\mu}{2}, \frac{\sigma^2}{2}\right)$, and hence Gamma distributed.

Consider the diffusion process on the space $(x,t) \in (-\infty, \infty) \times (0, \infty)$ given by the differential equation:

$$dX_t = \left(\mu + \frac{1}{2}\sigma^2\right)X_t dt - \lambda dt + \sigma X_t dB_t, \quad (5)$$

where $\mu > 0$, $\sigma > 0$, $\lambda > 0$ are constants and the initial value of the process at time $t = 0.$ Solving the differential equation in (5) yields the solution;

$$X_t = \exp(\mu s + \sigma B_t)\left[x - \lambda \int_0^t \exp(-(\mu s + \sigma B_s)ds)\right]. (6)$$

The process defined by equation (6) will be less than or equal to zero, at a future time t^* if and only if

$$\left[x - \lambda \int_0^t \exp(-(\mu s + \sigma B_s)ds)\right] \le 0$$

That is $X_{t^*} \le 0$ iff $\frac{x}{\lambda} \le \int_0^{t^*} \exp(-(\mu s + \sigma B_s))ds$ (7)

Another property of X_t is that, it crosses zero not more than once; which imply that the integral $\int_0^{t^*} \exp-(\mu s + \sigma B_s)ds$ is monotonic with respect to t^*. This can be written in probability form as

$$P(\inf[(X_s : 0 \le s \le t^*) \le 0]) = P(X_{t^*} \le 0). (8)$$

When $t^* = \infty$ equations (7) and (8), becomes

$$P(\inf[(X_s : 0 \le s \le \infty) \le 0]) = P\left(\frac{x}{\lambda} \le \int_0^\infty \exp-(\mu s + \sigma B_s)ds\right) \quad (9)$$

$$P\left[\left(\int_0^\infty \exp-(\mu s + \sigma B_s)ds\right)^{-1} \le \frac{\lambda}{x}\right] = P\left[I_\infty^{-1} \le \frac{\lambda}{x}\right]. (10)$$

Next is to show using the *scale function* that the probability of the differential equation given in equation (5) with initial value $X_0 = x$ ever crosses zero is given by:

$P[\inf(X_s : 0 \le s \le \infty) \le 0] = G\left(\frac{\lambda}{x} \mid \frac{2\mu}{\sigma^2}, \frac{\sigma^2}{2}\right)$ (11)**Definition:**The scale function is the mapping that transforms the original function into a martingale. We will use the scale function in our derivation.

Let's assume another diffusion process defined by $Y_t = f(X_t)$,then, applying Ito's lemma this new diffusion process will obey the stochastic differential equation:

$$df(X_t) = f'(X_t)^{'} dX_t + \frac{1}{2} f''(X_t) d\langle X \rangle_t, \quad (12)$$

where $d\langle X \rangle_t = \sigma^2 X_t^2 dt$

$$df(X_t) = \left[f'(X_t)\left(\left(\mu + \frac{1}{2\sigma^2}\right)X_t - \lambda + \right) + \frac{1}{2} f''(X_t)\sigma^2 X_t^2 \right] dt + f'(X_t)\sigma X_t dB_t \quad (13)$$

The diffusion process Y_t will be a martingale if the drift term $\left[f'(X_t)\left(\left(\mu + \frac{1}{2}\sigma^2\right)X_t - \lambda + \right) + \frac{1}{2} f''(X_t)\sigma^2 X_t^2 \right]$ collapses to zero. That is the process Y_t will be a martingale if:

$$f'(X_t)\left(\left(\mu + \frac{1}{2}\sigma^2\right)X_t - \lambda\right) + \frac{1}{2} f''(X_t)\sigma^2 X_t^2 = 0 \quad (14)$$

the scale function $f(x)$ must satisfy the differential equation

$$f'(x)\left(\left(\mu + \frac{1}{2}\sigma^2\right)x - \lambda\right) = -\frac{1}{2} f''(x)\sigma^2 x^2 \quad (15)$$

Subject to the given initial condition.

Equation (15) is written in simplified form as:

$$\frac{f''(x)}{f'(x)} = \frac{d}{dx}\left(\ln(f'(x))\right) = \frac{\left(\mu + \frac{1}{2}\sigma^2\right)}{-\frac{1}{2}\sigma^2 x} - \frac{\lambda}{-\frac{1}{2}\sigma^2 x^2}. \quad (16)$$

Integrating equation (16) we obtain the solution:

$$\ln(f'(x)) = \frac{\mu + \frac{1}{2}\sigma^2}{-\frac{1}{2}\sigma^2} \ln(x) + \frac{\lambda}{\frac{1}{2}\sigma^2 x} + c_1 \quad (17)$$

where c_1 is an arbitrary constant.

Writing equation (17) in exponential form and simplifying gives us the equation

$$f'(k) = k^{(\mu+\sigma^2/2)/(-\sigma^2/2)} \exp\left(\frac{\lambda}{-\frac{1}{2}\sigma^2 k}\right) c_2 \quad (18)$$

where c_2 is a constant. We set c_2 to be equal to one for simplicity inequation (18) and integrating we obtain

$$f(k) = \int_{c_3}^{x} k^{(\mu+\sigma^2/2)/(-\sigma^2/2)} \exp\left(\frac{\lambda}{-\frac{1}{2}\sigma^2 k}\right) dk \quad (19)$$

with c_3 greater than zero. We now re-arrange equation (19) by conveniently setting $w = \frac{2\lambda}{\sigma^2 k}$;

$dw = \frac{2\lambda}{\sigma^2 k^2} dk$ and $k = \frac{2\lambda}{\sigma^2 w}$, and substituting to obtain

$$f(x) = \int_{2\lambda/x\sigma^2}^{\infty} w^{(2\mu/\sigma^2 - 1)} \exp(-w) dw \quad (20)$$

by re-scaling the constant c_3. The integral in equation (20)can also be written as:

$$f(x) = \Gamma\left(\frac{2\mu}{\sigma^2}, \frac{2\lambda}{x\sigma^2}\right). \quad (21)$$

The original diffusion process X_t has been mapped into the martingale Y_t, via the scale function $f(x)$. Hence by mapping X_t into a martingale, the probability that the process reaches zero is:

$$P(\inf[X_s : 0 \le s \le \infty] \le 0) = 1 - \frac{f(x)}{f(\infty)}$$

$$P(\inf[X_s : 0 \le s \le \infty] \le 0) = \frac{\Gamma(2\mu/\sigma^2, 0) - \Gamma(2\mu/\sigma^2, 2\lambda/x\sigma^2)}{\Gamma(2\mu/\sigma \ \ , 0)} \quad (22)$$

that is:

$$P(\inf[X_s : 0 \le s \le \infty] \le 0) = \frac{\int_0^{2\lambda/x\sigma^2} w^{(2\mu/\sigma^2 - 1)} \exp(-w) dw}{\Gamma(2\mu/\sigma^2)} \quad (23)$$

And dividing by $\frac{1}{2}\sigma^2$ we arrived at the equation below which is the required result:

$$P(\inf[X_s : 0 \le s \le \infty] \le 0) = \int_0^{\lambda/x} \frac{1}{\frac{1}{2}\sigma^2 \Gamma(2\mu/\sigma^2)} \exp\left(\frac{-w}{-\frac{1}{2}\sigma^2}\right)\left(\frac{w}{\frac{1}{2}\sigma^2}\right)^{2\mu/\sigma^2 - 1} dw \quad (24)$$

Hence $P(I_\infty^{-1} \le w) = G(w \mid 2\mu/\sigma^2, \sigma^2/2)$.

III. Conclusion

In this paper we obtained an analytical expression for the probability distribution of the present value of a continuous perpetuity subject to a stochastic geometric Brownian motion rate of return. The result was obtained using the scale function and martingale principle from stochastic ordinary differential equation. The integral approximates to the infinite sum of lognormal variates that has no known standard density function. Thus,the result can be used both as a pedagogical tool and as a more precise method for computing confidence levels. It is of interest to note that the integral in our theorem approximates to the infinite sum of lognormal variates

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