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Properties of Square Roots of the Jacobsthal Matrix of Order 3x3

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Abstract: *In this study, a certain Jacobsthal matrix* J *of order* 3×3 *is taken, and two complex functions are defined with* $G(z)=z^n,\ n\in\mathbb{Z}$*, and* $G^{(n,2)}(z)\equiv z^{n/2},\ \gcd(n,2)=1$*. We obtain any integer* n^{th} *power of the Jacobsthal matrix* J *generated from the theory of functions of matrices. Also, the matrix functions of the Jacobsthal matrix* J *are given by* $G^{(n,2)}(J)\equiv J^{n/2},\ n\in\mathbb{Z}-\{0\}$*. Since the square of these matrices is equal to the Jacobsthal matrix* J*, for the Jacobsthal and Jacobsthal-Lucas numbers with integral or rational subscripts, some fundamental properties are acquired by matrix methods.*

Keywords – *Jacobsthal number, Jacobsthal Lucas number, Jacobsthal Matrices, Square Root Matrices*

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I. Introduction

The Jacobsthal number J_n and Jacobsthal Lucas number j_n are defined by the recurrence formulas $J_n=J_{n-1}+2J_{n-2},\ (J_0=0,\ J_1=1)$ and $j_n=j_{n-1}+2j_{n-2},\ (j_0=2,\ j_1=1)$ for all integers n [1-3]. The recurrence relation gives us a rule for obtaining the number J_n from the numbers J_{n-1} and J_{n-2} or the number j_n from the numbers j_{n-1} and j_{n-2}. In other words, the ordered pair (J_n,J_{n-1}) have been obtained by the previous ordered pair (J_{n-1},J_{n-2}). These are phrased in terms of matrix arithmetic for all integer n. Also, the numbers J_n and j_n are given with the Binet's forms, $J_n=\left(2^n-(-1)^n\right)/3$ and $j_n=2^n+(-1)^n,\ n\in\mathbb{Z}$ [1-5].

Several authors have considered the complex Jacobsthal J_x and Jacobsthal Lucas numbers j_x, where the subscript x is an arbitrary real number, and showed that these numbers enjoy most of the properties of the J_n and j_n, $n\in\mathbb{Z}$ numbers [6]. The modified Binet's forms of the numbers J_x and j_x yield

$$J_x=\frac{2^x-(-1)^x}{3}\ \text{and}\ j_x=2^x+(-1)^x,\ x\in\mathbb{R}. \tag{1}$$

As the modified Binet's forms of the numbers J_x and j_x can be considered to generate the usual Binet's forms, we generalized the Jacobsthal matrices given in [4], [5] from all integer n to special rational numbers, such that $n/2$ and r/s, according to the modified Binet's forms and by finding of the square root matrices for the Jacobsthal and Jacobsthal Lucas matrices [7]. The authors have given main result $F_i^{n/2}$ $(i=1-4)$ as the square roots of the Jacobsthal matrix F^n:

$$F_{1,2}^{n/2}=\pm\begin{bmatrix} J_{(n+2)/2} & 2J_{n/2} \\ J_{n/2} & 2J_{(n-2)/2}\end{bmatrix},\ F_{3,4}^{n/2}=\frac{\pm 1}{3}\begin{bmatrix} j_{(n+2)/2} & 2j_{n/2} \\ j_{n/2} & 2j_{(n-2)/2}\end{bmatrix}.$$

It is seen that entries of these matrices are either the Jacobsthal or the Jacobsthal Lucas numbers with rational subscripts. The authors have investigated properties of the complex Jacobsthal and Jacobsthal Lucas numbers. Now, let us consider a Jacobsthal matrix J^n of order 3×3 for nonnegative integers n, such as

$$J^n=\begin{pmatrix} 4J_{n-1}^2 & 4J_{n-1}J_n & 4J_n^2 \\ 4J_{n-1}J_n & 2\left(J_n^2+J_{n-1}J_{n+1}\right) & 4J_nJ_{n+1} \\ J_n^2 & J_nJ_{n+1} & J_{n+1}^2 \end{pmatrix}, \tag{2}$$

which is situation selected for $p=1$ and $q=-2$ in the matrix R^n [8]. There are many equivalent ways of defining the matrix J^n for any values of all integer n, and one of these definitions is a complex scalar functions. Now, we suppose a complex function $G(z)=z^n$, $n\in\mathbb{Z}$, and note that the matrix function $G(J)$ gives all integer n powers of the Jacobsthal matrix J. It is seen that the J^n is only given for the positive values of n [8]. Since $J_0=0$ and $J_1=1$ and $J_{-1}=\frac{1}{2}$, there is valid: $J^0=I_3$, for $n=0$.

We think that these matrices are not given for negative. Now, since the eigenvalues of the matrix J are nonzero, the eigenvalues -2, 1 and 4, there are matrices J^{-n}, $n\in\mathbb{Z}^+$. The function $G(J^{-1})$ is expressed using the Langrange-Sylvester interpolation polynomial under the polynomial expressions [9], [10],

$$G(J^{-1})=\sum_{i=1}^{3}G(\lambda_i^{-1})\prod_{\substack{j=1\\ i\neq j}}^{3}\frac{\left[J-\lambda_j^{-1}I\right]}{\lambda_i^{-1}-\lambda_j^{-1}},$$

the matrices J^{-n}, $n\in\mathbb{Z}^+$ are obtained as

$$J^{-n}=\begin{pmatrix} 4J_{-n-1}^2 & 4J_{-n-1}J_{-n} & 4J_{-n}^2 \\ 4J_{-n-1}J_{-n} & 2\left(J_{-n}^2+J_{-n-1}J_{-n+1}\right) & 4J_{-n}J_{-n+1} \\ J_{-n}^2 & J_{-n}J_{-n+1} & J_{-n+1}^2 \end{pmatrix}.$$

Also, last equation can be changed with $J_n=-(-2)^nJ_{-n}$, $n\in\mathbb{Z}^+$ into positive indices.

Let us consider that the function $G(z)=z^n$, $n\in\mathbb{Z}$ is analytic in some simply connected region of the complex plane that contains the eigenvalues $\lambda_i=\{-2,1,4\}$, and there is an unique annihilating polynomial $q(\lambda)$ of degree two. Then we define the matrix function $G(J)$ of the matrix J to be $q(J)$. The polynomial $q(\lambda)$ gives;

Theorem 1. *Let $G(J)=J^n$ be the matrix function mentioned above for $n\in\mathbb{Z}$, then*

$$J^n=J_nJ_{n-1}J^2+2J_nJ_{n-2}J-8J_{n-1}J_{n-2}I. \tag{3}$$

Proof Let $q(\lambda)=b_0+b_1\lambda+b_2\lambda^2$ be an annihilating polynomial for $G(\lambda)=\lambda^n$. From $G(\lambda_i)=q(\lambda_i), i=1,2,3$ we have

$$G(-2)=(-2)^n=b_0-2b_1+4b_2=q(-2),$$
$$G(1)=1^n=b_0+b_1+b_2=q(1),$$
$$G(4)=4^n=b_0+4b_1+16b_2=q(4).$$

If the system of linear equation is solved, then

$$\begin{bmatrix} b_0\\ b_1\\ b_2\end{bmatrix}=\begin{bmatrix} \frac{8}{9}(-1)^{2n}-\frac{1}{9}2^{2n}+\frac{2}{9}(-2)^n \\ \frac{2}{9}(-1)^{2n}+\frac{1}{18}2^{2n}-\frac{5}{18}(-2)^n \\ \frac{1}{18}2^{2n}-\frac{1}{9}(-1)^{2n}+\frac{1}{18}(-2)^n \end{bmatrix}=\begin{bmatrix} -8J_{n-1}J_{n-2} \\ 2J_nJ_{n-2} \\ J_nJ_{n-1}\end{bmatrix},$$

the desired result is obtained.

II. Properties of Square Roots of the Jacobsthal Matrix

Let us consider the scalar complex function $G^{(n,2)}(z)\equiv z^{n/2}$ $(-\pi<\arg(z)\leq\pi)$, n is an odd integer. Now, it is clear that $G^{(n,2)}(z)\equiv z^{n/2}$ is defined on the spectrum of the matrix J since the J admits three distinct eigenvalues, -2, 1 and 4, any square root matrices $G^{(n,2)}(J)\equiv J^{n/2}$ are computed by using the Lagrange-Sylvester interpolation polynomial [9], [10]. The function $G^{(n,2)}(z)$ is a double-valued function, giving rise to 2 branches:

$$G^{(n,2)}(z)\equiv z^{n/2}=g_k^{(n,2)}(z),\ k\in\{0,1\},$$

where $g_k^{(n,2)}(z)=|z|^{n/2}\exp\left[i\left(\frac{n}{2}\arg(z)+k\pi n\right)\right]$. If we denote principal root by $g_0^{(n,2)}(z)=z^{n/2}$, is called as principal root, then we can write

$$g_k^{(n,2)}(z) = \exp(ink\pi)z^{n/2},\ k \in \{0,1\}. \tag{4}$$

Therefore, there exist 2^3 matrix function $J^{n/2}$ derived from the 2 branches, these matrix functions are determined by the expression,

$$G_{(k_1,k_2,k_3)}^{(n,2)}(J) = \sum_{i=1}^{3} g_{k_i}^{(n,2)}(\lambda_i) \prod_{\substack{j=1 \\ i\neq j}}^{3} \frac{J-\lambda_j I}{\lambda_i - \lambda_j},\ (k_1,k_2,k_3) \in \{0,1\}^3. \tag{5}$$

Theorem 3. *For every odd number* $n \in \mathbb{Z}$ *,*

$$G_{(0,0,0)}^{(n,2)}(J) = -G_{(1,1,1)}^{(n,2)}(J) = \begin{pmatrix} 4J_{\frac{n}{2}-1}^2 & 4J_{\frac{n}{2}-1}J_{\frac{n}{2}} & 4J_{\frac{n}{2}}^2 \\ 4J_{\frac{n}{2}-1}J_{\frac{n}{2}} & 2\left(J_{\frac{n}{2}}^2 + J_{\frac{n}{2}-1}J_{\frac{n}{2}+1}\right) & 4J_{\frac{n}{2}}J_{\frac{n}{2}+1} \\ J_{\frac{n}{2}}^2 & J_{\frac{n}{2}}J_{\frac{n}{2}+1} & J_{\frac{n}{2}+1}^2 \end{pmatrix},$$

$$G_{(0,1,1)}^{(n,2)}(J) = -G_{(1,0,0)}^{(n,2)}(J) = \frac{1}{9}\begin{pmatrix} 4j_{\frac{n}{2}-1}^2 & 4j_{\frac{n}{2}-1}j_{\frac{n}{2}} & 4j_{\frac{n}{2}}^2 \\ 4j_{\frac{n}{2}-1}j_{\frac{n}{2}} & 2\left(j_{\frac{n}{2}}^2 + j_{\frac{n}{2}-1}j_{\frac{n}{2}+1}\right) & 4j_{\frac{n}{2}}j_{\frac{n}{2}+1} \\ j_{\frac{n}{2}}^2 & j_{\frac{n}{2}}j_{\frac{n}{2}+1} & j_{\frac{n}{2}+1}^2 \end{pmatrix},$$

$$G_{(1,0,1)}^{(n,2)}(J) = -G_{(0,1,0)}^{(n,2)}(J) = \begin{pmatrix} \frac{4}{3}\left(J_{\frac{n}{2}-1}j_{\frac{n}{2}-1} + \frac{2(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{4}{3}\left(J_{\frac{n-1}{2}}j_{\frac{n-1}{2}} + \frac{(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{4}{3}\left(J_{\frac{n}{2}}j_{\frac{n}{2}} + \frac{2(-2)^{\frac{n}{2}}}{3}\right) \\ \frac{4}{3}\left(J_{\frac{n-1}{2}}j_{\frac{n-1}{2}} + \frac{(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{2}{3}\left(2J_{\frac{n}{2}}j_{\frac{n}{2}} + \frac{(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{4}{3}\left(J_{\frac{n+1}{2}}j_{\frac{n+1}{2}} + \frac{(-2)^{\frac{n}{2}}}{3}\right) \\ \frac{1}{3}\left(J_{\frac{n}{2}}j_{\frac{n}{2}} + \frac{2(-2)^{\frac{n}{2}}}{3}\right) & \frac{1}{3}\left(J_{\frac{n+1}{2}}j_{\frac{n+1}{2}} + \frac{(-2)^{\frac{n}{2}}}{3}\right) & \frac{1}{3}\left(J_{\frac{n}{2}+1}j_{\frac{n}{2}+1} + \frac{2(-2)^{\frac{n}{2}+1}}{3}\right) \end{pmatrix}$$

$$G_{(0,0,1)}^{(n,2)}(J) = -G_{(1,1,0)}^{(n,2)}(J) = \begin{pmatrix} \frac{4}{3}\left(J_{\frac{n}{2}-1}j_{\frac{n}{2}-1} - \frac{2(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{4}{3}\left(J_{\frac{n-1}{2}}j_{\frac{n-1}{2}} - \frac{(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{4}{3}\left(J_{\frac{n}{2}}j_{\frac{n}{2}} - \frac{2(-2)^{\frac{n}{2}}}{3}\right) \\ \frac{4}{3}\left(J_{\frac{n-1}{2}}j_{\frac{n-1}{2}} - \frac{(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{2}{3}\left(2J_{\frac{n}{2}}j_{\frac{n}{2}} - \frac{(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{4}{3}\left(J_{\frac{n+1}{2}}j_{\frac{n+1}{2}} - \frac{(-2)^{\frac{n}{2}}}{3}\right) \\ \frac{1}{3}\left(J_{\frac{n}{2}}j_{\frac{n}{2}} - \frac{2(-2)^{\frac{n}{2}}}{3}\right) & \frac{4}{3}\left(J_{\frac{n+1}{2}}j_{\frac{n+1}{2}} - \frac{(-2)^{\frac{n}{2}}}{3}\right) & \frac{1}{3}\left(J_{\frac{n}{2}+1}j_{\frac{n}{2}+1} - \frac{2(-2)^{\frac{n}{2}+1}}{3}\right) \end{pmatrix}.$$

Proof If matrix functions $G_{(k_1,k_2,k_3)}^{(n,2)}(J)$, $(k_1,k_2,k_3) \in \{0,1\}^3$ are given with formula (5) for an odd integer n, then

$$\begin{aligned} G_{(k_1,k_2,k_3)}^{(n,2)}(J) &= \sum_{i=1}^{3} g_{k_i}^{(n,2)}(\lambda_i) \prod_{\substack{j=1 \\ i\neq j}}^{3} \frac{J-\lambda_j I}{\lambda_i-\lambda_j},\ k_i \in \{0,1\},\ \lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 4, \\ &= \tfrac{1}{9}\left[g_{k_1}^{(n,2)}(-2)\left[J^2 - 5J + 4I\right] - g_{k_2}^{(n,2)}(1)\left[J^2 - 2J - 8I\right] + g_{k_3}^{(n,2)}(4)\left[J^2 + J - 2I\right]\right]. \end{aligned}$$

The matrix functions $G_{(k_1,k_2,k_3)}^{(n,2)}(J)$ are computed with

$$G_{(k_1,k_2,k_3)}^{(n,2)}(J) = \frac{1}{9}\left[(-2)^{n/2} e^{nk_1\pi i}\begin{pmatrix} 4 & 2 & -8 \\ 2 & 1 & -4 \\ -2 & -1 & 4 \end{pmatrix} + 1^{n/2} e^{nk_2\pi i}\begin{pmatrix} 4 & -4 & 4 \\ -4 & 4 & -4 \\ 1 & -1 & 1 \end{pmatrix} + 4^{n/2} e^{nk_3\pi i}\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 1 & 2 & 4 \end{pmatrix}\right] \tag{6}$$

by using the expression (4). We carry on the proof for the some special cases, such that $(k_1,k_2,k_3) \in \{0,1\}^3$, using the generalized Binet's formulas (1). Since the all cases have exactly the same computation ways, the matrix functions $G_{(k_1,k_2,k_3)}^{(n,2)}(J)$ are obtained by doing similar calculation for the other cases. For simplicity, we omit the details.

Let $k_1=0,k_2=0,k_3=0,$ be in (6), the matrix form of the function $G_{(0,0,0)}^{(n,2)}(J)=\left[a_{ij}\right]_{3\times3}$ is given as

$$G_{(0,0,0)}^{(n,2)}(J)=\begin{cases} a_{11}=\frac{4}{9}\left[\left(2^{\frac{n}{2}-1}\right)^2-2(-2)^{\frac{n}{2}-1}+\left((-1)^{\frac{n}{2}-1}\right)^2\right] & a_{12}=a_{21}=\frac{4}{9}\left(2^{-1}\left(2^{\frac{n}{2}}\right)^2+2^{-1}(-2)^{\frac{n}{2}}-\left((-1)^{\frac{n}{2}}\right)^2\right) \\ a_{22}=\frac{2}{9}\left(2\left(2^{\frac{n}{2}}\right)^2+2^{-1}(-2)^{\frac{n}{2}}+2\left((-1)^{\frac{n}{2}}\right)^2\right) & a_{23}=4a_{32}=\frac{4}{9}\left(2\left(2^2\right)^{\frac{n}{2}}-(-2)^{\frac{n}{2}}-\left((-1)^{\frac{n}{2}}\right)^2\right) \\ a_{33}=\frac{1}{9}\left(\left(2^{\frac{n}{2}+1}\right)^2-2(-2)^{\frac{n}{2}+1}+\left((-1)^{\frac{n}{2}+1}\right)^2\right) & a_{13}=4a_{31}=\frac{4}{9}\left(\left(2^{\frac{n}{2}}\right)^2-2(-2)^{\frac{n}{2}}+\left((-1)^{\frac{n}{2}}\right)^2\right) \end{cases}$$

$$G_{(0,0,0)}^{(n,2)}(J)=\begin{cases} a_{11}=4\left(\frac{2^{\frac{n}{2}-1}-(-1)^{\frac{n}{2}-1}}{3}\right)^2 & a_{12}=a_{21}=4\left(\frac{2^{\frac{n}{2}-1}-(-1)^{\frac{n}{2}-1}}{3}\right)\left(\frac{2^{\frac{n}{2}}-(-1)^{\frac{n}{2}}}{3}\right) \\ a_{33}=\left(\frac{2^{\frac{n}{2}+1}-(-1)^{\frac{n}{2}+1}}{3}\right)^2 & a_{22}=2\left(\left(\frac{2^{\frac{n}{2}}-(-1)^{\frac{n}{2}}}{3}\right)^2+\left(\frac{2^{\frac{n}{2}-1}-(-1)^{\frac{n}{2}-1}}{3}\right)\left(\frac{2^{\frac{n}{2}+1}-(-1)^{\frac{n}{2}+1}}{3}\right)\right) \\ a_{13}=4a_{31}=4\left(\frac{2^{\frac{n}{2}}-(-1)^{\frac{n}{2}}}{3}\right)^2 & a_{23}=4a_{32}=4\left(\frac{2^{\frac{n}{2}}-(-1)^{\frac{n}{2}}}{3}\right)\left(\frac{2^{\frac{n}{2}+1}-(-1)^{\frac{n}{2}+1}}{3}\right) \end{cases}$$

$$G_{(0,0,0)}^{(n,2)}(J)=\begin{pmatrix} 4J_{\frac{n}{2}-1}^2 & 4J_{\frac{n}{2}-1}J_{\frac{n}{2}} & J_{\frac{n}{2}}^2 \\ 4J_{\frac{n}{2}-1}J_{\frac{n}{2}} & 2\left(J_{\frac{n}{2}}^2+J_{\frac{n}{2}-1}J_{\frac{n}{2}+1}\right) & 4J_{\frac{n}{2}}J_{\frac{n}{2}+1} \\ J_{\frac{n}{2}}^2 & J_{\frac{n}{2}}J_{\frac{n}{2}+1} & J_{\frac{n}{2}+1}^2 \end{pmatrix}.$$

It is seen that the $G_{(1,1,1)}^{(n,2)}(J)=-G_{(0,0,0)}^{(n,2)}(J)$ in the case $k_1=1,\ k_2=1,\ k_3=1$.

Secondly, let $k_1=0,\ k_2=k_3=1$ be in (6), then the $G_{(0,1,1)}^{(n,2)}(J)=\left[b_{ij}\right]_{3\times3}$

$$G_{(0,1,1)}^{(n,2)}(J)=\begin{cases} b_{11}=\frac{4}{9}\left(2^{-2}\left(2^{\frac{n}{2}}\right)^2+2(-2)^{\frac{n}{2}-1}+\left((-1)^{\frac{n}{2}-1}\right)^2\right) & b_{12}=b_{21}=\frac{4}{9}\left(2^{-1}\left(2^{n/2}\right)^2-2^{-1}(-2)^{n/2}-\left((-1)^{n/2}\right)^2\right) \\ b_{22}=\frac{2}{9}\left(2\left(2^{n/2}\right)^2-2^{-1}(-2)^{n/2}+2\left((-1)^{n/2}\right)^2\right) & b_{13}=4b_{31}=\frac{4}{9}\left(\left(2^{n/2}\right)^2+2(-2)^{n/2}+\left((-1)^{n/2}\right)^2\right) \\ b_{33}=\frac{1}{9}\left(4\left(2^{n/2}\right)^2-4(-2)^{n/2}+\left((-1)^{n/2}\right)^2\right) & b_{23}=4b_{32}=\frac{4}{9}\left(2\left(2^{n/2}\right)^2+(-2)^{n/2}-\left((-1)^{n/2}\right)^2\right) \end{cases},$$

$$G_{(0,1,1)}^{(n,2)}(J)=\begin{cases} b_{11}=4\left(\frac{2^{\frac{n}{2}-1}+(-1)^{\frac{n}{2}-1}}{3}\right)^2 & b_{12}=b_{21}=4\left(\frac{2^{\frac{n}{2}-1}+(-1)^{\frac{n}{2}-1}}{3}\right)\left(\frac{2^{\frac{n}{2}}+(-1)^{\frac{n}{2}}}{3}\right) \\ b_{33}=\left(\frac{2^{\frac{n}{2}+1}+(-1)^{\frac{n}{2}+1}}{3}\right)^2 & b_{22}=2\left(\left(\frac{2^{\frac{n}{2}}+(-1)^{\frac{n}{2}}}{3}\right)^2+\left(\frac{2^{\frac{n}{2}-1}+(-1)^{\frac{n}{2}-1}}{3}\right)\left(\frac{2^{\frac{n}{2}+1}+(-1)^{\frac{n}{2}+1}}{3}\right)\right), \\ b_{13}=4b_{31}=4\left(\frac{2^{\frac{n}{2}}+(-1)^{\frac{n}{2}}}{3}\right)^2 & b_{23}=4b_{32}=4\left(\frac{2^{\frac{n}{2}}+(-1)^{\frac{n}{2}}}{3}\right)\left(\frac{2^{\frac{n}{2}+1}+(-1)^{\frac{n}{2}+1}}{3}\right) \end{cases}$$

$$G_{(0,1,1)}^{(n,2)}(J)=\frac{1}{9}\begin{pmatrix} 4j_{\frac{n}{2}-1}^2 & 4j_{\frac{n}{2}-1}j_{\frac{n}{2}} & 4j_{\frac{n}{2}}^2 \\ 4j_{\frac{n}{2}-1}j_{\frac{n}{2}} & 2\left(j_{\frac{n}{2}}^2+j_{\frac{n}{2}-1}j_{\frac{n}{2}+1}\right) & 4j_{\frac{n}{2}}j_{\frac{n}{2}+1} \\ j_{\frac{n}{2}}^2 & j_{\frac{n}{2}}j_{\frac{n}{2}+1} & j_{\frac{n}{2}+1}^2 \end{pmatrix}.$$

It is seen that the $G_{(1,0,0)}^{(n,2)}(J)=-G_{(0,1,1)}^{(n,2)}(J)$ in the case $k_1=1,\ k_2=k_3=0$.

Thirdly, let $k_1 = 1,\ k_2 = 0,\ k_3 = 1$ be in (6), the $G_{(1,0,1)}^{(n,2)}(J) = \left[c_{ij}\right]_{3\times 3}$ is obtained with

$$G_{(1,0,1)}^{(n,2)}(J) = \begin{cases} c_{11} = \frac{4}{3}\left(\frac{2^{\frac{n}{2}-1}-(-1)^{\frac{n}{2}-1}}{3}\left(2^{\frac{n}{2}-1}+(-1)^{\frac{n}{2}-1}\right)-\frac{2(-2)^{\frac{n}{2}-1}}{3}\right) & c_{12} = c_{21} = \frac{4}{3}\left(\frac{2^{\frac{n-1}{2}}-(-1)^{\frac{n-1}{2}}}{3}\left(2^{\frac{n-1}{2}}+(-1)^{\frac{n-1}{2}}\right)-\frac{(-2)^{\frac{n}{2}-1}}{3}\right) \\ c_{22} = \frac{2}{3}\left(2\frac{2^{\frac{n}{2}}-(-1)^{\frac{n}{2}}}{3}\left(2^{\frac{n}{2}}+(-1)^{\frac{n}{2}}\right)-\frac{(-2)^{\frac{n}{2}-1}}{3}\right) & c_{23} = 4c_{32} = \frac{4}{3}\left(\frac{2^{\frac{n+1}{2}}-(-1)^{\frac{n+1}{2}}}{3}\left(2^{\frac{n+1}{2}}+(-1)^{\frac{n+1}{2}}\right)-\frac{(-2)^{\frac{n}{2}}}{3}\right), \\ c_{33} = \frac{1}{3}\left(\frac{2^{\frac{n}{2}+1}-(-1)^{\frac{n}{2}+1}}{3}\left(2^{\frac{n}{2}+1}+(-1)^{\frac{n}{2}+1}\right)-\frac{2(-2)^{\frac{n}{2}+1}}{3}\right) & c_{13} = 4c_{31} = \frac{4}{3}\left(\frac{2^{\frac{n}{2}}-(-1)^{\frac{n}{2}}}{3}\left(2^{\frac{n}{2}}+(-1)^{\frac{n}{2}}\right)-\frac{2(-2)^{\frac{n}{2}}}{3}\right) \end{cases}$$

$$G_{(1,0,1)}^{(n,2)}(J) = \begin{bmatrix} \frac{4}{3}\left(J_{\frac{n}{2}-1}j_{\frac{n}{2}-1}-\frac{2(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{4}{3}\left(J_{\frac{n-1}{2}}j_{\frac{n-1}{2}}-\frac{(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{4}{3}\left(J_{\frac{n}{2}}j_{\frac{n}{2}}-\frac{2(-2)^{\frac{n}{2}}}{3}\right) \\ \frac{4}{3}\left(J_{\frac{n-1}{2}}j_{\frac{n-1}{2}}-\frac{(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{2}{3}\left(2J_{\frac{n}{2}}j_{\frac{n}{2}}-\frac{(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{4}{3}\left(J_{\frac{n+1}{2}}j_{\frac{n+1}{2}}-\frac{(-2)^{\frac{n}{2}}}{3}\right) \\ \frac{1}{3}\left(J_{\frac{n}{2}}j_{\frac{n}{2}}-\frac{2(-2)^{\frac{n}{2}}}{3}\right) & \frac{1}{3}\left(J_{\frac{n+1}{2}}j_{\frac{n+1}{2}}-\frac{(-2)^{\frac{n}{2}}}{3}\right) & \frac{1}{3}\left(J_{\frac{n}{2}+1}j_{\frac{n}{2}+1}-\frac{2(-2)^{\frac{n}{2}+1}}{3}\right) \end{bmatrix}.$$

In the case $k_1 = 0,\ k_2 = 1,\ k_3 = 0$, it is the $G_{(0,1,0)}^{(n,2)}(J) = -G_{(1,0,1)}^{(n,2)}(J)$.

Finally, let $k_1 = 0,\ k_2 = 0,\ k_3 = 1$ be in (6), then the $G_{(0,0,1)}^{(n,2)}(J) = \left[d_{ij}\right]_{3\times 3}$

$$G_{(0,0,1)}^{(n,2)}(J) = \begin{cases} d_{11} = \frac{4}{3}\left(\frac{2^{\frac{n}{2}-1}-(-1)^{\frac{n}{2}-1}}{3}\left(2^{\frac{n}{2}-1}+(-1)^{\frac{n}{2}-1}\right)+\frac{2(-2)^{\frac{n}{2}-1}}{3}\right) & d_{13} = 4d_{31} = \frac{4}{3}\left(\frac{2^{\frac{n}{2}}-(-1)^{\frac{n}{2}}}{3}\left(2^{\frac{n}{2}}+(-1)^{\frac{n}{2}}\right)+\frac{2(-2)^{\frac{n}{2}}}{3}\right) \\ d_{22} = \frac{2}{3}\left(2\frac{2^{\frac{n}{2}}-(-1)^{\frac{n}{2}}}{3}\left(2^{\frac{n}{2}}+(-1)^{\frac{n}{2}}\right)+\frac{(-2)^{\frac{n}{2}-1}}{3}\right) & d_{12} = d_{21} = \frac{4}{3}\left(\frac{2^{\frac{n-1}{2}}-(-1)^{\frac{n-1}{2}}}{3}\left(2^{\frac{n-1}{2}}+(-1)^{\frac{n-1}{2}}\right)+\frac{(-2)^{\frac{n}{2}-1}}{3}\right), \\ d_{33} = \frac{1}{3}\left(\frac{2^{\frac{n}{2}+1}-(-1)^{\frac{n}{2}+1}}{3}\left(2^{\frac{n}{2}+1}+(-1)^{\frac{n}{2}+1}\right)+\frac{2(-2)^{\frac{n}{2}+1}}{3}\right) & d_{23} = 4d_{32} = \frac{4}{3}\left(\frac{2^{\frac{n+1}{2}}-(-1)^{\frac{n+1}{2}}}{3}\left(2^{\frac{n+1}{2}}+(-1)^{\frac{n+1}{2}}\right)+\frac{(-2)^{\frac{n}{2}}}{3}\right) \end{cases}$$

$$G_{(0,0,1)}^{(n,2)}(J) = \begin{bmatrix} \frac{4}{3}\left(J_{\frac{n}{2}-1}j_{\frac{n}{2}-1}+\frac{2(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{4}{3}\left(J_{\frac{n-1}{2}}j_{\frac{n-1}{2}}+\frac{(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{4}{3}\left(J_{\frac{n}{2}}j_{\frac{n}{2}}+\frac{2(-2)^{\frac{n}{2}}}{3}\right) \\ \frac{4}{3}\left(J_{\frac{n-1}{2}}j_{\frac{n-1}{2}}+\frac{(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{2}{3}\left(2J_{\frac{n}{2}}j_{\frac{n}{2}}+\frac{(-2)^{\frac{n}{2}-1}}{3}\right) & \frac{4}{3}\left(J_{\frac{n+1}{2}}j_{\frac{n+1}{2}}+\frac{(-2)^{\frac{n}{2}}}{3}\right) \\ \frac{1}{3}\left(J_{\frac{n}{2}}j_{\frac{n}{2}}+\frac{2(-2)^{\frac{n}{2}}}{3}\right) & \frac{1}{3}\left(J_{\frac{n+1}{2}}j_{\frac{n+1}{2}}+\frac{(-2)^{\frac{n}{2}}}{3}\right) & \frac{1}{3}\left(J_{\frac{n}{2}+1}j_{\frac{n}{2}+1}+\frac{2(-2)^{\frac{n}{2}+1}}{3}\right) \end{bmatrix}.$$

It is seen that $G_{(1,1,0)}^{(n,2)}(J) = -G_{(0,0,1)}^{(n,2)}(J)$ is in the case $k_1 = 1,\ k_2 = 1,\ k_3 = 0$.

Now, let us consider the scalar complex function

$$G^{(p,q)}(z) \equiv z^{p/q} = g_k^{(p,q)}(z),\ k \in \{0,1,...,q-1\},$$

where $(p,q) \in \left(\mathbb{Z} - \{0\} \times \mathbb{N}^+\right)$, such that p and q are relatively prime. The $G^{(p,q)}(z)$ are q-valued functions:

$$g_k^{(p,q)}(z) = |z|^{p/q} \exp\left[i\left(\frac{p}{q}\arg(z) + k\pi n\right)\right], k \in \{0,1,...,q-1\},$$

$$g_k^{(p,q)}(z) = \exp\left(\frac{2ipk\pi}{q}\right)z^{p/q}, \text{ and } g_0^{(p,q)}(z) = z^{p/q} = |z|^{p/q} \exp\left[i\frac{p}{q}\arg(z)\right].$$

The matrix functions are given as follows:

$$G_{(k_1,k_2,k_3)}^{(p,q)}(J) = \sum_{i=1}^{3} g_{k_i}^{(p,q)}(\lambda_i) \prod_{\substack{j=1 \\ i \neq j}}^{3} \frac{J - \lambda_j I}{\lambda_i - \lambda_j},\ k_i \in \{0,1...q-1\}.$$

Hence, there is a unique matrix polynomial $q(J)$ of degree 2 for every the matrix function $J^{p/q}$ such that

$$J^{p/q} = c_0 I + c_1 J + c_2 J^2. \tag{7}$$

Notice that the polynomial $q(J)$ mentioned above is nothing else but the Lagrange-Sylvester polynomial [9], [10]. Therefore,

$$\begin{aligned}
G^{(p,q)}_{(k_1,k_2,k_3)}(J) &= \sum_{i=1}^{3} g^{(p,q)}_{k_i}(\lambda_i) \prod_{\substack{j=1 \\ i\neq j}}^{3} \frac{J-\lambda_j I}{\lambda_i - \lambda_j}, k_i \in \{0,1...q-1\}, \\
&= \frac{g^{(p,q)}_{k_i}(-2)}{18}(J-I)(J-4I) - \frac{g^{(p,q)}_{k_i}(1)}{9}(J+2I)(J-4I) + \frac{g^{(p,q)}_{k_i}(4)}{18}(J+2I)(J-I) \\
&= \frac{1}{18}\left[4^{p/q} e^{\frac{2ipk_3\pi}{q}} + (-2)^{p/q} e^{\frac{2ipk_1\pi}{q}} - 2e^{\frac{2ipk_2\pi}{q}}\right] J^2 + \frac{1}{18}\left[4^{p/q} e^{\frac{2ipk_3\pi}{q}} - 5(-2)^{p/q} e^{\frac{2ipk_1\pi}{q}} + 4e^{\frac{2ipk_2\pi}{q}}\right] J \\
&\quad - \frac{1}{9}\left[4^{p/q} e^{\frac{2ipk_3\pi}{q}} - 2(-2)^{p/q} e^{\frac{2ipk_1\pi}{q}} - 8e^{\frac{2ipk_2\pi}{q}}\right] I
\end{aligned} \tag{8}$$

Therefore the scalars c_0, c_1 and c_2 in (7) are determined by

$$c_0 = -\tfrac{1}{9}\left[(4)^{\frac{p}{q}} e^{\frac{2ipk_3\pi}{q}} - 2(-2)^{\frac{p}{q}} e^{\frac{2ipk_1\pi}{q}} - 8e^{\frac{2ipk_2\pi}{q}}\right],$$

$$c_1 = \tfrac{1}{9}\left[\tfrac{1}{2}4^{\frac{p}{q}} e^{\frac{2ipk_3\pi}{q}} - \tfrac{5}{2}(-2)^{\frac{p}{q}} e^{\frac{2ipk_1\pi}{q}} + 2e^{\frac{2ipk_2\pi}{q}}\right],$$

$$c_2 = \tfrac{1}{9}\left[\tfrac{1}{2}4^{\frac{p}{q}} e^{\frac{2ipk_3\pi}{q}} + \tfrac{1}{2}(-2)^{\frac{p}{q}} e^{\frac{2ipk_1\pi}{q}} - e^{\frac{2ipk_2\pi}{q}}\right].$$

The case $k_1 = k_2 = k_3 = k$ values are rewritten in the equation (8), it follows that

$$c_0 = -8e^{\frac{2ipk\pi}{q}} \frac{2^{\frac{p}{q}-2} - e^{i\pi\left(\frac{p}{q}-2\right)}}{3} \frac{2^{\frac{p}{q}-1} - e^{i\pi\left(\frac{p}{q}-1\right)}}{3} = -8e^{\frac{2ipk\pi}{q}} J_{\frac{p}{q}-2} J_{\frac{p}{q}-1}$$

$$c_1 = 2e^{\frac{2ipk\pi}{q}} \frac{2^{\frac{p}{q}-2} - e^{i\pi\left(\frac{p}{q}-2\right)}}{3} \frac{2^{\frac{p}{q}} - e^{i\pi\frac{p}{q}}}{3} = 2e^{\frac{2ipk\pi}{q}} J_{\frac{p}{q}-2} J_{\frac{p}{q}}$$

$$c_2 = e^{\frac{2ipk\pi}{q}} \frac{2^{\frac{p}{q}-1} - e^{i\pi\left(\frac{p}{q}-1\right)}}{3} \frac{2^{\frac{p}{q}} - e^{i\pi\frac{p}{q}}}{3} = e^{\frac{2ipk\pi}{q}} J_{\frac{p}{q}-1} J_{\frac{p}{q}}$$

Theorem 4. The matrix functions $G^{(p,q)}_{(k,k,k)}(J) = \left[f_{ij}\right]_{3\times 3}$ are again obtained matrix equations with

$$G^{(p,q)}_{(k,k,k)}(J) = \exp\left(\tfrac{2ipk\pi}{q}\right)\left(J_{\frac{p}{q}-1} J_{\frac{p}{q}} J^2 + 2J_{\frac{p}{q}-2} J_{\frac{p}{q}} J - 8J_{\frac{p}{q}-2} J_{\frac{p}{q}-1} I\right).$$

If we think about in the case $p := n$ and $q := 2$ in (8), then we have the similar matrix equations given in the Theorem 3

$$\begin{aligned}
G^{(n,2)}_{(k_1,k_2,k_3)}(J) = \frac{1}{9}\Big[&\left(\tfrac{1}{2}4^{\frac{n}{2}} e^{nk_3\pi i} + \tfrac{1}{2}(-2)^{\frac{n}{2}} e^{nk_1\pi i} - e^{nk_2\pi i}\right) J^2 + \left(\tfrac{1}{2}4^{\frac{n}{2}} e^{nk_3\pi i} - \tfrac{5}{2}(-2)^{\frac{n}{2}} e^{nk_1\pi i} + 2e^{nk_2\pi i}\right) J \\
&- \left(4^{\frac{n}{2}} e^{nk_3\pi i} - 2(-2)^{\frac{n}{2}} e^{nk_1\pi i} - 8e^{nk_2\pi i}\right) I\Big].
\end{aligned}$$

As a result, if some special cases for $(k_1, k_2, k_3) \in \{0,1\}^3$ are evaluated, then

$$\begin{aligned}
G^{(n,2)}_{(0,0,0)}(J) &= \frac{1}{9}\left[\left(2^{\frac{n}{2}-1} 2^{\frac{n}{2}} + \tfrac{1}{2}(-2)^{\frac{n}{2}} - 1\right) J^2 + \left(\tfrac{1}{2}4^{\frac{n}{2}} - \tfrac{1}{2}5(-2)^{\frac{n}{2}} + 2\right) J - \left(4^{\frac{n}{2}} - 2.(-2)^{\frac{n}{2}} - 8\right)\right] \\
&= \frac{2^{\frac{n}{2}} - (-1)^{\frac{n}{2}}}{3} \frac{2^{\frac{n}{2}-1} - (-1)^{\frac{n}{2}-1}}{3} J^2 + \frac{2^{\frac{n}{2}} - (-1)^{\frac{n}{2}}}{3} \frac{2^{\frac{n}{2}-2} - (-1)^{\frac{n}{2}-2}}{3} J - 8 \frac{2^{\frac{n}{2}-1} - (-1)^{\frac{n}{2}-1}}{3} \frac{2^{\frac{n}{2}-2} - (-1)^{\frac{n}{2}-2}}{3} I
\end{aligned}$$

and for $k_1 = 1, k_2 = 1, k_3 = 1$, this case is negative of the other case. The two cases can be examined with similar way, and achieved as

$$G^{(n,2)}_{(k_1,k_2,k_3)}(J) = \begin{cases} \pm\left(J_{\frac{n}{2}} J_{\frac{n}{2}-1} J^2 + 2J_{\frac{n}{2}} J_{\frac{n}{2}-2} J - 8J_{\frac{n}{2}-1} J_{\frac{n}{2}-2} I\right), & \begin{cases} if & k_1 = 0, k_2 = 0, k_3 = 0 \\ if & k_1 = 1, k_2 = 1, k_3 = 1 \end{cases} \\ \pm\frac{}{9}\left(j_{\frac{n}{2}} j_{\frac{n}{2}-1} J^2 + 2j_{\frac{n}{2}} j_{\frac{n}{2}-2} J - 8j_{\frac{n}{2}-1} j_{\frac{n}{2}-2} I\right), & \begin{cases} if & k_1 = 0, k_2 = 1, k_3 = 1 \\ if & k_1 = 1, k_2 = 0, k_3 = 0 \end{cases} \end{cases} \tag{9}$$

Also, by equating corresponding elements of matrices in equalities (9) and the Theorem 3, we can obtain the following identities. Let us consider that equating of the matrix $G_{(0,0,0)}^{(n,2)}(J)$, then

$i)\quad J_{\frac{n}{2}-1}^{2}=J_{\frac{n}{2}}J_{\frac{n}{2}-1}-2J_{\frac{n}{2}-1}J_{\frac{n}{2}-2}=J_{\frac{n}{2}-1}\left(J_{\frac{n}{2}}-2J_{\frac{n}{2}-2}\right),\quad ii)\quad J_{\frac{n}{2}+1}^{2}=9J_{\frac{n}{2}}J_{\frac{n}{2}-1}+2J_{\frac{n}{2}}J_{\frac{n}{2}-2}-8J_{\frac{n}{2}-1}J_{\frac{n}{2}-2},$

$iii)\quad J_{\frac{n}{2}}^{2}=J_{\frac{n}{2}}J_{\frac{n}{2}-1}+2J_{\frac{n}{2}}J_{\frac{n}{2}-2}=J_{\frac{n}{2}}\left(J_{\frac{n}{2}-1}+2J_{\frac{n}{2}-2}\right),\qquad iv)\quad J_{\frac{n}{2}}J_{\frac{n}{2}+1}=3J_{\frac{n}{2}}J_{\frac{n}{2}-1}+2J_{\frac{n}{2}}J_{\frac{n}{2}-2}.$

or equating of the matrices $G_{(0,1,1)}^{(n,2)}(J)$ gives

$i)\quad j_{\frac{n}{2}-1}^{2}=j_{\frac{n}{2}}j_{\frac{n}{2}-1}-2j_{\frac{n}{2}-1}j_{\frac{n}{2}-2}=j_{\frac{n}{2}-1}\left(j_{\frac{n}{2}}-2j_{\frac{n}{2}-2}\right),\quad ii)\quad j_{\frac{n}{2}+1}^{2}=9j_{\frac{n}{2}}j_{\frac{n}{2}-1}+2j_{\frac{n}{2}}j_{\frac{n}{2}-2}-8j_{\frac{n}{2}-1}j_{\frac{n}{2}-2},$

$iii)\quad j_{\frac{n}{2}}^{2}=j_{\frac{n}{2}}j_{\frac{n}{2}-1}+2j_{\frac{n}{2}}j_{\frac{n}{2}-2}=j_{\frac{n}{2}}\left(j_{\frac{n}{2}-1}+2j_{\frac{n}{2}-2}\right),\qquad iv)\quad j_{\frac{n}{2}}j_{\frac{n}{2}+1}=3j_{\frac{n}{2}}j_{\frac{n}{2}-1}+2j_{\frac{n}{2}}j_{\frac{n}{2}-2}.$

By definition, an alternative way to obtain a square root of the matrix J is to solve the matrix equation $G^{(n,2)}(J)\times G^{(n,2)}(J)=J^n$, that is, the square roots of the matrix J^n are the matrices $G^{(n,2)}(J)$. Then, for each $G_{(k_1,k_2,k_3)}^{(n,2)}(J)$ in given Theorem 3 and the matrices in (9), this states $G_{(k_1,k_2,k_3)}^{(n,2)}(J)\times G_{(k_1,k_2,k_3)}^{(n,2)}(J)=J^n$.

Let $J_{\frac{n}{2}}$ denotes the complex Jacobsthal number in branches $G_{(0,0,0)}^{(n,2)}(J)$, and $j_{\frac{n}{2}}$ denotes the complex Jacobsthal Lucas number in branches $G_{(0,1,1)}^{(n,2)}(J)$. Therefore, by equating corresponding elements for matrix equation $G_{(0,0,0)}^{(n,2)}(J)\times G_{(0,0,0)}^{(n,2)}(J)=J^n$, we achieve some complex Jacobsthal identities, and from matrix equation $G_{(0,1,1)}^{(n,2)}(J)\times G_{(0,1,1)}^{(n,2)}(J)=J^n$, some complex Jacobsthal Lucas identities are achieved:

$i)\ J_n^2+J_{n-1}J_{n+1}=2\left(J_{\frac{n}{2}}^2+J_{\frac{n}{2}-1}J_{\frac{n}{2}+1}\right)^2+8J_{\frac{n}{2}}^2J_{\frac{n}{2}-1}^2+2J_{\frac{n}{2}}^2J_{\frac{n}{2}+1}^2,\ ii)\ J_nJ_{n+1}=J_{\frac{n}{2}}J_{\frac{n}{2}+1}^3+4J_{\frac{n}{2}}^3J_{\frac{n}{2}-1}+2\left(J_{\frac{n}{2}}^2+J_{\frac{n}{2}-1}J_{\frac{n}{2}+1}\right)J_{\frac{n}{2}}J_{\frac{n}{2}+1},$

$iii)\ J_{n-1}^2=J_{\frac{n}{2}}^4+4J_{\frac{n}{2}}^2J_{\frac{n}{2}-1}^2+4J_{\frac{n}{2}-1}^4=\left(J_{\frac{n}{2}}^2+2J_{\frac{n}{2}-1}^2\right)^2,\qquad iv)\ 81J_{n+1}^2=4j_{\frac{n}{2}}^4+4j_{\frac{n}{2}}^2j_{\frac{n}{2}+1}^2+j_{\frac{n}{2}+1}^4=\left(j_{\frac{n}{2}+1}^2+2j_{\frac{n}{2}}^2\right)^2,$

$v)\ 81J_nJ_{n+1}=j_{\frac{n}{2}}j_{\frac{n}{2}+1}^3+4j_{\frac{n}{2}}^3j_{\frac{n}{2}-1}+2\left(j_{\frac{n}{2}}^2+j_{\frac{n}{2}-1}j_{\frac{n}{2}+1}\right)j_{\frac{n}{2}}j_{\frac{n}{2}+1},\quad vi)\ J_n^2+J_{n-1}J_{n+1}=\frac{2}{81}\left(j_{\frac{n}{2}}^2+j_{\frac{n}{2}-1}j_{\frac{n}{2}+1}\right)^2+\frac{8}{81}j_{\frac{n}{2}}^2j_{\frac{n}{2}-1}^2+\frac{2}{81}j_{\frac{n}{2}}^2j_{\frac{n}{2}+1}^2.$

III. Conclusion

The matrix function $G(J)=b_0I+b_1J+b_2J^2$ is well known from the theory of functions of matrices where I is the 3×3 identity matrix. Therefore, any power n^{th} of the Jacobsthal matrix J was generated by solving the system $G(\lambda_i)=b_0+b_1\lambda_i+b_2\lambda_i^2\quad i=1,2,3$. This matrix contains the different multiplication and squares of the n^{th} term of the Jacobsthal sequence, and also is obtained as matrix formula. Furthermore, the functions $G^n(J)=J^n, n\in\mathbb{Z}$ and $G^{(p,q)}(J)\equiv J^{p/q}, \gcd(p,q)=1$ were defined on the spectrum of the Jacobsthal matrix J. Then, the matrix functions of the matrix J are defined by these complex functions, and the technique mentioned above were applied to compute any square root of the Jacobsthal matrix J^n.

References

[1] N. J. A. Sloane, *A Handbook of Integer Sequences*, (New York: Academic Press, 1973).
[2] A.F. Horadam, *Jacobsthal representation numbers*, *The Fibonacci Quarterly, 34, No. 1*, 1996, 40 - 54.
[3] G. Berzsenyi. Solution to Problem H-274. *The Fibonacci Quarterly 17. No. 1*, 1979, 95.
[4] F. Koken and D. Bozkurt, On The Jacobsthal numbers by matrix methods, *Int. J. Cont. Math. Sci., V.3, N.13*, 2008, 605-614.
[5] F. Koken, D. Bozkurt, On The Jacobsthal-Lucas Numbers by Matrix Methods, *Int. J. Cont.. Math. Sci., V.3, N.33*, 2008, 1629-1633.
[6] A.F. Horadam, Jacobsthal and Pell Curves, *The Fibonacci Quarterly, 26, No. 1*, 1988, 77 - 83.
[7] S. Arslan and F. Koken, The Jacobsthal and Jacobsthal-Lucas Numbers via Square Roots of Matrices, *Int. Math. Forum, Vol. 11, no. 11*, 2016, 513 - 520.
[8] R. S. Melham, Lucas Sequences and Functions of a 3-by-3 Matrix, *The Fibonacci Quarterly 37. No. 2*, 1999, 111-16.
[9] F. R. Gantmacher, *The Theory of Matrices*, (Chelsea, New York, 1960).
[10] N. J. Higham, *Functions of Matrices: Theory and Computation*, (Soc. for In. and App. Math., Philadelphia, PA, USA, 2008).

Fikri Köken, Saadet Arslan, "Properties of Square Roots of the Jacobsthal Matrix of Order 3x3. "(IOSR-JM) 14.2 (2018): 61-67.