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Common Fixed Point Results in Fuzzy Metric Spaces and Occasionally Weakly Compatible Mappings

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Abstract: *in this paper,we introduce a new concept of common fixed point theorem in fuzzy metric spaces and occasionally weakly compatible mappings .*

Key Words: *fixed point , common fixed point theorem , fuzzy metric space , occasionally weakly compatible mappings AMS subject classification : Primary :47H10, Secondary :45G10.*

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I. Introduction and Preliminaries

The concept of Fuzzy sets was introduced by Zadeh [101] in 1965 at the University of California. Zadeh was almost single-handedly responsible for the early development in this field. In mathematical programming, problems are expressed as optimizing some goal function given certain constraints, and there are real life problems that consider multiple objectives. Generally, it is very difficult to get a feasible solution that brings us to the optimum of all objective functions.

Since then, there were many authors who studied the fuzzy sets with applications, especially George and Veeramani [35] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [57]. A. George and P. Veermani [35] revised the notion of fuzzy metric spaces with the help of continuous t-norm in 1994. As a result of many fixed point theorem for various forms of mapping are obtained in fuzzy metric spaces. Recently, many researchers have proved common fixed point theorems involving fuzzy sets. Pant [77] introduced the new concept of reciprocally continuous mappings and established some common fixed point theorems. Vasuki [99] and Singh and Chouhan [96] also introduced some fixed point theorems in fuzzy metric spaces for R-weakly commuting and compatible mappings respectively.

Balasubramaniam et al. [14] proved the open problem of Rhoades [86] on the existence of a contractive definition which generates a fixed point but does not force the mapping to be continuous at the fixed point, posses an affirmative answer. Pant and Jha [76] proved an anologus of the result given by Balasubramaniam et al., [14]. Recent work on fixed point theorems in fuzzy metric space can be viewed in references [1, 37, 53, 57].

Our aim in this chapter is to prove some fixed and common fixed point results using fuzzy metric spaces.

To prove of our results we need some definitions which are as follows;

Definition 1.1: *A fuzzy set A in X is a function with domain X and values in* $[0,1]$.

Definition 1.2: *A triangular norm* $*$ *(shortly t− norm) is a binary operation on the unit interval* $[0,1]$ *such that for all* $a,b,c,d \in [0,1]$ *the following conditions are satisfied:*
1.2 (a) $*$ *is commutative and associative;*
1.2 (b) $*$ *is continuous;*
1.2 (c) $a * 1 = a, \quad \forall\ a \in [0,1]$;
1.2 (d) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$.
Two typical examples of continuous t-norm are $a * b = ab$ *and* $a * b = min\{a,b\}$.

Definition 1.3: *A 3-tuple* $(X, M, *)$ *is said to be a fuzzy metric space, if* X *is an arbitrary set,* $*$ *is a continuous t-norm and M is a fuzzy set on* $X^2 \times (0,\infty)$ *satisfying the following conditions: for all* $x,y,z \in X$ *and* $s,t >$
1.3 (a) $M(x,y,0) > 0$;
1.3 (b) $M(x,y,t) = 1$, *for all* $t > 0$, *if and only if* $x = y$;
1.3 (c) $M(x,y,t) = M(y,x,t)$;
1.3 (d) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
1.3 (e) $M(x,y,*): (0,\infty) \to (0,1]$ *is continuous,*

where $M(x, y, t)$ denote the degree of nearness between x and y with respect to t. Then M is called a fuzzy metric on X.

Definition 1.4: *Let $(X, M,*)$ be a fuzzy metric space. Then*
1.4 (a) *a sequence $\{x_n\}$ in X is said to converges to x in X if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in N$ such that*

$$M(x_n, x, t) > 1 - \varepsilon,$$

for all $n \geq n_0$.
1.4 (b) *a sequence $\{x_n\}$ in X is said to Cauchy if for each $\varepsilon > 0$ and each $t > 0$,there exists $n_0 \in N$ such that*

$$M(x_n, x_m, t) > 1 - \varepsilon,$$

for all $m > n$ and $m, n \geq n_0$.
1.4 (c) *a fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*

Definition 1.5: *Two mappings f and g of a fuzzy metric space $(X, M,*)$ into itself are said to be weakly commuting if*

$$M(fgx, gfx, t) \geq M(fx, gx, t),$$

for all $x \in X$ and $t > 0$.

Definition 1.6: *Two mappings f and g of a fuzzy metric space $(X, M,*)$ into itself are R- weakly commuting provided there exists some positive real number R such that*

$$M(fgx, gfx, t) \geq M\left(fx, gx, \frac{t}{R}\right),$$

for all $x \in X, R > 0$ and $t > 0$.

Remark 1: *Clearly point R- weakly commutativity implies weak commutativity only when $R \leq 1$.*

Definition 1.7: *Two self maps f and g of a fuzzy metric space $(X, M,*)$ are called reciprocally continuous on X if*

$$lim_{n\to\infty} fgx_n = fx \text{ and } lim_{n\to\infty} gfx_n = gx,$$

whenever $\{x_n\}$ is a sequence in X such that

$$lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = x$$

or some x in X.

Definition 1.8: *Two self mappings f and g of a fuzzy metric space $(X, M,*)$ are called compatible if*

$$lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that

$$lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = x$$

for some x in X.

Definition 1.9: *Two self maps f and g of a fuzzy metric space $(X, M,*)$ are called reciprocally continuous on X, if*

$$lim_{n\to\infty} fgx_n = fx \text{ and } lim_{n\to\infty} gfx_n = gx,$$

whenever $\{x_n\}$ is a sequence in X such that

$$lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = x$$

for some x in X.

Definition 1.10: *Let $(X, M,*)$ be a fuzzy metric space.If there exists $q \in (0,1)$, such that*

$$M(x, y, qt) \geq M(x, y, t)$$

for all $x, y \in X$ and $t > 0$.

Definition 1.11: *Let X be a set and f, g self maps of X. A point x in X is called coincidence point of f and g , iff $fx = gx$.*
we say $w = fx = gx$, a point of coincidence of f and g.

Definition 1.12: *Two self maps A and B of a fuzzy metric space $(X, M,*)$ are called weak-compatible (or coincidentally commuting) if they commute at their coincidence point, i.e., if $Ax = Bx$ then $ABx = BAx$ for some $x \in X$.*

Lemma 1.13: *Let X be a set f, g OWC self maps of X. If f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.*

II. Main Theorem

4.2: Common Fixed Point Theorem for Occasionally Weakly Compatible Mappings

In this section we prove some common fixed point theorems satisfying occasionally weakly compatible mappings in fuzzy metric spaces. In fact we prove following results.

Theorem 2.1: *Let $(X, M,*)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ are OWC. If there exists a point $q \in (0,1)$, for all $x, y \in X$ and $t > 0$, such that*

$$\begin{aligned} M(Ax, By, qt) \geq \alpha_1 \ & min\{ M(Sx, Ty, t), M(Sx, Ax, t)\} \\ & +\alpha_2\, min\{ M(By, Ty, t), M(Ax, Ty, t)\} \\ & +\alpha_3\, M(By, Sx, t) + \alpha_4 M(Ax, By, t) \qquad 2.1(i) \end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3 > 0$, and $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 1$.

Then there exists a unique point of $w \in X$, such that $Aw = Sw = w$ and a unique point $z \in X$, such that $Bz = Tz = z$.Moreover $z = w$, so that there is a unique common fixed point of A, B, S and T.

Proof: *Let the pairs $\{A, S\}$ and $\{B, T\}$ be OWC, there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$, we claim that $Ax = By$. From $4.2.1(i)$ we have*

$$\begin{aligned} M(Ax, By, qt) & \geq \alpha_1 \ min\{ M(Sx, Ty, t), M(Sx, Ax, t)\} \\ & +\alpha_2\, min\{ M(By, Ty, t), M(Ax, Ty, t)\} \\ & + \alpha_3\, M\big(By, Sx, t) + \alpha_4 M(Ax, By, t)\big) \\ & = \alpha_1\, min\{ M(Ax, By, t), M(Ax, Ax, t)\} \\ & +\alpha_2\ min\{M(By, By, t), M(Ax, By, t)\} \\ & +\alpha_3 M(By, Ax, t)+\alpha_4 M(Ax, By, t)+ \\ & = \alpha_1\, min\{ M(Ax, By, t), 1\} \\ & +\alpha_2\ min\{1, M(Ax, By, t)\} \\ & +\alpha_3 M(By, Ax, t)+\alpha_4 M(Ax, By, t) \\ & = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\, M(Ax, By, t) \end{aligned}$$

A contradicition, since $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 1$. Therefore $Ax = By$, i.e., $Ax = Sx = By = Ty$. Suppose that there is a another point z such that $Az = Sz$ then by $2.1(i)$ we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. Using Lemma 1.13, we get w is the only common fixed point of A and S. i.e., $w = Aw = Sw$.

Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$.

Assume that $w \neq z$. We have

$$\begin{aligned} M(w, z, qt) = M(Aw, Bz, qt) & \\ & \geq \alpha_1 min\{M(Sw, Tz, t), M(Sw, Az, t)\} \\ & +\alpha_2 min\{M(Bz, Tz, t), M(Aw, Tz, t)\} \\ & +\alpha_3 M(Bz, Sw, t) + \alpha_4\, M(w, z, t) \\ & = \alpha_1 min\{M(w, z, t), M(w, z, t)\} \\ & +\alpha_2\, min\{M(z, z, t), M(w, z, t)\} \\ & +\alpha_3 M(z, w, t) + \alpha_4\, M(w, z, t) \\ & = \alpha_1 min\{M(w, z, t), 1\} \\ & +\alpha_2\, min\{1, M(w, z, t)\} \\ & +\alpha_3 M(z, w, t) + \alpha_4\, M(w, z, t) \\ & = \alpha_1 M(w, z, t) + \alpha_2\, M(w, z, t) + \alpha_3\, M(z, w, t) \\ & = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\, M(w, z, t) \end{aligned}$$

Since $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 1$.Therefore, a contradiction, we have $z = w$ also z is a common fixed point of A, B, S and T.

The uniqueness of the fixed point holds from $4.2.1(i)$.

Corroloary 2.2: *Let $(X, M,*)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ are OWC. If there exists a point $q \in (0,1)$, for all $x, y \in X$, such that*

$$\begin{aligned} M(Ax, By, qt) \geq \alpha\, & M(Sx, Ty, t) \\ & +\beta min\,\{ M(Sx, Ax, t), M(Ax, Ty, t)\} \\ & +\gamma mi\,n\{M(By, Ty, t), M(By, Sx, t)\} + \delta M(Ax, By, t) \qquad 2.2(i) \end{aligned}$$

where $\alpha, \beta, \gamma > 0,$ *and* $(\alpha + \beta + \gamma + \delta) > 1$ *then there exist a unique point* $w \in X,$ *such that* $Aw = Sw = w,$ *and a unique point* $z \in X,$ *such that* $Bz = Tz = z.$ *Moreover* $z = w,$ *so that it is a unique common fixed point of* A, B, S *and* $T.$

Corroloary 2.3*: Let* $(X, M,*)$ *be a complete fuzzy metric space and let A and S be selfmap of* $X.$ *Let the A and S are OWC. If there exists a point* $q \in (0,1)$ *for all* $x, y \in X$ *and* $t > 0,$ *such that*

$$M(Sx, Sy, qt) \geq \alpha_1 M(Ax, Ay, t) + \alpha_2 M(Ax, Ay, t) + \alpha_3 \min\{M(Sy, Ay, t), M(Sx, Ay, t)\} + \alpha_4 M(Sy, Ax, t), \qquad 4.2.3(i)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0,$ *and* $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 1.$ *Then A and S have a unique common fixed point.*

Corollary 2.4*: Let* $(X, M,*)$ *be a complete fuzzy metric space and let* A, B, S *and* T *be self-mappings of* $X.$ *Let the pairs* (A, S) *and* (B, T) *are OWC. If there exists a point* $q \in (0,1),$ *for all* $x, y \in X,$ *such that*

$$M(Ax, By, qt) \geq \delta\left(\min\left\{\begin{matrix} M(Sx, Ty, t), \\ M(Sx, Ax, t), M(By, Ty, t), \\ [M(Ax, Ty, t) + M(By, Sx, t)] + M(Ax, By, t)\end{matrix}\right\}\right) \qquad 4.2.5(i)$$

Such that $\delta(t) > t$ *for all* $0 < t < 1,$ *and* $\delta: [0,1] \rightarrow [0,1],$ *then there exists a unique common fixed point of* A, B, S *and* $T.$

Corollary 2.6*: Let* $(X, M,*)$ *be a complete fuzzy metric space and let* A, B, S *and* T *be self-mappings of* $X.$ *Let the pairs* (A, S) *and* (B, T) *are OWC. If there exists a point* $q \in (0,1),$*such that*

$$M(Ax, By, qt) \geq \delta\left(\begin{matrix} M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ [M(Ax, Ty, t) + M(By, Sx, t) + M(Ax, By, t)]\end{matrix}\right)$$

for all $x, y \in X$ *and* $\delta: [0,1]^5 \rightarrow [0,1]$ *such that* $\delta(t, 1, 1, t, t) > t$ *for all* $0 < t < 1,$ *then there exists a unique common fixed point of* A, B, S *and* $T.$

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