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Extension of Some Theoremsin General Metric Spaces

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Abstract:Abstract: *We prove a version ofCaristi-Kirk - Browder Theorem and Park's Theorem (Park, 198) and (Park and Rhoades, 1983) in G-metric space. And then give some corollaries.*
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I. Introduction

In 2006, a general metric space was introduced by Mustafa and Sims, as appropriate notion of generalized metric space called G-metric spaces as follows.

Definition (1.1)(Mustafa, Sims, 2006)
(1) 'Let X be a non-empty set and G:X×X×X→R^+ be a function for all x,y,z,a in X satisfying the following conditions: $(x, y, z) = 0$ if $x = y = z$
(2) $0 < G(x, x, y)$ with $x \neq y$
(3) $G(x, x, y) \leq G(x, y, z)$ with $y \neq z$
(4) $G(x, y, z) = G\big(P(x, y, z)\big), P(x, y, z)$ is permutation of x, y, z
(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$

Then the ordered pair (X, G) is called a generalized metric or G-metric space. X is said to be symmetric if for all x, y in X

$G(x, y \quad , y) = G(y, x, x)$."

Proposition (1. 3) (Mustafa, Sims, 2006)
"Let (X, G) be a G-metric space Then for any $u, v, w, and\ b \in X$, the following are satisfies
(1)if$G(u, v, w) = 0$ Then $u = v = w$
(2)$G(u, v, w) \leq G(u, u, v) + G(u, u, w)$
(3)$G(u, v, v) \leq 2G(v, u, u)$
(4)$G(u, v, w) \leq G(u, b, w) + G(b, v, w)$
(5)$G(u, v, w) \leq 2/3\big(G(u, v, b) + G(u, b, w) + G(b, w, y)\big)$
(6)$G(u, v, w) \leq G(u, b, b) + G(v, b, b) + G(w, b, b)$"

Definition (1.4) (Mustafa, Obiedat, Awawdeh**,**2008)
"Let (X, G) be a G-metric space, let (x_n) be a sequence of points of X a point $x \in X$ is said to be the limit of the sequence (x_n) if

$$\lim_{n,m\to\infty} G(x, x_n , x_m) = 0$$

Thus, that if $x_n \to x_0$ in a G-metric space (X, G),then for any $\varepsilon > 0$ there exists $K \in N$ such that $G(x, x_n, x_m) < \varepsilon$ for all $n, m \geq K$."

Definition (1.5) (Mustafa, Obiedat, Awawdeh, 2008)
"Let (X, G) be a G–metric space a sequence (x_n) is called G –Cauchy if given $\varepsilon > 0$, there is $K \in N$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq K$,that is , if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$."

Definition (1.6) (Mustafa, Obiedat, Awawdeh, 2008)
"A G-metric space (X, G) is said to be G-complete or (complete G-metric) if every G-cauchy sequence in (X, G) is convergent in (X, G)."

Definition (1.7)(Mustafa, Sims, 2006)

"Let (X, G) and (X', G') be two G-metric spaces, and let $f: (X, G) \to (X', G')$ be a function .Then f is said to be G-continuaus at a point $a \in X$ if and only if given $\varepsilon > 0$,there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta \Rightarrow G'(f(a), f(x), f(y)) < \varepsilon$

A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$."

Definition (1.8) (Mustafa, Obiedat, Awawdeh,2008)

"Let (X, G) be a G-metric spase, the mapping $T: X \to X$ then for all $x, y, z \in X$

i-T is called $G -$ contraction mapping if

$G(T(x), T(y), T(z)) \leq k\, G(x, y, z)$,forsome $k \in (0,1)$

ii-.T is called a G - contractive if

$G(T(x), T(y), T(z)) < G(x, y, z)$**,** for all x, y, z in X with $x \neq y \neq z$

iii-T is called G -expansive mapping if

$G(T(x), T(y), T(z)) \geq a\, G(x, y, z)$, for some $a > 1$"

"The version of Banach's fixed point Theorem in G-metric space is

Theorem (1.10) (Mustafa, Obiedat, Awawdeh,2008)

"If (X, G) be a complete G-metric space and $T: X \to X$ be a G – contraction mapping, then T has unique fixed point z in$X$, and $\lim_{n\to\infty} T^n(x) = z,$ for any intial point $x$in X."

II. Method

We begin with following

Theorem (2.1): Let Mbe a subset of a complete G-metric space and $T : X \to X$ be a mapping such that $\emptyset : X \to R^+ G(x, x, Tx) \leq \emptyset(x) - \emptyset(T(x))$, for all $x \in X$.

where$\emptyset$ is lower semi continuous function

Proof:

For $x_0 \in X\ and\ n, m \in N\ with\ n < m$, we have$\emptyset: X \to R$, then , by similar argument of proof of Theorem (2.1) in [2]

$$G\big(T^n(x_0), T^n(x_0), T^{m+1}(x_0)\big) \leq \sum_{i=n}^{m} G\left(T^i(x_0), T^i(x_0), T^{i+1}(x_0)\right)$$
$$\leq \emptyset\big(T^n(x_0)\big) - T^{m+1}(x_0)$$

In particular,

$$\sum_{i=0}^{\infty} G\left(T^i(x_0), T^i(x_0), T^{i+1}(x_0)\right) < \infty$$

Therefore, $(T^n(x_0))$is Cauchy sequence. Since T is continuous, then $(T^n(x_0))$ converges to a fixed point of T.

Definition(2.2):

A real valued function $\emptyset$ on X has a $G -$point $p \in X$ if

$$\emptyset(p) - \emptyset(x) < G(p, p, x), \quad \text{forother point } x \in X, x \neq p.$$

Proposition (2.3) :

Every lower semi continuous function $\emptyset: X \to R^+$on a complete X has a $G -$point p in X.

Proof:

By putting $T = I$ and $T(x) = p$ in theorem (2.1)**.**

Theorem (2.4)

Let M be a subset of a complete G-metric space X and $f, g : M \to X$ be maps such that

(i) f is surjective

(ii) There exist a lower semi continuous function $\emptyset : X \to R^+$ satisfying

$G(f(x), f(x), g(x)) \leq \emptyset(f(x)) - \emptyset(g(x)) \quad \ldots \quad (2.1)$

for each $x \epsilon M$. Then f and g have a coincidence point.

Proof:

By proposition **(**2.3**)**, then $\emptyset$ has a G-point $p \epsilon X$, means that

$$\emptyset(p) - \emptyset(x) < G(p, x, x)$$

Now, let $x \epsilon f^{-1}p$, suppose $fx = gx$ since $p = fx$ and $gx \epsilon X$, we have

$$\emptyset(f(x)) - \emptyset(g(x)) < G(f(x), f(x), g(x))$$

which contradicts (ii).

By putting $X = M$and $= I$, Theorem (2.1) reduces to the version of Caristi-Kirk Theorem in G-metric space: Consequently, we obtain the following:

Corollary (1)

(i) If $M = X$ and $f = I_x$,then the above theorem reduces to the version Caristi-Kirk -Browder theorem in this case ,if g is G-continuous then for any $x \in X$ the sequence $\{g_x^n\}$ G-converges to a fixed point of g

(ii) If $M = X$ and $g = 1_x$ .then f has a fixed point

Corollary (2)

Let X be a G-metric space and $f: X \to X$ be onto mapping such that for all x, y in X if there is a constant $a > 1$ such that

$G(f(x), f(x), f(y)) \geq a\, G(x, x, y)$ …. (2.2)

then$f$ has a unique fixed point

Proof:

From (2.2) f is clearly injective. Since f is also surjective, $g = f^{-1}$ exists and is surjective for any x, y in X we obtain, from (2.3)

$$G(x, x, y) \geq a\, G(gx, gx, gy)$$

and$g$ is G-continuous. One could use Theorem (1.10) at this point to prove that g has a unique fixed point. Adding $(a-1)G(x, x, y)$ to each side of the above inequality to get

$$a\, G(x, x, y) - a\, G(gx, gx, gy) \geq (a-1)G(x, x, y)$$

Now, put $y = gx$ to get

$G(x, x, gx) \leq \emptyset(x) - \emptyset(gx)$,

where, define $\emptyset$ as

$$\emptyset(x) = \frac{a\, G(x, x, gx)}{(a-1)}$$

since$g$ is G-continuous, $\emptyset$ is lower semi continuous , and g has a fixed point by Corollary (1-i). For any $x \in X$,the sequence $\{g^n x\}$$G$-converges to a fixed point of g , that is ,of f. From (2.2) the fixed point is unique.

Corollary (4)

Let X be a G-metric space and $f: X \to X$ be onto mapping such that for all x, y in X if there exist $a, b, c \geq 0$ with $a + b + c > 1$ and $a < 1$ such that

$G(f(x), f(x), f(y)) \geq a\, G(x, x, f(x)) + b\, G(y, y, f(y)) + c\, G(x, x, y)$….(2.3)

with$x \neq y$,then f has a fixed point

Proof:

Since (2.4) is symmetric in x and $y$assume that$a = b < 1$. Adding, a $G(fx, fx, fy)$ to both sides of (2.3) we have

$$(1+a)\, G(fx, fx, fy) \geq a[G(x, x, fx) + G(fx, fx, fy) + G(fy, y, y)] + c\, G(x, x, y) \geq (a+c)G(x, x, y)$$

or

$$G(fx, fx, fy) \geq \frac{a+c}{1+a} G(x, x, y)$$

since$a + c = 0$ implies $a = b > 1$, $(a+c)/(1+a) > 0$

and$f$ is injective. Since f is also surjective $g = f^{-1}$ exists. Also, since

$G(x, x, y) \geq \frac{a+c}{1+a} G(gx, gx, gy)$, for all $x, y \in X$,

and hence g is G-continuous. (2.3) will be in the form

$$G(x, x, y) \geq a\, G(gx, gx, x) + b\, G(gy, gy, y) + c\, G(gx, gx, gy)$$

set $y = gx$ and then add $(b + c + a - 1)G(x, x, gx)$

to each side to get

$$(b+c)[\, G(x, x, gx) - G(gx, gx, g^2x)] \geq (b + c + a - 1)G(x, x, gx)$$

or

$$G(x, x, gx) \leq \emptyset(x) - \emptyset(gx)$$

where defined

$$\emptyset(x) = \frac{(b+c)G(x, x, gx)}{(a+b+c-1)}$$

Since g is G-continuous, $\emptyset$ is (lsc) and g has a fixed point by Corollary (1-i). Moreover for $\forall x \in X$ the sequence$\{g^n x\}$$G$-converges to a fixed point of g, that is ,of$f$

Corollary (5)

Let X be a G-metric space and $f: X \to X$ be onto and G-continuous mapping such that for all $x \in X$ and if$\exists a > 1$ satisfying

$$G(f(x), f(x), f^2(x)) \geq a\, G(x, x, f(x)) \qquad \ldots.(2.4)$$

then f has a fixed point

Proof:

Adding $-G(x, x, f(x))$ to condition (2.4) yields

$$G(x, x, fx) \leq [G(fx, fx, f^2x) - G(x, x, fx)]/(a-1)$$

Define, $\emptyset: X \to R^+$ by $\emptyset(x) = \frac{G(x,x,f(x))}{a-1}$

since$f$ is G-continuous, $\emptyset$ is (lsc) and by corollary (1-ii) , f has fixed point

Corollary (6)

Let X be a G-metric space and $f: X \to X$ be onto and G-continuous mapping such that for all x, y in X and if there exists a real constant $a > 1$ such that

$$G(f(x), f(x), f(y)) \geq a\, min[G(x, x, f(x)), G(y, y, fy), G(x, x, y)] \ldots.(2.5)$$

then$f$ has a fixed point

Proof:

Set $y = f(x)$ in condition (2.5)

$$G(f(x), f(x), f(y)) \geq a\, min[G(x, x, fx), G(y, y, fy), G(x, x, y)]$$

yield to

$$G(f(x), f(x), f^2(x)) \geq a\, G\,(x, x, f(x)) \ldots..(2.4)$$

then$f$ has a fixed point .

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